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AN INVERSION FORMULA FOR HANKEL TRANSFORM

C. NASIM

In the present note we propose a procedure for inverting the Hankel transform of arbitrary order. The technique we use is similar to the one used by Widder, inasmuch as the inversion is accomplished in two steps.

The inversion of the sine transform and cosine transform become special cases of our result. In these instances the operators employed for inversion are different from those used in the known results.

Methods for inverting the Fourier sine and Fourier cosine transform using operators of differential and integral types, have been known for sometime now. Widder [5 (Th. 8), 6], has used techniques whereby the inversion can be accomplished in two steps; namely, first by the application of an integral operator, in this case the Laplace transform, followed by differential operator, which is the limit of a polynomial in $\theta(\theta \equiv -x(d/dx))$ or $D(D \equiv (d/dx))$.

2. Preliminaries.

DEFINITION. A function $E(s)$, belongs to the class E_0, if and only if

$$E(s) = e^{bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) \exp\left(\frac{s}{a_k}\right),$$

where α_k and b are real and finite, $\sum 1/a_k^2 < \infty$ and s, a complex number, [5; p. 174].

Some examples of the class E_0 are $\cos s$, $(\sin s)/s$, $(1/\Gamma(1-s))$. Note that $E(0) = 1$. We state a known results [4] for reference.

LEMMA. *If* (i) $E(s) = 1 \Big/ \int_0^\infty K(x) x^{s-1} dx$,

(ii) $E(s) \in E_0$,

(iii) *$\phi(t)$ is bounded and continuous in* $(0, \infty)$,

$$\text{(iv)} \quad f(x) = \int_0^\infty \frac{1}{t} K\left(\frac{x}{t}\right) \phi(t) dt, \quad x > 0 \tag{2.1}$$

then for almost all x,

$$E(\theta) f(x) = \phi(x), \quad \theta = -x \frac{d}{dx}.$$

Now, a few remarks about the algebra of operators involving θ.

Let $\theta = -x(d/dx)$, then $\theta^m[x^\alpha] = (-\alpha)^m x^\alpha$, $m = 0, 1, 2, \cdots$, whence

$$n^\theta[x^\alpha] = n^{-\alpha}x^\alpha . \tag{2.2}$$

Thus for any polynomial $P(\theta)$, $P(\theta)x^\alpha = P(-\alpha)x^\alpha$ and hence for $E(\theta)$, the limit of polynomials, $E(\theta)x^\alpha = E(-\alpha)x^\alpha$. Similarly, by using Gauss' formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} ,$$

one can deduce that

$$\frac{1}{\Gamma(a - b\theta)}[x^\alpha] = \frac{1}{\Gamma(\alpha + ba)}x^\alpha . \tag{2.3}$$

3. The main result.

THEOREM. *If*

(i) $\phi(x)$ *is Lebesgue integrable and continuous in* $(0, \infty)$

$$\text{(ii)}\quad f(x) = \int_0^\infty (xt)^{1/2}J_\nu(xt)\phi(t)dt , \left(\nu \geqq -\frac{1}{2}\right) \tag{3.1}$$

and

$$\text{(iii)}\quad R(x) = 2^{1/4-1/2\nu}\int_0^\infty (xy)^{\nu+1/2}e^{-1/2x^2y^2}f(y)dy , \tag{3.2}$$

then

$$\frac{2^{1/2\theta}}{\Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\theta + \frac{3}{4}\right)}R(x) = \phi(x) . \tag{3.3}$$

Proof. Consider the equation (3.1),

$$f(x) = \int_0^\infty (xt)^{1/2}J_\nu(xt)\phi(t)dt = \left(\int_0^\delta + \int_0^\infty\right)(xt)^{1/2}J_\nu(xt)\phi(t)dt$$
$$= x^{1/2}0\left(\int_0^\delta t^{\nu+1/2}\phi(t)dt\right) + x^{1/2}0\left(\int_\delta^\infty \phi(t)dt\right) .$$

By hypothesis (i), $\phi(t) \in L(0, \infty)$, therefore $t^{\nu+1/2}\phi(t) \in L(0, \delta)$, $\delta > 0$, $\nu \geqq -1/2$ and $\phi(t) \in L(\delta, \infty)$. Hence the above integral is absolutely convergent and $f(x)$ exists for $x \geqq 0$ as defined by (3.1). Also it can be verified that

$f(x) = O(x^\alpha)$, $\alpha > 0$ as $x \to 0$ and $f(x) = O(x^\beta)$, $\beta < 0$ as $x \to \infty$, hence $f(x)$ is bounded and continuous in $(0, \infty)$ and therefore $R(x)$ of (3.2) is well defined for $x \geqq 0$. Now

$$
\begin{aligned}
R(x) &= 2^{1/4-1/2\nu}\int_0^\infty (xy)^{\nu+1/2}e^{-1/2x^2y^2}f(y)dy \\
&= 2^{1/4-1/2\nu}\int_0^\infty (xy)^{\nu+1/2}e^{-1/2x^2y^2}dy\int_0^\infty (yt)^{1/2}J_\nu(yt)\phi(t)dt \\
&= 2^{1/4-1/2\nu}x^{\nu+1/2}\int_0^\infty t^{1/2}\phi(t)dt\int_0^\infty y^{\nu+1}e^{-1/2x^2y^2}J_\nu(yt)dy\ ,
\end{aligned}
\tag{3.4}
$$

by Fubini's theorem since $\phi(t) \in L(0, \infty)$.

The inner integral can be evaluated [2; p. 29 (10)] to give

$$
\int_0^\infty y^{\nu+1}e^{-1/2x^2y^2}J_\nu(yt)dy = x^{-2(\nu+1)}t^\nu e^{-t^2/2x^2}\ , \quad (R(\nu) > -1)\ .
$$

Thus

$$
\begin{aligned}
R(x) &= 2^{1/4-1/2\nu}x^{-(\nu+3/2)}\int_0^\infty t^{\nu+1/2}e^{-t^2/2x^2}\phi(t)dt \\
&= \int_0^\infty \frac{1}{t}K\Big(\frac{x}{t}\Big)\phi(t)dt\ , \quad \text{say}
\end{aligned}
\tag{3.5}
$$

where

$$
K(x) = 2^{1/4-1/2\nu}x^{-(\nu+3/2)}e^{-1/2x^2}\ ,
$$

and therefore

$$
1/E(s) = \int_0^\infty K(x)x^{s-1}dx = 2^{-1/2s}\Gamma\Big(\frac{1}{2}\nu - \frac{1}{2}s + \frac{3}{4}\Big), \Big(R(s) \leqq R(\nu) + \frac{3}{2}\Big)\ .
$$

It is clear that $E(s) \in E_0$ and hence applying the above lemma to (3.5), we have

$$
E(\theta)R(x) = \phi(x)\ ,
$$

where

$$
E(\theta) = \frac{2^{1/2\theta}}{\Gamma\Big(\frac{1}{2}\nu - \frac{1}{2}\theta + \frac{3}{4}\Big)} \tag{3.6}
$$

If we put $\nu = -1/2$ and $1/2$ in the above theorem, we obtain the inversion for cosine transform and sine transform respectively.

The scope of the main result is illustrated by applying it to $\phi(x)$, where, for example, $\phi(x)$ is $x^{\nu+1/2}e^{-ax^2}$, $(a > 0, R(\nu) > -1)$ or $x^{\nu+1/2}(1 + x^2)^{-1}$, $(R(\nu) > -1)$. The assumption on $\phi(x)$, is by no means best possible, since one can verify that the result still holds if we take, $\phi(x)$ to be x^μ or $x^{-1/2}J_{\nu+1}(ax)$, with appropriate restrictions on μ, ν and a, although it does not satisfy assumption (i) of the theorem.

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