PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 1, 1975

ALMOST PERIODIC COMPACTIFICATIONS OF TRANSFORMATION SEMIGROUPS

H. D. JUNGHENN

In this paper we generalize the notion of (weakly) almost periodic compactification of a semitopological semigroup to the corresponding notion for transformation semigroups. The properties of these compactifications are studied and applications are made to semidirect products.

0. **Introduction.** Let X be a Hausdorff topological space and S a semitopological semigroup [2]. The pair (S, X) is a *semitopological transformation semigroup* (abbr. s.t.t.s.) if there exists a separately continuous mapping $(s, x) \rightarrow sx$ of $S \times X$ into X (called an *action*) such that $s(tx) = (st)x(s, t \in S; x \in X)$. If S has an identity 1 and $1x = x(x \in X)$ we say that (S, X) *has an identity*. If S is a topological semigroup and the action is jointly continuous, then (S, X) is called a *topological transformation semigroup* (abbr. t.t.s.). (S, X) is *compact* if both S and X are compact. Any semitopological semigroup may be considered to be a s.t.t.s., where the action is left multiplication.

Let (S, X) be a s.t.t.s. and denote by $C(X)$ the Banach space of all bounded continuous complex-valued functions on X. A function f in $C(X)$ is (*weakly*) *almost periodic* if $O(f) = \{{}_sf : s \in S\}$ is relatively (weakly) compact in $C(X)$, where ${}_sf(x) = f(sx)(x \in X)$. The set of all almost periodic (resp. weakly almost periodic) functions in $C(X)$ is denoted by $A(X)$ (resp. $W(X)$). Both $A(X)$ and $W(X)$ are C^*-subalgebras of $C(X)$ which are invariant under the action $s \rightarrow {}_sf$ [2].

A *homomorphism* of a s t.t.s. (S, X) into a s.t.t.s. (T, Y) is a pair (σ, ξ), where $\sigma: S \rightarrow T$ is a continuous homomorphism and $\xi: X \rightarrow Y$ a continuous map such that $\xi(sx) = \sigma(s)\xi(x)(x \in X, s \in S)$. The *dual* of ξ is the map $\tilde{\xi}: C(Y) \rightarrow C(X)$ defined by $\tilde{\xi}(f) = f \circ \xi$. Clearly $\tilde{\xi}$ is a bounded linear operator.

Recall that a weakly almost periodic (resp. almost periodic) compactification of a semitopological semigroup S may be defined as a pair $(\bar{S}, \rho)$, where $\bar{S}$ is a compact semitopological (resp. compact topological) semigroup and $\rho: S \rightarrow \bar{S}$ is a continuous homomorphism such that $\rho(S)$ is dense in $\bar{S}$ and $\tilde{\rho}(C(\bar{S})) = W(S)$ (resp. $\tilde{\rho}(C(\bar{S})) = A(S)$) (see [6], [8]). Motivated by this we define a *weakly almost periodic* (resp. *almost periodic*) *compactification* of a s.t.t.s. (S, X) as a compact s.t.t.s. (resp. compact t.t.s.) $(\bar{S}, \bar{X})$ and a homomorphism (ρ, η) of (S, X) into $(\bar{S}, \bar{X})$ such that $(\bar{S}, \rho)$ is a weakly almost periodic (resp. almost periodic) compactification of S, $\eta(X)$ is dense in $\bar{X}$, and $\tilde{\eta}(C(\bar{X})) = W(X)$ (resp. $\tilde{\eta}(C(\bar{X})) = A(X)$). Clearly $\tilde{\eta}$ (and $\tilde{\rho}$) is an isometric isomorphism. We shall write $\hat{f}$ for $\tilde{\eta}^{-1}(f)$. We shall also occasionally use the notation (S^w, X^w) and (S^a, X^a) for weakly almost periodic and almost periodic compactifications respectively.

In §1 we shall show that if S has a (weakly) almost periodic compactification then (S, X) has a (weakly) almost periodic compactification. Furthermore, a compactification is unique up to isomorphism and satisfies a universal factorization property analogous to that satisfied by a (weakly) almost periodic compactification of S. These results are extensions of results in [11].

In §2 some specialized lemmas are proved, and in §3 we use these to characterize almost periodic compactifications of semidirect products. It is shown that if S is a topological group, then for a large class of semitopological semigroups X (including abelian semigroups, compact topological semigroups, and topological groups), the almost periodic compactification of a semidirect product $S \circledS X$ of S and X is a semidirect product of $\bar{S}$ and $\bar{X}$, where $\bar{S}$ is the almost periodic compactification of S, and $\bar{X}$ is a certain compactification of X. These results generalize Theorem 4 of [11]; for other results along this line see [5]. We also show that the kernel of an almost periodic compactification of $S \circledS X$ may be expressed as a semidirect product of the kernels of $\bar{S}$ and $\bar{X}$. A similar result is obtained for the weakly almost periodic case.

1. Existence and uniqueness of compactifications.

LEMMA 1.1. *Let (S, X) be a s.t.t.s.*

(a) *If (S, X) is compact then $W(X) = C(X)$.*

(b) *If (S, X) is a compact t.t.s. then $A(X) = C(X)$.*

(c) *If $f \in W(X)$ then the map $s \to {}_sf$, $S \to W(X)$ is continuous in the weak topology.*

(d) *If $f \in A(X)$ then the map $s \to {}_sf$, $S \to A(X)$ is continuous in the norm topology.*

Proof. We prove only (a) and (c), the proofs of (b) and (d) being similar. Let (S, X) be compact and $f \in C(X)$. Since $s \to {}_sf$ is pointwise continuous, $O(f)$ is compact in the pointwise topology of $C(X)$. Since this topology agrees with the weak topology on norm-bounded pointwise compact subsets of $C(X)$ [10], $O(f)$ is weakly compact, proving (a). For (c) simply observe that if (s_α) is a net in S and $s_\alpha \to s$, then $({}_{s_\alpha}f)$ has a unique weak limit point in $C(X)$.

LEMMA 1.2. *Let* (σ, ξ) *be a homomorphism of* (S, X) *into* (T, Y). *Then*

$$\tilde{\xi}(W(Y)) \subset W(X) \cap \tilde{\xi}(C(Y)) , \tag{1}$$

and equality holds if $\xi(X)$ *is dense in* Y *and* $\sigma(S)$ *is dense in* T. *The analogous statement holds for the almost periodic case.*

Proof. If $f \in C(Y)$ and $g = \tilde{\xi}(f)$, then

$${}_s g = \tilde{\xi}({}_{\sigma(s)}f) \quad (s \in S) . \tag{2}$$

If $f \in W(Y)$ then $\tilde{\xi}(O(f))$ is relatively weakly compact, and (2) shows that $g \in W(X)$, verifying (1). If $\xi(X)$ and $\sigma(S)$ are dense in Y and T respectively, then $\tilde{\xi}$ is an isometry and (2) implies that $\tilde{\xi}^{-1}(\bar{O}(g)) = \bar{O}(f)$ (bars denote weak closures). Hence if $g \in W(X)$, then $f \in W(Y)$, verifying equality in (1).

THEOREM 1.3. *If* (S, X) *is a s.t.t.s. and if* S *has a weakly almost periodic* (resp. almost periodic) *compactification, then* (S, X) *has a weakly almost periodic* (resp. almost periodic) *compactification. Moreover, any weakly almost periodic* (resp. almost periodic) *compactification* $(\bar{S}, \bar{X}, \rho, \eta)$ *satisfies the following universal property: Given any homomorphism* (σ, ξ) *of* (S, X) *into a compact s.t.t.s.* (resp. compact t.t.s.) (T, Y), *there exists a homomorphism* $(\bar{\sigma}, \bar{\xi})$ *of* $(\bar{S}, \bar{X})$ *into* (T, Y) *such that* $\bar{\sigma} \circ \rho = \sigma$ *and* $\bar{\xi} \circ \eta = \xi$.

The following corollaries are immediate.

COROLLARY 1.4. *Let* (S_i, X_i) *be a s.t.t.s. with weakly almost periodic compactification* $(\bar{S}_i, \bar{X}_i, \rho_i, \eta_i)(i = 1, 2)$. *If* (σ, ξ) *is a homomorphism of* (S_1, X_1) *into* (S_2, X_2) *then there exists a homomorphism* $(\bar{\sigma}, \bar{\xi})$ *of* $(\bar{S}_1, \bar{X}_1)$ *into* $(\bar{S}_2, \bar{X}_2)$ *such that* $\bar{\sigma} \circ \rho_1 = \rho_2 \circ \sigma$ *and* $\bar{\xi} \circ \eta_1 = \eta_2 \circ \xi$. *A similar statement holds for the almost periodic case.*

COROLLARY 1.5. *Weakly almost periodic compactifications and almost periodic compactifications are unique* (up to isomorphism).

Proof of Theorem 1.3. Let $(\bar{S}, \rho)$ be a weakly almost periodic compactification of S and let $\bar{X}$ be the maximal ideal space of $W(X)$. Define $\eta: X \to \bar{X}$ by $\eta(x)(f) = f(x)(x \in X, f \in W(X))$. Then X is compact, η is continuous and $\eta(X)$ is dense in $\bar{X}$. Furthermore, if $\hat{f} \in C(\bar{X})$ denotes the Gelfand transform of $f \in W(X)$, then $\tilde{\eta}(\hat{f}) = f$.

Define $\pi: S \times \bar{X} \to \bar{X}$ by $\pi(s, \theta)(f) = \theta({}_s f)(s \in S, \theta \in \bar{X}, f \in W(X))$. Clearly, $\pi(s, \cdot)$ is continuous, and $s \to \pi(s, \cdot)$ is a continuous homomorphism from S into $\bar{X}^{\bar{X}}$ (Lemma 1.1(c)), where $\bar{X}^{\bar{X}}$ carries the product topology. Let E denote the closure in $\bar{X}^{\bar{X}}$ of $\{\pi(s, \cdot): s \in S\}$. E will be a compact semitopological semigroup if we show that each $v \in E$ is continuous. Since $\bar{X}$ is compact it suffices to show that $g \circ v$ is continuous for arbitrary $g \in C(\bar{X})$. Let (s_α) be a net in S such that $\pi(s_\alpha, \cdot) \to v$, and let $g = \hat{f}$, where $f \in W(X)$. For each $\theta \in \bar{X}$, $\theta({}_{s_\alpha}f) = \hat{f}(\pi(s_\alpha, \theta)) \to g(v(\theta))$. Also, there exists some $h \in W(X)$ and a subnet (s_β) such that ${}_{s_\beta}f \to h$ weakly. Thus $g \circ v = \hat{h} \in C(\bar{X})$.

By Theorem 2.2 of [8] there exists a continuous homomorphism $u \to \bar{\pi}(u, \cdot)$ of $\bar{S}$ onto E such that $\bar{\pi}(\rho(s), \cdot) = \pi(s, \cdot)(s \in S)$. Define an action of $\bar{S}$ on $\bar{X}$ by $u\theta = \bar{\pi}(u, \theta)(u \in \bar{S}, \theta \in \bar{X})$. With this action $(\bar{S}, \bar{X}, \rho, \eta)$ is a weakly almost periodic compactification of (S, X).

Now let $(\bar{S}, \bar{X}, \rho, \eta)$ be any weakly almost periodic compactification of (S, X), and let (σ, ξ) be a homomorphism into the compact s.t.t.s. (T, Y). Let $\bar{\sigma}: \bar{S} \to T$ be a continuous homomorphism such that $\bar{\sigma} \circ \rho = \sigma$. Define $\bar{\xi}: \eta(X) \to Y$ by $\bar{\xi}(\eta(x)) = \xi(x)$. If $\xi(x_1) \neq \xi(x_2)$ choose $g \in C(Y)$ such that $g(\xi(x_1)) \neq g(\xi(x_2))$. Then $f = \xi(g) \in W(X)$ (Lemmas 1.1, 1.2) and $\hat{f}(\eta(x_1)) = f(x_1) \neq f(x_2) = \hat{f}(\eta(x_2))$, so $\eta(x_1) \neq \eta(x_2)$. Thus $\bar{\xi}$ is well defined. Now for *any* $g \in C(Y)$, if $f = \xi(g)$ then $\hat{f}|\eta(X) = g \circ \bar{\xi}$, hence $g \circ \bar{\xi}$ is uniformly continuous. Since Y is compact, its uniform structure is defined by $C(Y)$, hence $\bar{\xi}$ is uniformly continuous.

We may now extend $\bar{\xi}$ continuously to $\bar{X}$. Since $\bar{\xi}(\rho(s)\eta(x)) = \bar{\xi}(\eta(sx)) = \xi(sx) = \sigma(s)\xi(x) = \bar{\sigma}(\rho(s))\bar{\xi}(\eta(x))(s \in S, x \in X)$, $(\bar{\sigma}, \bar{\xi})$ is a homomorphism.

The almost periodic case is proved similarly except that the set E must be shown to be a topological semigroup. This follows readily from the equicontinuity of $\{\pi(s, \cdot): s \in S\}$.

REMARK 1.6. In the almost periodic case of Corollary 1.4 it may be shown that if X_2 is compact and if the action of S_2 on X_2 is equicontinuous, then $\eta_2: X_2 \to \bar{X}_2$ is a homeomorphism and therefore $\bar{X}_2$ may be replaced by X_2 in the conclusion of the corollary. This means that $\bar{S}_2$ acts on X_2 such that $\rho_2(s)x = sx(s \in S_2, x \in X_2)$, and that $\bar{\xi} \circ \eta_1 = \xi$.

The following theorem exhibits the connection between our approach to almost periodic compactifications and that of Landstad [11].

THEOREM 1.7. *Let $(\bar{S}, \bar{X}, \rho, \eta)$ be an almost periodic compactification of the s.t.t.s. (S, X) and let $\mathscr{U}$ denote the coarsest uniform structure on X relative to which each $f \in A(X)$ is uniformly continuous. Then $(\bar{X}, \eta)$ is a Hausdorff completion of $(X, \mathscr{U})$. Furthermore, $\mathscr{U}$ is the finest uniform structure $\mathscr{V}$ on X satisfying the following properties*:

(a) *$\mathscr{V}$ defines a topology $\mathscr{T}(\mathscr{V})$ on X coarser than the given* *one,*

(b) $\mathscr{V}$ *is totally bounded, and*

(c) *the family of mappings* $x \to sx$ $(x \in X, s \in S)$ *is* $\mathscr{V}$ *uniformly equicontinuous (equivalently,* $\mathscr{V}$ *has a base consisting of those* $V \in \mathscr{V}$ *such that* $(x, y) \in V$ *implies that* $(sx, sy) \in V$ *for all* $s \in S$).

Proof. Clearly $\mathscr{U}$ satisfies (a). $\mathscr{U}$ satisfies (b) because $\eta(X)$ is totally bounded and $\mathscr{U}$ is the coarsest uniform structure on X making η uniformly continuous. These facts also imply that (X, η) is a Hausdorff completion of $(X, \mathscr{U})$. (See for example [3].) That $\mathscr{U}$ satisfies (c) follows from the total boundedness of $O(f)$ in $C(X)$ for each $f \in A(X)$.

Now let $\mathscr{V}$ be a uniform structure on X satisfying (a), (b), and (c), and let (Y, ξ) be the Hausdorff completion of $(X, \mathscr{V})$. By (b), Y is compact. For each $s \in S$, $x \to \xi(sx)$ is $\mathscr{V}$-uniformly continuous, hence there exists a uniformly continuous function $\sigma(s): Y \to Y$ such that $\sigma(s)(\xi(x)) = \xi(sx)$ $(x \in X)$. From (a) and (c) it follows that $\sigma: S \to Y^Y$ is a continuous homomorphism and that $F = \{\sigma(s): s \in S\}$ is uniformly equicontinuous. Let T be the closure of F in the product space Y^Y. Then T is a compact topological semigroup and $vy = v(y)(v \in T, y \in Y)$ defines a jointly continuous action of T on Y. Furthermore, (σ, ξ) is a homomorphism of (S, X) into (T, Y), so there exists a uniformly continuous map $\bar{\xi}: \bar{X} \to Y$ such that $\bar{\xi} \circ \eta = \xi$ (Theorem 1.3). Thus ξ is $\mathscr{U}$-uniformly continuous, and it follows that $\mathscr{V}$ is coarser than $\mathscr{U}$.

2. More lemmas.

We shall assume throughout this section that (S, X) is a s.t.t.s. with identity 1, that X is a semitopological semigroup with identity 1, and that there exists a continuous homomorphism $\phi: X \to S$ such that $\phi(1) = 1$ and $\phi(x)y = xy(x, y \in X)$. If $f \in C(X)$ and $y \in X$, f_y shall denote the function $x \to f(xy)$. A subspace L of $C(X)$ is said to be *right translation invariant* if $f \in L$ implies that $f_y \in L$ for all $y \in X$.

We shall denote by $K(Q)$ the minimal ideal of the compact semitopological semigroup Q (see [6]). If B is a Banach space, $L(B)$ and B^* are respectively the space of continuous linear operators and the space of continuous linear functionals on B.

LEMMA 2.1. *Let* $(\bar{S}, \bar{X}, \rho, \eta)$ *be an almost periodic* (resp. weakly almost periodic) *compactification of* (S, X). *Then in order for* $\bar{X}$ *to be a topological* (resp. semitopological) *semigroup and* η *a homomorphism it is necessary and sufficient that* $A(X)$ (resp. $W(X)$) *be right translation invariant.*

Proof. The necessity is clear. For the sufficiency we consider only the weakly almost periodic case. By Corollary 1.5 and the proof of Theorem 1.3 we may suppose that $\bar{X}$ is the maximal ideal space of $W(X)$ and that η is the map defined by $\eta(x)(f) = f(x)(x \in X, f \in W(X))$. Define $u(s) \in L(W(X))(s \in S)$ *by* $u(s)(f) = {}_sf$, and let T denote the closure of $\{u(s): s \in \phi(X)\}$ in the weak operator topology of $L(W(X))$. Then T with this topology is a compact semitopological semigroup (under the operation composition) [6; Theorem 3.1]. Given $v \in T$ define $\Psi(v) \in W(X)^*$ by $\Psi(v)(f) = (vf)(1)(f \in W(X))$. Clearly, Ψ is continuous in the weak* topology of $W(X)^*$, and $\Psi(u(\phi(x))) = \eta(x)(x \in X)$. It follows that $\Psi(T) = \bar{X}$. If $v, w \in T$ and $\Psi(v) = \Psi(w)$, then $(vf)(x) = \Psi(v)(f_x) = \Psi(w)(f_x) = (wf)(x)(x \in X, f \in W(X))$, so $v = w$. Therefore Ψ is a homeomorphism of T onto $\bar{X}$ and hence induces a multiplication on $\bar{X}$ making the latter a semitopological semigroup and Ψ an anti-isomorphism. Finally, $\eta(xy) = \Psi(u(\phi(xy))) = \Psi(u(\phi(y))u(\phi(x))) = \eta(x)\eta(y)$, completing proof.

The proofs of the following two lemmas are straightforward and therefore omitted.

LEMMA 2.2. *Let $(\bar{S}, \bar{X}, \rho, \eta)$ be a weakly almost periodic compactification of (S, X) and let $W(X)$ be right translation invariant. If $\bar{X}$ has a unique minimal left ideal and if $K(\bar{S})$ is a group with identity e, then $e\theta = \theta$ for all $\theta \in K(\bar{X})$.*

LEMMA 2.3. *Let (S^a, X^a, ρ, η) and (S^w, X^w, ρ', η') denote respectively almost periodic and weakly almost periodic compactifications of (S, X). Suppose $K(S^w)$ is a group with identity e, $W(X)$ and $A(X)$ are right translation invariant, X^w has a unique minimal left ideal, and for some idempotent $d \in K(X^w)$, $u(\theta d) = (u\theta)d(\theta \in X^w, u \in S^w)$. Then $K(X^a)$ and $K(X^w)$ are canonically isomorphic as semitopological semigroups.*

3. Applications to semidirect products. Let S and X be semitopological semigroups with identities, and let $\tau: S \times X \to X$ satisfy $\tau(s, xy) = \tau(s, x)\tau(s, y)$, $\tau(st, x) = \tau(s, \tau(t, x))$, and $\tau(1, x) = x$, $(x, y \in X; s, t \in S)$. Thus $s \to \tau(s, \cdot)$ is a homomorphism from S into $\operatorname{Hom}(X)$, the semigroup of all homomorphisms from X to X. We shall assume that $\tau(s, \cdot)$ is continuous for each $s \in S$ and that $(s, x) \to x\tau(s, y)$ is continuous for each $y \in X$. The *semidirect product* $S \textcircled{\tau} X$ *of* S and X is the topological space $S \times X$ with multiplication defined by $(s, x)(t, y) = (st, x\tau(s, y))(s, t \in S; x, y \in X)$. The above assumptions on τ imply that $S \textcircled{\tau} X$ is a semitopological semigroup with identity $(1, 1)$.

Following Landstad [11] we define an action of $S \textcircled{\tau} X$ on X by $(s, x)y = x\tau(s, y)(x, y \in X; s \in S)$. Let $A(X)$ (resp. $W(X)$) denote the almost periodic (resp. weakly almost periodic) functions on X relative to this action, and let $((S \textcircled{\tau} X)^a, X^a, \rho_0, \eta)$ be an almost periodic compactification of $(S \textcircled{\tau} X, X)$ and (S^α, ρ) an almost periodic compactification of S. Both of these compactification exist because $S \textcircled{\tau} X$ and S have identities. The map $\phi: X \to S \textcircled{\tau} X$, $\phi(x) = (1, x)$, is a homomorphism satisfying $\phi(x)y = xy$, hence the results of §2 apply. Thus, assuming $A(X)$ is right translation invariant, X^a is a topological semigroup and η a continuous homomorphism.

Define $\pi: S \times X^a \to X^a$ by $\pi(s, \theta) = \rho_0(s, 1)\theta$. Then $s \to \pi(s, \cdot)$ is a continuous homomorphism of S into $\mathrm{Hom}(X^a)$. Furthermore, $\{\pi(s, \cdot): s \in S\}$ is equicontinuous hence its closure E in the product space $X^{a^{X^a}}$ is a compact topological semigroup contained in $\mathrm{Hom}(X^a)$. Thus there exists a continuous homomorphism $u \to \bar{\tau}(u, \cdot)$ of S^a into E such that $\bar{\tau}(\rho(s), \cdot) = \pi(s, \cdot)(s \in S)$. Note that

$$\bar{\tau}(\rho(s), \eta(x)) = \eta(\tau(s, x)) \qquad (s \in S,\ x \in X)\ . \tag{1}$$

We may now form the semidirect product $S^a \textcircled{\bar{\tau}} X^a$, where multiplication is defined by

$$(u, \theta)(v, \psi) = (uv, \theta\bar{\tau}(u, \psi)) \quad (u, v \in S^a;\ \theta, \psi \in X^a)\ . \tag{2}$$

It follows from (1) that the map $\beta: S \textcircled{\tau} X \to S^a \textcircled{\bar{\tau}} X^a$ defined by $\beta(s, x) = (\rho(s), \eta(x))$ is a homomorphism.

THEOREM 3.1. *If S is a topological group and $A(X)$ is right translation invariant, then $(S^a \textcircled{\bar{\tau}} X^a, \beta)$ is an almost periodic compactification of $S \textcircled{\tau} X$, and*

$$K(S^a \textcircled{\bar{\tau}} X^a) = S^a \textcircled{\bar{\tau}} K(X^a)\ . \tag{3}$$

Proof. For the first part it suffices to show that given any continuous homomorphism α from $S \textcircled{\tau} X$ into a compact topological semigroup T, there exists a continuous homomorphism $\bar{\alpha}: S^a \textcircled{\bar{\tau}} X^a \to T$ such that $\bar{\alpha} \circ \beta = \alpha$. Following Landstad we define an action of $S \textcircled{\tau} X$ on T by

$$(s, x)t = \alpha(s, x)t\alpha(s^{-1}, 1) \qquad (s \in S,\ x \in X,\ t \in T).$$

Define $\alpha_1: S \to T$ by $\alpha_1(s) = \alpha(s, 1)$ and $\alpha_2: X \to T$ by $\alpha_2(x) = \alpha(1, x)$. Then α_1 is a continuous homomorphism, hence there exists a continuous homomorphism $\bar{\alpha}_1: S^a \to T$ such that $\bar{\alpha}_1 \circ \rho = \alpha_1$. Also, (ι, α_2) is a homomorphism of $(S \textcircled{\tau} X, X)$ into $(S \textcircled{\tau} X, T)$, where $\iota: S \textcircled{\tau} X \to S \textcircled{\tau} X$ is the identity map. Since the action on T is equicontinuous, Remark 1.6 implies the existence of a continuous map $\bar{\alpha}_2: X^a \to T$ such that $\bar{\alpha}_2 \circ \eta = \alpha_2$. The required map $\bar{\alpha}$ is defined by $\bar{\alpha}(u, \theta) = \bar{\alpha}_2(\theta)\bar{\alpha}_1(u)$, $(u \in S^a,\ \theta \in X^a)$.

To verify (3) note first that $\bar{\tau}(u, K(X^a)) \subset K(X^a)(u \in S^a)$ so that $S^a \textcircled{\bar{\tau}} K(X^a)$ is defined and is an ideal, and $K(S^a \textcircled{\bar{\tau}} X^a) \subset S^a \textcircled{\bar{\tau}} K(X^a)$. Now let $(u, \theta) \in S^a \times K(X^a)$ and choose $e \in K(X^a)$ such that $e^2 = e$ and $e\theta = \theta$ [6; Theorem 2.3]. Since $(\rho(1), e)(u, \theta) = (u, \theta)$, it suffices to show that $(\rho(1), e) \in K(S^a \textcircled{\bar{\tau}} X^a)$. Now, $S^a \times eX^a$ is a right ideal in $S^a \textcircled{\bar{\tau}} X^a$ and hence contains an idempotent $d \in K(S^a \textcircled{\bar{\tau}} X^a)$ [6; Lemma 2.2, Theorem 2.3]. It is easily seen that $d = (\rho(1), e_1)$, where $e_1^2 = e_1 \in K(X^a)$. Then $S^a \times e_1X^a = d(S^a \textcircled{\bar{\tau}} X^a) \subset S^a \times eX^a$, and by the minimality of eX^a, $e_1X^a = eX^a$. Therefore $(\rho(1), e) \in d(S^a \textcircled{\bar{\tau}} X^a) \subset K(S^a \textcircled{\bar{\tau}} X^a)$.

The next result shows that the conclusions of Theorem 3.1 are valid for a large class of semitopological semigroups X. Theorems 3.1 and 3.2 generalize Theorem 4 of [11].

THEOREM 3.2. *If S is a topological group then $A(X)$ is right translation invariant if any one of the following conditions holds*:
(i) *X is abelian.*
(ii) *X is a compact topological semigroup.*
(iii) *X is a topological group.*

Proof. Clearly (i) implies that $A(X)$ is right translation invariant. Suppose (ii) holds and let $f \in A(X)$, $z \in X$ and (s_α, x_α) a net in $S \times X$. There exists $g \in A(X)$, $y \in X$ and a subnet (s_β, x_β) such that $z_\beta = \tau(s_\beta^{-1}, z) \to y$ and $f(x_\beta \tau(s_\beta, x)) \to g(x)$ uniformly in $x \in X$. Then $g(xz_\beta) \to g(xy)$ uniformly in $x \in X$, hence by the triangle inequality $f_z(x_\beta \tau(s_\beta, x)) = f(x_\beta \tau(s_\beta, xz_\beta)) \to g(xy)$ uniformly in $x \in X$. Therefore $f_z \in A(X)$.

Now suppose (iii) holds. Let $\mathscr{U}$ denote the coarsest uniform structure on X making each $f \in A(X)$ uniformly continuous. We shall show that for each $x_0 \in X$, $x \to xx_0$ is $\mathscr{U}$-uniformly continuous. Let $\mathscr{B}$ denote the base for $\mathscr{U}$ consisting of all symmetric $U \in \mathscr{U}$ such that $(x, y) \in U$ implies that $((s, z)x, (s, z)y) \in U$ for all $(s, z) \in S \textcircled{\tau} X$ (see Theorem 1.7). If $U \in \mathscr{B}$ and $X = \bigcup_{i=1}^n U[x_i]$, define

$$[U] = [U; x_1, \cdots, x_n] = \{(x, y) : (xx_i, yx_i) \in U^2 (1 \leqq i \leqq n)\} .$$

Let $\mathscr{C}$ be the set of all $[U]$ $(U \in \mathscr{B})$. Claim that $\mathscr{C}$ is a base for a uniform structure $\mathscr{V}$ on X. For it is clear that $(x, x) \in [U] (x \in X)$ and $[U]^{-1} = [U]$, and the remaining axioms are easily verified with the help of the inclusion $[U; x_1, \cdots, x_n]^2 \subset [U^2; x_1, \cdots, x_n]$ and the following fact (whose simple proof we omit):

(4) $U, V \in \mathscr{B}$ and $V^2 \subset U$ implies $[V; y_1, \cdots, y_m] \subset [U; x_1, \cdots, x_n]$.

We shall show that $\mathscr{V}$ is coarser than $\mathscr{U}$ by verifying that (a), (b), and (c) of Theorem 1.7 hold. Let $\mathscr{T}$ denote the given topology of X and $\mathscr{T}(\mathscr{U})$ and $\mathscr{T}(\mathscr{V})$ the topologies induced by the uniform structures $\mathscr{U}$ and $\mathscr{V}$ respectively. If $U \in \mathscr{B}$ then for any $x \in X$, $[U; x_1, \cdots, x_n][x] = \bigcap_{i=1}^{n} xU^2[x_i]x_i^{-1}$ is a $\mathscr{T}$-neighborhood of x since $\mathscr{T}(\mathscr{U})$ is coarser than $\mathscr{T}$ and $\mathscr{T}$ is a group topology. Thus $\mathscr{T}(\mathscr{V})$ is coarser than $\mathscr{T}$, verifying (a). Call a neighborhood N of 1 in $(X, \mathscr{T})$ *left relatively dense* with respect to $y_1, \cdots, y_k \in X$ if $X = \bigcup_{i=1}^{k} y_i N$. Let $[U] = [U; x_1, \cdots, x_n] \in \mathscr{C}$. For each $j = 1, \cdots, n$, $U[x_j]x_j^{-1} = x_j U[1]x_j^{-1}$ is left relatively dense with respect to $x_i x_j^{-1} (1 \leqq i \leqq n)$, and since $(x_j U[1]x_j^{-1})^{-1}(x_j U[1]x_j^{-1}) = x_j U[1]^{-1} U[1] x_j^{-1} \subset x_j U^2[1]x_j^{-1} = U^2[x_j]x_j^{-1}$, it follows from Proposition 3 of [1] that $\bigcap_{j=1}^{n} U^2[x_j]x_j^{-1}$ is left relatively dense with respect to say $y_1, \cdots, y_k$. Thus $X = \bigcup_{i=1}^{k} [U][y_i]$, verifying (b). To verify (c) let $[U] \in \mathscr{C}$ and choose $[V] = [V; x_1, \cdots, x_n] \in \mathscr{C}$ such that $V^4 \subset U$. Let $(x, y) \in [V]$, $z \in X$ and $s \in S$. For each $1 \leqq i \leqq n$ there exists j such that $(x_j, \tau(s^{-1}, x_i)) \in V$. Then $(xx_j, x\tau(s^{-1}, x_i))$, $(yx_j, y\tau(s^{-1}, x_i)) \in V$ hence $(x\tau(s^{-1}, x_i), y\tau(s^{-1}, x_i)) \in V^4$ and so $(z\tau(s, x)x_i, z\tau(s, y)x_i) \in V^4$. By (4) then, $((s, z)x, (s, z)y) \in [U]$. Therefore $\mathscr{V}$ is coarser than $\mathscr{U}$.

Now let $x_0 \in X$ and $U \in \mathscr{B}$. If $X = \bigcup_{i=1}^{n} U[x_i]$, then $[U] = [U; x_0, x_1, \cdots, x_n] \in \mathscr{U}$, and $(x, y) \in [U]$ implies that $(xx_0, yx_0) \in U^2$. Therefore $x \to xx_0$ is $\mathscr{U}$-uniformly continuous and it follows from Theorem 1.7 that there exists a uniformly continuous function $F: X^a \to X^a$ such that $F(\eta(x)) = \eta(xx_0)(x \in X)$. Thus if $f \in A(X)$ then $\hat{f} \circ F \in C(X^a)$, hence $f_{x_0} = \tilde{\eta}(\hat{f} \circ F) \in A(X)$.

Now let $((S \textcircled{\tau} X)^w, X^w, \rho_0', \eta')$ and (S^w, ρ') be weakly almost periodic compactifications of $(S \textcircled{\tau} X, X)$ and S respectively, and assume that $W(X)$ is right translation invariant. As in the almost periodic case, we may form the semidirect product $S^w \textcircled{\bar{\tau}} X^w$, and the analogs of (1) and (2) are valid. However, $S^w \textcircled{\bar{\tau}} X^w$ need not be a weakly almost periodic compactification of $S \textcircled{\tau} X$, even in the case of a direct product [7]. We can, however, express $K((S \textcircled{\tau} X)^w)$ under certain conditions as a semidirect product.

THEOREM 3.3. *Let S be a topological group, $A(X)$ and $W(X)$ right translation invariant, and assume that $K((S \textcircled{\tau} X)^w)$ and $K(X^w)$ are groups. Then $K((S \textcircled{\tau} X)^w)$ and $K(S^w) \textcircled{\bar{\tau}} K(X^w)$ are canonically isomorphic.*

Proof. Note first that since S is a topological group, $W(S)$ has an invariant mean [12], hence $K(S^w)$ is a group [6]. Let $\gamma: S^w \to S^a$ be a continuous homomorphism such that $\gamma \circ \rho' = \rho$, and let (σ, ξ) be a homomorphism of $((S \textcircled{\tau} X)^w, X^w)$ onto $((S \textcircled{\tau} X)^a, X^a)$ such that $\sigma \circ \rho_0' = \rho_0$ and $\xi \circ \eta' = \eta$. If d denotes the identity of $K(X^w)$, then by the analog of (1), $u(\theta d) = (u\theta)d (u \in (S \textcircled{\tau} X)^w, \theta \in X^w)$, hence by

Lemma 2.3, $\xi_0 = \xi \mid K(X^w)$ is an isomorphism of $K(X^w)$ onto $K(X^a)$. By the same lemma $\gamma_0 = \gamma \mid K(S^w)$ and $\sigma_0 = \sigma \mid K((S \textcircled{\tau} X)^w)$ are isomorphisms onto $K(S^a) = S^a$ and $K((S \textcircled{\tau} X)^a)$ respectively. Define δ: $K(S^w) \textcircled{\tau} K(X^w) \rightarrow S^a \textcircled{\tau} K(X^a)$ by $\delta(u, \theta) = (\gamma_0(u), \xi_0(\theta))$. Then δ is an isomorphism, and since $K((S \textcircled{\tau} X)^a) = S^a \textcircled{\tau} K(X^a)$ (Theorem 3.1), $\mu = \sigma_0^{-1} \circ \delta$ is the required isomorphism of $K(S^w) \textcircled{\tau} K(X^w)$ onto $K((S \textcircled{\tau} X)^w)$.

REMARK 3.4. Let $W(S \textcircled{\tau} X)_p$ denote the closed subspace of $W(S \textcircled{\tau} X)$ spanned by the coefficients of the finite dimensional unitary representations of $S \textcircled{\tau} X$ (see [6, p. 85]). Under the conditions of the previous theorem, $W(S \textcircled{\tau} X)_p$ is the completed ε-tensor product of $A(S)$ and the space of all $f \in W(X)$ such that $f(d\theta) = f(\theta)$ for all $\theta \in X^w$, where d is the identity of $K(X^w)$. This follows from Theorem 3.3 together with Theorem 5.7 and Lemma 5.10 of [6] and the identity $\mu(e_1\rho'(s), d\eta'(x)) = e\rho_0'(s, x)(s \in S, x \in X)$, where e is the identity of $K((S \textcircled{\tau} X)^w)$, e_1 the identity of $K(S^w)$, and μ the isomorphism obtained in the proof of Theorem 3.3.

REFERENCES

1. E. M. Alfsen and P. Holm, *A note on compact representations and almost periodicity in topological groups*, Math. Scand., **10** (1962), 127-136.
2. J. Berglund and K. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture Notes in Mathematics 42, Springer-Verlag, Berlin, 1967.
3. N. Bourbaki, *Elements of Mathematics, General Topology, Part* I, Addison-Wesley, Reading, Mass., and Hermann, Paris, 1966.
4. R. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.
5. F. Dangello and R. Lindahl, *Almost periodic functions on a semidirect product of two semigroups*, preprint.
6. K. deLeeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta. Math., **105** (1961), 63-97.
7. ———, *Almost periodic functions on semigroups*, Acta. Math., **105** (1961), 99-140.
8. J. de Vries, *Equivalence of almost periodic compactifications*, Compositio Mathematica, **22** (1970), 453-456.
9. R. Ellis, *Locally compact transformation groups*, Duke Math. J., **24** (1957), 119-126.
10. A. Grothendieck, *Criteres de compacite' dans les espaces fonctionnels généraux*, Amer. J. Math., **74** (1952), 168-186.
11. M. Landstad, *On the Bohr compactification of a transformation group*, Math. Z., **127** (1972), 167-178.
12. C. Ryll-Nardzewski, *On fixed points of semigroups of endomorphisms of linear spaces*, Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability (1965) vol. II Part I, Theory of Probability, Univ. of Cal. Press, Berkeley.

Received July 23, 1974 and in revised form November 4, 1974.

THE GEORGE WASHINGTON UNIVERSITY