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CONVEX HULLS AND EXTREME POINTS OF FAMILIES OF STARLIKE AND CLOSE-TO-CONVEX MAPPINGS

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The closed convex hull is obtained for the functions which are starlike of order α, k-fold symmetric, and real on $(-1, 1)$. The same result for the close-to-convex functions which are k-fold symmetric is obtained. Integral representations are given for the hulls of these and other families in terms of probability measures on suitable sets. These results are used to solve extremal problems.

Introdution. Let $\varDelta$ denote the unit disk and let A denote the set of functions analytic in $\varDelta$. Then A is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $\varDelta$.

We consider the family, denoted by $St_R(\alpha, k)$ of starlike function of order α which are real on $(-1, 1)$ and have power series developments which are k-fold symmetric. We recall that a function f analytic in $\varDelta$ is called k-fold symmetric $(k = 1, 2, \cdots)$ if its power series has the form

$$f(z) = \sum_{m=0}^{\infty} a_{mk+1} z^{mk+1} .$$

We also consider the family, denoted by C_k of close-to-convex functions which are k-fold symmetric. We also consider the class of functions denoted by $K_R(\beta)$ of functions which are close-to-convex of order β and real on $(-1, 1)$. These functions were introduced by Pommerenke in [8].

Let

$$F_p = \left\{\int_X \frac{1}{[(1 - xz)(1 - \bar{x}z)]^p} d\mu(x) \colon \mu \in \mathscr{P}\right\}$$

where $p > 0$ and $\mathscr{P}$ is the set of probability measures on $X = \{x\colon |x| = 1 \text{ and } \operatorname{Im} x \geqq 0\}$. We prove that $F_p \cdot F_q \subset F_{p+q}$.

We use this result to obtain the closed convex hull and extreme points of $St_R(\alpha, k)$ which we denote $\mathscr{H} St_R(\alpha, k)$ and $\mathscr{E}\mathscr{H} St_R(\alpha, k)$ respectively. We also use it to obtain the closed convex hull of $K_R(\beta)$ for $\beta \geqq 1$ which we denote $\mathscr{H} K_R(\beta)$ We recall that in [2] $\mathscr{H} K(\beta)$ for $\beta \geqq 1$ was determined.

By way of application of our results we prove that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is subordinate to a close-to-convex odd function, then $|a_n| <$

$\sqrt{2}$ for $n = 1, 2, \cdots$. We recall that an analytic function f is said to be subordinate to an analytic function F if $f(z) = F(\phi(z))$ where $\phi(z)$ is analytic in Δ, $\phi(0) = 0$. and $|\phi(z)| < 1$. We write this relationship $f \prec F$.

1. A product theorem and a geometric mean theorem for F_p.

THEOREM 1. *Let $X = \{x: |x| = 1$ and $\operatorname{Im} x \geqq 0\}$, $\alpha \geqq 1$, and $\mathscr{P}$ denote the set of probability measures on X. Given $v \in \mathscr{P}$ there exists a $\mu \in \mathscr{P}$ so that*

$$\left[\int_X \frac{1 - z^2}{(1 - xz)(1 - \bar{x}z)} dv(x)\right]^\alpha = \int_X \left[\frac{1 - z^2}{(1 - xz)(1 - \bar{x}z)}\right]^\alpha d\mu(x) .$$

Proof. This result follows from obvious modifications of the Herglotz representation for functions of positive real part and the same type of arguments made in the proof of Theorem 2.2 in [2].

THEOREM 2. *Let*

$$F_p = \left\{\int_X \frac{1}{[(1 - xz)(1 - \bar{x}z)]^p} d\mu(x) \colon \mu \in \mathscr{P}\right\}$$

where X and $\mathscr{P}$ are as in Theorem 1. If $p > 0$ and $q > 0$ then $F_p \cdot F_q \subset F_{q+q}$ where $F_p \cdot F_q = [f\colon f = gh$ and $g \in F_p, h \in F_q\}$.

Proof. If $p + q \geqq 1$, then proof of this therem follows from Theorem 1 by the same arguments used to prove Theorem 1 in [3].

Now suppose $p + q \geqq 1$. Consider the linear operator L defined by

$$(\mathrm{L}f)(z) = \frac{1}{p} \frac{z}{1 - z^2} f'(z) + \frac{1}{1 - z^2} f(z) .$$

It is easily verified that L is a linear map from F_p onto F_{p+1} since L applied to $1/[(1 - xz)(1 - \bar{x}z)]^p$ yields $1/[(1 - xz)(1 - \bar{x}z)]^{p+1}$.

Let

$$h(z) = \frac{1}{[(1 - xz)(1 - \bar{x}z)]^p} \frac{1}{[(1 - yz)(1 - \bar{y}x)]^q} .$$

A computation shows that

$$\frac{1}{p + q} \frac{z}{1 - z^2} h'(z) + \frac{1}{1 - z^2} h(z)$$

$$= \frac{p}{p+q} \frac{1}{[(1-xz)(1-\bar{x}z)]^{p+1}} \frac{1}{[(1-yz)(1-\bar{y}z)]^{q}} + \frac{q}{p+q} \frac{1}{[(1-xz)(1-\bar{x}z)]^{p}} \frac{1}{[(1-yz)(1-\bar{y}z)]^{q+1}} .$$

Applying the case of the theorem when $p + q \geqq 1$ and the convexity of F_{p+q+1}, we conclude that the left hand side of the above equation is a member of F_{p+q+1}. It now follows that $F_p \cdot F_q \subset F_{p+q}$ if $0 < p + q < 1$.

THEOREM 3. *Let X and $\mathscr{P}$ be as in Theorem 1. Then given $\mu \in \mathscr{P}$, $\exists v \in \mathscr{P}$ such that*

$$\exp\left\{\int_X -p \log(1-xz)(1-\bar{x}z)\,d\mu(x)\right\} = \int_X [(1-xz)(1-\bar{x}z)]^{-p} dv(x) .$$

Proof. The proof of this theorem follows from Theorem 2 in a direct and obvious way.

2. The convex hull of $St_R(\alpha, k)$

THEOREM 4. *Let X and $\mathscr{P}$ be as in Theorem 1, $\alpha < 1$, k be any positive integer and $\mathscr{F}$ be the set of functions f_μ on Δ defined by*

$$f_u(z) = \int_X \frac{z}{(1-xz^k)^{(1-\alpha)/k}(1-\bar{x}z^k)^{(1-\alpha)/k}} \qquad (\mu \in \mathscr{P})$$

then $\mathscr{F} = \mathscr{H} St_R(\alpha, k)$ and

$$\mathscr{E}\mathscr{H} St_N(\alpha, k) \subset \left\{\frac{z}{(1-xz^k)^{(1-\alpha)/k}(1-\bar{x}z^k)^{(1-\alpha)/k}} : |x| = 1, \operatorname{Im} x \geqq 0\right\} .$$

Proof. It is easy to show that each $f(z)$ in $St_R(\alpha, k)$ can be represented by

$$f(z) = z \exp\left\{-\left(\frac{1-\alpha}{k}\right)\int_X \log(1-xz^k)(1-\bar{x}z^k)dv(x)\right\}$$

where $X = \{x: |x| = 1 \text{ and } \operatorname{Im} x \geqq 0\}$ and v is a probability measure on X. The result now follows by direct application of Theorem 3 and standard arguments.

REMARK 1. This result generalizes Theorem 1 of [5] and Theorem 3 of [3].

2. The problem of determing $\mathscr{H}T_k$ when T_k denotes the typically real k-fold symmetric functions seems more difficult. Our next theorem settles the case $k = 2$, i.e., typically real odd functions.

3. The question of whether each kernel function is an extreme point remains to be decided. It is known to be true when $\alpha = 0$ and $k = 1$.

LEMMA 1. *Suppose* $\operatorname{Re} p(z) > 0$, $p(0) = 1$, $p(z)$ *is even, and* $p(z)$ *is real on* $(-1, 1)$.
Then

$$p(z) = \int_X \frac{1 - z^4}{(1 - xz^2)(1 - \bar{x}z^2)} d\mu(x)$$

where $X = \{x: |x| = 1 \text{ and } \operatorname{Im} x \geqq 0\}$ *and* μ *is a probability measure on* X.

Proof. The result follows from an obvious modification of the Herglotz formula.

THEOREM 5. *Let* $X = \{x: |x| = 1, \operatorname{Im} x \geqq 0\}$, $\mathscr{P}$ *be the set of probability measures on* X, *and* $\mathscr{F}$ *be the set of functions* f_μ *on* $\varDelta$ *defined by*

$$f_\mu(z) = \int_X \frac{z(1 + z^2)}{(1 - xz^2)(1 - \bar{x}z^2)} d\mu(x) \quad (\mu \in \mathscr{P}) .$$

Let T_2 *be the set of typically real odd functions on* $\varDelta$. *Then* $\mathscr{H}T_2 = T_2 = \mathscr{F}$, *the map* $\mu \rightarrow f_\mu$ *is one-to-one, and each function*

$$z \longrightarrow \frac{z(1 + z^2)}{(1 - xz^2)(1 - \bar{x}z^2)}$$

is an extreme point of $\mathscr{H}T_2$.

Proof. This result follows in a very direct way from the previous lemma and a classical result of W. Rogosinski [9] which states that if $f(z)$ is typically real then

$$f(z) = \frac{z}{1 - z^2} p(z)$$

where $\operatorname{Re} p(z) > 0$, $p(0) = 1$ and $p(z)$ is real on $(-1, 1)$. The fact that $\mu \rightarrow f_\mu$ is one-to-one follows by direct appeal to Theorem 4 in [4, p. 95].

3. The convex hull of $K_R(\beta)$.

In [2] D. A. Brannan, J. G. Clunie and W. E. Kirwan determined $\mathscr{H}K(\beta)$ where $\beta \geqq 1$. We now turn our attention to $K_R(\beta)$ for $\beta \geqq 1$, i.e., those functions in $K(\beta)$ which are real on $(-1, 1)$. We recall that in [2] the above authors showed that $f(z) \in K(\beta)$ if and only if there exists a function $p(z)$ satisfying $\operatorname{Re} p(z) > 0$ and a starlike function $s(z)$ so that $zf'(z) = a(p(z))^\beta s(z)$ where $|a| = 1$. This is equivalent to the original definition given by Ch. Pommerenke in [8].

THEOREM 6. *Let $\{X = x: |x| = 1, \operatorname{Im} x \geqq 0\}$, $\mathscr{P}$ be the set of probability measures on X, and $\mathscr{F}$ be the class of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_X \left[\frac{1 - z^2}{(1 - xz)(1 - \bar{x}z)}\right]^\beta \frac{1}{(1 - xz)(1 - \bar{x}z)} d\mu(x) \quad (\mu \in \mathscr{P}) .$$

Then $\mathscr{F} = \mathscr{H}K'_R(\beta)$, where $\beta \geqq 1$ and $K'_R(\beta)$. Also

$$\mathscr{E}\mathscr{H}K'_R(\beta) \subset \left\{\left[\frac{1 - z^2}{(1 - xz)(1 - \bar{x}z)}\right]^\beta \frac{1}{(1 - xz)(1 - \bar{x}z)} : |x| = 1, \operatorname{Im} x \geqq 0\right\} .$$

Proof. We assume that $f(z) \in K_R(\beta)$ for $\beta \geqq 1$. In [2] the authors showed that

$$zf'(z) = \left[\left[p(z) \cdot \overline{p(\bar{z})}\right]^{1/2}\right]^\beta [s(z) \cdot \overline{s(\bar{z})}]^{1/2}$$

where $[p(z)\overline{p(\bar{z})}]^{1/2}$ has positive real part and is real on $(-1, 1)$ and $[s(z)s(\bar{z})]^{1/2}$ is univalent, starlike, and real on $(-1, 1)$. The result now follows with the usual arguments by appealing to Theorems 1 and 2.

REMARK 1. The question of whether each kernel function is an extreme point remains undecided.

2. We introduce some useful notation at this point. Suppose $f(z) = \sum_{n=1}^\infty a_n z^n$ and $F(z) = \sum_{n=1}^\infty a_n z^n$ are such that $|a_n| \leqq |A_n|$ for $n = 1, 2, \cdots$. We then write $f(z) \ll F(z)$. In [2] the authors proved that $f(z) \in K_R(\beta)$ implies $f'(z) \ll f'_k(z)$ where $p = (1/2)k - 1$ and

$$f_k(z) = \frac{1}{k}\left[\left(\frac{1 + z}{1 - z}\right)^{k/2} - 1\right] .$$

They proved this with no restriction on β. If $\beta \geqq 1$, the previous theorem gives an easy proof of this result. The argument needed is essentially the same one used by D. A. Brannan, J. G. Clunie, and W. E. Kirwan in [2] to prove the coefficient conjecture for $K(\beta)$ when $\beta \geqq 1$. We recall that D. Aharonov and S. Friedland settled that coefficient conjecture for $K(\beta)$ in [1]. We next prove a generalized coefficient conjecture for $K(\beta)$.

THEOREM 7. *Suppose* $F(z) \in K(\beta)$ *where* $\beta \geqq 1$. *If* $f(z) \prec F(z)$ *then* $f(z) \ll f_k(z)$ *where*

$$f_k(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] \quad and \quad \beta = \frac{1}{2}k - 1 .$$

Proof. It suffices to consider $F(z)$ which are in $\mathscr{E}\mathscr{H}K(\beta)$. So we may assume by Theorem 4.1 in [2] that for $f(z) = F(\phi(z))$ we have

$$\begin{aligned} f'(z) &= F'(\phi(z))\phi'(z) \\ &= \left[\frac{1 + x\phi(z)}{1 - y\phi(z)}\right]^\beta \frac{1}{(1 - y\phi(z))^2}\phi'(z) . \end{aligned}$$

We recall that D. Aharonov and S. Friedland in [1] proved that

$$\left[\frac{1 + cz}{1 - z}\right]^\beta \ll \left[\frac{1 + z}{1 - z}\right]^\beta \quad \text{for} \quad \beta \geqq 1 \quad \text{and} \quad |c| \leqq 1 .$$

Since

$$\left[\frac{1 + x\phi(z)}{1 - y\phi(z)}\right]^\beta = \int_X \left[\frac{1 + cxz}{1 - xz}\right]^\beta d\mu(x)$$

for some μ a probability measure on $X = \{x: |x| = 1\}$ by Theorem 2.2 in [2] we see that

$$\left[\frac{1 + x\phi(z)}{1 - y\phi(z)}\right]^\beta \ll \left[\frac{1 + z}{1 - z}\right]^\beta .$$

Since

$$\frac{\phi'(z)}{(1 - y\phi(z))^2} = \frac{1}{y}\left[\frac{1}{1 - y\phi(z)}\right]' \quad \text{and} \quad \frac{1}{1 - y\phi(z)} \ll \frac{1}{1 - z}$$

we conclude that

$$\frac{\phi'(z)}{(1 - y\phi(z))^2} \ll \frac{1}{(1 - z)^2} .$$

Hence we see that

$$f'(z) \ll \left[\frac{1+z}{1-z}\right]^{\beta} \frac{1}{(1-z)^2} = f_k'(z)$$

where

$$f_k(z) = \frac{1}{k}\left[\left[\frac{1+z}{1-z}\right]^{k/2} - 1\right] \quad \text{and} \quad \beta = \frac{1}{2}k - 1\,.$$

The theorem now follows directly.

4. The convex hull of C_k.

THEOREM 8. *Let $X^2 = \{(x, y): |x| = |y| = 1\}$, $\mathscr{P}$ be the set of probability measures on X^2, and $\mathscr{F}$ be the class of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_{X^2} \frac{1 - xz^k}{(1 - yz^k)^{2/k+1}}\, d\mu(x, y) \quad (\mu \in \mathscr{P})$$

then $\mathscr{F} = \mathscr{H}C_k'$, where C_k' is the set of derivatives of functions in C_k and $k = 1, 2, \cdots$. Furthermore

$$\mathscr{E}\mathscr{H}C_k' \subset \left\{\frac{1 + xz^k}{(1 - yz^k)^{2/k+1}}: |x| = |y| = 1\right\}\,.$$

Proof. Let $f(z)$ be in C_k. An inspection of the proof of Theorem 2 in [6], as noted by Ch. Pommerenke in [7, p. 263], shows that we can choose a starlike function $s(z)$ with k-fold symmetry and a function of positive real part with k-fold symmetry so that

$$zf'(z) = p(z)s(z)\,.$$

The result now follows appeal to Theorem 3 in [4, p. 95] and the same arguments in [4, Theorem 6].

The next result was proven earlier in [7, p. 266] by Ch. Pommerenke.

COROLLARY. (1) *If $f \in C_k$, then $f(z) \ll z/(1 - z^k)^{2/k}$.*

Proof. We prove $f'(z) \ll [z/((1 - z^k)^{2/k})]'$ which is equivalent to the above statement. It suffices to prove the result for f in $\mathscr{E}\mathscr{H}C_k$, i.e., for functions

$$\frac{1 + xz^k}{(1 - yz^k)^{2/k+1}} \quad \text{where} \quad |x| = |y| = 1\,.$$

It is easy to see that

$$\frac{1+xz^k}{1-yz^k} \ll \frac{1+z^k}{1-z^k} \quad \text{and} \quad \frac{1}{(1-yz^k)^{2/k}} \ll \frac{1}{(1-z^k)^{2/k}} .$$

Hence we have

$$\frac{1+xz^k}{(1-yz^k)^{2/k+1}} \ll \frac{1+z^k}{(1-z^k)^{2/k+1}} = \left[\frac{z}{(1-z^k)^{2/k}}\right]' .$$

The result now follows.

COROLLARY. (2) *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *be subordinate to* $F(z)$ *where* $F(z)$ *is in* C_2. *Then* $|a_n| < \sqrt{2}$ *for* $n = 1, 2, \cdots$.

Proof. Applying Theorem 8 when k = 2, integrating and using the fact that $f_\mu(0) = 0$ we find that the extreme points of C_2 are given by the collection

$$\left\{\left(\frac{1}{2} + \frac{1}{2}\frac{x}{y}\right)\frac{z}{1-yz^2} + \left(\frac{1}{2} - \frac{1}{2}\frac{x}{y}\right)\frac{1}{2\sqrt{y}}\log\frac{1+\sqrt{y}z}{1-\sqrt{y}z} : |x| \right.$$
$$\left. = |y| = 1\right\} .$$

It suffices to prove the above theorem for $F(z)$ one of these extreme points. So we have

$$f(z) = \left[\frac{1}{2} + \frac{1}{2}\frac{x}{y}\right]\frac{\phi(z)}{1-y\phi^2(z)} + \left[\frac{1}{2} - \frac{1}{2}\frac{x}{y}\right]\frac{1}{2\sqrt{y}}\log\frac{1+\sqrt{y}\phi(z)}{1-\sqrt{y}\phi(x)}$$

where $\phi(0) = 0$, $|\phi(z)| < 1$. Let $\phi(z)/(1 - y\phi^2(z)) = \sum_{n=1}^{\infty} b_n z^n$. Since $z/(1 - yz^2)$ is starlike and odd, we have by Theorem 9 in [3] the inequality $|b_n| \leqq 1$ for $n = 1, 2, , \cdots$. Let

$$\frac{1}{2\sqrt{y}}\log\frac{1+\sqrt{y}\phi(z)}{1-\sqrt{y}\phi(z)} = \sum_{n=1}^{\infty} c_n z^n .$$

Since $1/2 \log (1 + z)/(1 - z)$ is convex we have $|c_n| \leqq 1$ for $n = 1, 2, \cdots$ by the classical result of Rogosinski [10]. We have

$$a_n = \left(\frac{1}{2} + \frac{1}{2}\frac{x}{y}\right)b_n + \left[\frac{1}{2} - \frac{1}{2}\frac{x}{y}\right]c_n .$$

We conclude that

$$|a_n| \leqq \frac{1}{2}\left\{\left|1 + \frac{x}{y}\right| + \left|1 - \frac{x}{y}\right|\right\} \leqq \sqrt{2} \quad \text{for} \quad n = 1, 2 \cdots .$$

To see that we must have the strict inequality $|a_n| < \sqrt{2}$ for all n recall when $|b_n| = 1$ and $|c_n| = 1$. If equality occurs we must have $\phi(z) = \varepsilon z^n$ where $|\varepsilon| = 1$. However, for such a ϕ it is easy to see that $|a_n| \leqq 1$.

REMARKS. If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is in C_2, then $|a_n| \leqq 1$. This is the same bound which holds for odd starlike functions. Since it has been proven in [3] that a function subordinate to an odd starlike function also has coefficients bounded by 1, it is natural to conjecture that the correct bound in Corollary (2) is 1.

5. The hull of a class of close-to-convex functions.

In [11] K. Sukaguchi proved that the operator $(Lf)(z) = \int_0^z (f(w)/w)dw$ applied to a close-to-convex function produces a close-to-convex function. This result was proven again and generalized by Ch. Pommerenke in [8]. We will examine in this section the compact family of close-to-convex functions $L(C) = \{Lf : f \in C\}$.

We remark that since the operator L is linear the extreme points and closed convex hull of $L(C)$ can be precisely determined from Theorem 6 in [4, p. 97]. We find that

$$\begin{aligned}&\mathscr{E}\mathscr{H}L(C)\\ &= \left\{\frac{1}{2}\left(1 - \frac{x}{y}\right)\frac{z}{1 - yz} - \frac{1}{2}\left(1 + \frac{x}{y}\right)\frac{1}{y}\log(1 - yz) : |x| \right.\\ &\left. = |y| = 1\right\}.\end{aligned}$$

We also note that by applying the technique of proof used in Corollary (2) we can prove that if $f(z) = \sum_{n=1}^{\infty} a_n z^n \prec F(z)$ where $F(z) \in L(C)$ then $|a_n| < \sqrt{2}$ for $n = 1, 2, \cdots$. We remark that it natural to conjecture that $|a_n| \leqq 1$ is the correct inequality.

Added in proof. The extreme points of $\mathscr{H}\,\mathrm{St}_R(\alpha, k)$ are identical with the set of functions given in the inclusion in Theorem 4. In [4, p. 95], it was in effect proven that the map $\mu \to f\mu$ is one-to-one.

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