PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 1, 1975

THE CONVERSE TO A THEOREM OF CONNER AND FLOYD

A. L. EDELSON

If W^{2n} is a manifold with almost complex structure $J: \tau(W) \to \tau(W)$ on its tangent bundle, then a conjugation on W is a smooth involution $T: W^{2n} \to W^{2n}$ whose differential anti-commutes with J, i.e., $T_*J = -JT_*$. Examples of such actions are those induced by complex conjugation of coordinates in $P^n(C)$ and $H_{m,n}(C)$ having fixed point sets $P^n(R)$ and $H_{m,n}(R)$ respectively.

Conner and Floyd have proved that the fixed point set of a conjugation is always an n-dimensional submanifold if it is nonempty. Furthermore, they show that if F^n denotes the fixed point set of the conjugation $T: W^{2n} \to W^{2n}$ and $[\ ]_2$ denotes the nonoriented cobordism class, then $[W^{2n}]_2 = [F^n \times F^n]_2$. In this article we prove that every closed n-manifold is the fixed point set of a conjugation on a closed $2n$-dimensional almost complex manifold.

The technique of the proof involves modification of the authors previous work on the case of stable almost complex structures, that is a conjugation of an almost complex structure on the stable tangent bundle $\tau(W^{2n}) \oplus \theta^k$, $k > 0$. The proof consists of showing that if for every $n > 0$ the sphere S^n is fixed point set of a conjugation, then every closed n-manifold is also. This proof involves a suggestion made by R. Stong. Next we describe an almost complex manifold W^{2n} having conjugation fixing S^n. We use generalized equivariant surgery, and rely heavily on the fact that a regular neighborhood of the fixed point set is diffeomorphic to the tangent disc bundle. Note that every manifold is fixed point of a conjugation on an open manifold; namely, the bundle involution on its tangent disc bundle.

THEOREM. *Let M^n be a smooth closed n-manifold. Then there exists a smooth closed almost complex manifold W^{2n} with conjugation $T: W^{2n} \to W^{2n}$ having fixed point set M^n.*

Proof. It follows from [5] that the nonoriented cobordism ring can be generated by the manifolds $P^{2n}(R)$ and $H_{m,n}(R)$, where the latter is the hypersurface in $P^m(R) \times P^n(R)$ defined by $\sum_{i=0}^{\min(m,n)} x_i y_i = 0$. Complex conjugation of coordinates defines conjugations on the corresponding complex manifolds $P^{2n}(C)$ and $H_{m,n}(C)$, so it follows that the generators of η_* are fixed point sets of conjugations. It then follows from [3] that if M^n is any manifold, there is an almost complex manifold V^{2n} with conjugation $S: V^{2n} \to V^{2n}$ having fixed point set F^n, and such that M^n can be obtained from F^n by a sequence of surgeries. We will show that any such modification of F^n can be extended to an equivariant modification of V^{2n}, which preserves the almost complex structure and conjugation.

We now make the assumption that for every $n > 0$ there is a closed almost complex manifold W^{2n} with conjugation $T: W^{2n} \to W^{2n}$ having fixed point set S^n.

LEMMA. *If F^n is the fixed point set of the conjugation $S: V^{2n} \to V^{2n}$, then any manifold obtained from F^n by surgery on an imbedded sphere is also the fixed point set of a conjugation on some almost complex manifold.*

Proof. Let $f_0: S^p \to F^n$, $0 \leqq p < n$ be an imbedding with trivial normal bundle. Then f_0 extends to an imbedding $f: S^p \times D^{n-p} \to F^n$. The restriction to $f(S^p \times D^{n-p})$ of the tangent bundle $\tau(F^n)$ is trivial and again by [1: 24. 2] the almost complex structure on V^{2n} defines an isomorphism $\tau(F^n) \overset{=}{\to} \nu(F^n)$ where $\nu(F^n)$ denotes the normal bundle of F^n in V^{2n}. By this isomorphism we can extend f to an imbedding $F: S^p \times D^{n-p} \times D^n \to V^{2n}$, equivariant with respect to the involution given by -1 in the factor D^n. This follows since at a fixed point of the involution S, the representation is multiplication by -1 in $\nu(F^n)$. Similarly if $T: W^{2n} \to W^{2n}$ is a conjugation with fixed point set S^n, let $G: D^{p+1} \times S^{n-p-1} \times D^n \to W^{2n}$ be the equivariant imbedding induced by the standard inclusion $g: S^{n-p-1} \to S^n$. There is a diffeomorphism

$$h: F(S^p \times (D^{n-p} - \{0\}) \times D^n) \longrightarrow G((D^{p+1} - \{0\}) \times S^{n-p-1} \times D^n)$$

given by $h(F(u, tv, w)) = G(tu, v, w)$ for $0 < t \leqq 1$. It is clear that h is equivariant. The almost complex structures define isomorphisms between the tangent and normal bundles to the fixed point sets, so it follows that the differential h_* preserves the almost complex structure.

Now let M^{2n} be the manifold obtained from $V^{2n} - F(S^p \times \{0\} \times D^n)$ and $W^{2n} - G(\{0\} \times S^{n-p-1} \times D^n)$ by identifying the submanifolds $F(S^p \times (D^{n-p} - \{0\}) \times D^n)$ and $G((D^{p+1} - \{0\}) \times S^{n-p-1} \times D^n)$ using the diffeomorphism h. Then M^{2n} has an almost complex structure and conjugation induced by T and S. The fixed point set of this conjugation is obtained from $F^n - F(S^p \times (D^{n-p} - \{0\} \times \{0\}))$ and $S^n - G((D^{p+1} - \{0\}) \times S^{n-p-1} \times \{0\})$ by identifying the appropriate submanifolds using the restriction of h. This is precisely the manifold obtained from F^n by surgery on the imbedded sphere $f_0(S^p)$.

We will now construct for each S^n, an almost complex manifold W^{2n} with conjugation $T: W^{2n} \to W^{2n}$ having S^n as fixed point set.

Let $D(S^n)$ denote the tangent disc bundle to S^n and $\tau_1(S_n)$ its boundary, the unit tangent bundle. Then $D(S^n)$ can be described as the submanifold of S^{2n+1} consisting of vectors $\{(x, y) \in R^{n+1} \times R^{n+1}\}$ satisfying the conditions $x \cdot x + y \cdot y = 2$, $x \cdot y = 0$, $0 < x \cdot x \leqq 1$. We take the sphere of radius 2 for convenience. Identifying (x, y) with the complex vector $z = x + iy$ in C^{n+1}, the unit tangent bundle $\tau_1(S^n)$ is described by the equation $\sum_0^n Z_i^2 = 0$. Define involutions $T_j \colon D(S^n) \to D(S^n)$ for $j = 1, 2$, by $T_1(x, y) = (x, -y)$ and $T_2(x, y) = (-x, y)$. Then T_1 corresponds to multiplication by -1 in the fibers of $D(S^n)$ and so has fixed point set equal to S^n. T_2 reduces to the antipodal involution on S^n and has no fixed points.

We will now describe almost complex structures J_1 and J_2 on $D(S^n)$ with respect to which T_1 and T_2 are conjugations. At a point $(x, y) \in D(S^n)$ the tangent space $\tau_{(x,y)}(D(S^n))$ consists of all vectors $(u, v) \in R^{n+1} \times R^{n+1}$ satisfying the equations

(1) $x \cdot u + y \cdot v = 0$

(2) $y \cdot u + x \cdot v = 0$.

Define

$$J_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} |x|\left(-v + \dfrac{v \cdot x}{|x|^2}x\right) - \dfrac{y \cdot u}{|x|^3}x \\ \dfrac{v \cdot y}{|x|}x + \dfrac{u}{|x|} - \dfrac{x \cdot u}{|x|^3}x \end{pmatrix}$$

$$J_2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{u \cdot x}{|y|}y + \dfrac{v}{|y|} - \dfrac{y \cdot v}{|y|^3}y \\ |y|\left(-u + \dfrac{u \cdot y}{|y|^2}y\right) - \dfrac{x \cdot v}{|y|^3}y \end{pmatrix}.$$

It can be verified that $J_1^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix}$ and $J_2^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix}$ so that these formulae describe almost complex structures at the point (x, y). The maps T_1 and T_2 extend to $R^{n+1} \times R^{n+1}$ so their differentials are given by $T_{1*} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix}$ and $T_{2*}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ v \end{pmatrix}$. Again it can be verified that $T_{1*} \circ J_{1(x,y)} = -J_{1(x,-y)} \circ T_{1*}$ and $T_{2*} \circ J_{2(x,y)} = -J_{2(-x,y)} \circ T_{2*}$, so that the involutions T_1 and T_2 are in fact conjugate linear. Now define a diffeomorphism $h \colon \tau_1(S^n) \to \tau_1(S^n)$ by $h(x, y) = (y, x)$. Form a closed manifold W^{2n} from two copies of $D(S^n)$ by identifying them along $t_1(S^n)$ using h. Then W^{2n} can be made into a smooth manifold and since $h \circ T_1(x, y) = h(x, -y) = (-y, x)$, $T_2 h(x, y) = T_2(y, x) = (-y, x)$, it follows that W^{2n} can be given an involution $T \colon W^{2n} \to W^{2n}$ given by T_1 on the first copy of $D(S^n)$ and by T_2 on the second. It is clear that the fixed point set of T equals the fixed point set of T_1, which is S^n. It remains to show that W^{2n} is an almost complex manifold.

There are almost complex structures defined on each copy of $D(S^n)$ so W^{2n} is almost complex provided the identification map h has differential which commutes with J_1 and J_2. We note that there is a commutative diagram.

$$\begin{array}{ccc} t_{(x,y)}D(S^n) & \xrightarrow{h_*} & t_{(y,x)}D(S^n) \\ \downarrow J_1 & & \downarrow J_2 \\ t_{(x,y)}D(S^n) & \longrightarrow & t_{(y,x)}D(S^n) \end{array}$$

This follows since

$$h_* \circ J_{1(x,y)}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{r\cdot y}{|x|}x + \dfrac{u}{|x|} - \dfrac{x\cdot u}{|x|^3}x \\ |x|\left(-r + \dfrac{r\cdot x}{|x|^2}x\right) - \dfrac{y\cdot y}{|x|^3}x \end{pmatrix} = J_{2(y,x)}\circ h_*\begin{pmatrix} u \\ v \end{pmatrix}.$$

Then W^{2n} is an almost complex manifold and T is a conjugation which completes the proof.

References

1. P. E. Conner, and E. E. Floyd, *Differentiable Periodic Maps*, Springer, Berlin, 1964.
2. A. Edelson, *Conjugations on stably almost complex manifolds*, to appear, Pacific J. Math.
3. J. Milnor, *A procedure for killing the homotopy groups of differentiable manifolds*, Symposium in Pure Math., Amer. Math. Soc., (3), (1961).
4. ———, *On the cobordism ring and a complex analogue*, Amer. J. Math., (1960).
5. René Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv., (1954).

Received July 8, 1974.

UNIVERSITY OF CALIFORNIA, DAVIS