

ON LOCALLY COMPACT WREATH PRODUCTS

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The conditions are given under which an algebraic wreath product becomes a topological wreath product with respect to the natural product topology. Various standard topological properties are discussed, and the Haar integral is described. All irreducible representations of the separable locally compact wreath products having a type I group as first factor are determined.

1. **Introduction.** Wreath products have been an important source of examples for (discrete) group theory, as can be seen by their use throughout [7] and [15], for example. Recent work has also shown their usefulness outside group theory ([3], [4], [17]). The possibilities for the theory of arbitrary topological groups seem not to have been openly explored, however. In this paper we begin by recording the condition under which an algebraic wreath product becomes a topological wreath product—with respect to the natural product topology. We next note the conditions under which the product will have various standard topological properties, and we then look at the Haar integral. This brings us to the first problem of substance: to determine the irreducible representations of the locally compact wreath products. This problem has been solved for finite groups through the use of Clifford's work [2] on induced representations (see [9]), and it is natural to make use of induced representations in the more general case as well. The appropriate tool here is Mackey's extension [13] of Clifford's theory, and in the last section we apply this to determine, in particular, all irreducible representations of the separable locally compact wreath products having a type I group as first factor.

2. **The groups.** If A and B denote groups, then the wreath product of A and B , $A \wr B$, is constructed as follows. Let $F = A^B$ be the direct product of copies of A indexed by the set B . Explicitly, F is the set of all functions from B into A , made into a group by componentwise multiplication. For $f \in F$ and $b \in B$, define $f^b \in F$ by $f^b(y) = f(yb^{-1})$ for all $y \in B$. Then, for each $b \in B$, the mapping $f \mapsto f^b$ is an automorphism of F , and the group of all such automorphisms is isomorphic to B ; the *wreath product of A and B* , $A \wr B$, is the extension of F by this group of automorphisms. Each element of $A \wr B$ can be written uniquely as fb , with $f \in F$ and $b \in B$, and

these elements multiply according to the rule $f_1 b_1 f_2 b_2 = f_1 f_2^{-1} b_1 b_2$. So $A \wr B$ is a semi-direct product of F and B : $F \triangleleft A \wr B$, $F \cap B = \{e\}$, and $FB = A \wr B$. F is called the *base group* of the wreath product.

The wreath product defined here is sometimes called the *standard wreath product*, to distinguish it among wreath products in which the second factor is permitted to be an arbitrary permutation group; compare [9]. The standard case appears to bring out the essential ideas, and it avoids the slightly more cumbersome notation of the more general product. (See the remarks following Theorem 2.2, also.)

Now assume A and B to be topological groups; all topological groups will be Hausdorff, and, to avoid degeneracy, we always assume $|A| > 1$ and $|B| > 1$. Because $A \wr B$ is a semi-direct product of F and B , it is natural to inquire whether $A \wr B$ becomes a topological group with respect to the product topology on $F \times B$; in other words, whether it is a topological semi-direct product in the usual sense (as in [6, (6.20)]). Of course, we must first put a topology on $F = A^B$, and for this we take the product topology. It will be convenient to have the following notation: if S is a subset of A , and $b \in B$, then S_b will denote $\{f \in F: f(b) \in S \text{ and } f(x) = e \text{ for } x \neq b\}$.

THEOREM 2.1. *If A and B are topological groups, then $A \wr B$ is a topological group with respect to the product topologies on $F = A^B$ and on $F \times B$ iff B is discrete.*

Proof. For $A \wr B$ to be a topological group, it is necessary and sufficient that the map $(f, b) \mapsto f^b$ of $F \times B$ into F be continuous. For each $x \in B$, let $\varphi_x: F \times B \rightarrow A_x$ be defined by $\varphi_x(f, b) = f^b(x)$. Then $A \wr B$ will be a topological group iff each φ_x is continuous, and thus iff

$$(2.1) \quad \varphi_x^{-1}(U_x) = \bigcup_{b \in B} [(U_{xb^{-1}} \times A^{B \setminus \{xb^{-1}\}}) \times \{b\}]$$

is open for each open set U of A and each $x \in B$. This is certainly the case if B is discrete. In proving the converse it suffices to show that B must have a finite open subset, for B is Hausdorff and homogeneous. Suppose, then, that each φ_x is continuous, so that in particular φ_e is continuous. Choose a nonempty proper open subset U of A , and choose $fb \in \varphi_e^{-1}(U_e)$. Then there are open subsets V_1 in F and V_2 in B such that $fb \in V_1 \times V_2 \subseteq \varphi_e^{-1}(U_e)$. Furthermore, it can be assumed that there are elements b_1, \dots, b_k in B and open sets W_1, \dots, W_k in A such that $f(b_j) \in W_j$ for $1 \leq j \leq k$ and $V_1 = W_1 \times \dots \times W_k \times A^{B \setminus \{b_1, \dots, b_k\}}$. But then for each $b' \in V_2$ we must have $V_1 \times \{b'\} \subseteq V_1 \times V_2 \subseteq \varphi_e^{-1}(U_e)$, and thus $V_1 \subseteq U_{b'^{-1}} \times A^{B \setminus \{b'^{-1}\}}$. But the latter is impossible

for some b' if V_2 is infinite, as can be seen by choosing $b' \notin \{b_1^{-1}, \dots, b_k^{-1}\}$. Thus the open set V_2 of B must be finite. Finally, we note that $F \times B$ is Hausdorff since it is a product of Hausdorff spaces.

By the wreath product of topological groups we shall always mean the product referred to in Theorem 2.1. The next theorem follows readily from standard facts about product spaces and the topological properties involved.

THEOREM 2.2. *Let A and B denote topological groups. (a) $A \wr B$ is compact iff A is compact and B is finite. (b) $A \wr B$ is locally compact iff either A is locally compact and B is finite, or A is compact and B is arbitrary (but discrete). (c) $A \wr B$ is separable (has a countable open base) iff A is separable and B is countable.*

In (discrete) group theory it is often useful to consider, in place of $A \wr B$, the *restricted* (standard) wreath product. This is constructed by using $A^{(B)} = \{f: f \in A^B \text{ and } f(x) \neq e \text{ for at most finitely many } x \in B\}$, rather than A^B , for the base group. This restricted wreath product is a subgroup of $A \wr B$, and is, of course, a topological group with the relative topology. However, when one is interested only in locally compact groups, as we are in studying representations, such a restricted group yields nothing new, for it will be locally compact only if B is finite, in which case it is the same as $A \wr B$.

Philip Hall was the first to show the usefulness of wreath products for the theory of infinite discrete groups, making use of the "generalized" wreath product, $Wr\{H_\lambda: \lambda \in A\}$, constructed with arbitrary ordered set A and arbitrary transitive permutation groups H_λ ; see [5] as well as references to Hall's paper in [15]. It follows readily from the law of segmentation [5, p. 172] for general wreath products, together with Theorem 2.1, that such a wreath product will become a topological group in our context only if A has a least element μ and all H_λ for $\lambda \neq \mu$ are discrete. With $A = H_\mu$ and $B = Wr\{H_\lambda: \lambda \neq \mu\}$, the group will have the form $A \wr B$, and thus will, in fact, be in the class we are considering.

Any extension of a discrete group A by a discrete group B can be embedded in the discrete wreath product $A \wr B$. Lakshmi [10] considered the corresponding statement for topological groups, obtaining an analogous theorem with suitable conditions on the groups, and with the compact-open topology, rather than the product topology, on the base group of the wreath product. His theorem implies that if A is locally compact and B is discrete, then any topological extension of A by B can be embedded in (our) $A \wr B$.

Finally, we point out that an example of Mackey's [13, pp. 309-

310], showing that a semi-direct product need not have trivial “Mackey obstruction”, is a variation on a wreath product.

3. **The Haar integral.** We now describe how the Haar integral for locally compact $A \wr B$ can be given explicitly in terms of the Haar integrals for its component subgroups. We make use of the known characterizations of the Haar integral for arbitrary direct and semi-direct products.

First consider $F = A^B$, the base group. Because of Theorem 2.2(b), there are two cases to consider: (1) A locally compact and B finite, and (2) A compact and B arbitrary (discrete). In case (1), Fubini’s Theorem and induction yield that the right Haar measure on F is that corresponding to the product of $|B|$ copies of the right Haar measure on A :

$$(3.1) \quad \int g(f)df = \int \cdots \iint g(a_1, a_2, \dots, a_n) da_1 da_2 \cdots da_n,$$

where da_1, \dots, da_n correspond one-to-one with the elements of B , the order being irrelevant, and where each da_j designates Haar measure for the corresponding copy of A in the base group [6, (13.16) and (15.29)]. Case (2) is similar, although in general involving nets and thus slightly more complicated; the details follow from [6, (13.8) and (15.17)(j)].

With F taken care of, we are ready to look at $A \wr B$ considered as a semidirect product of F and B . It is easy to show [6, (15.29)] that the right Haar integral for a semi-direct product simply corresponds to the right Haar measures for the two component groups. We formalize this for $A \wr B$ as follows.

THEOREM 3.1. *Let dw, df, db designate Haar measure on $W = A \wr B, F = A^B$, and B , respectively. Then*

$$\int g(w)dw = \iint g(fb) df db$$

for each function g with compact support on W , with df as described accompanying (3.1).

There remains the question of the modular function. For μ and ν left and right invariant Haar measures on $W = A \wr B$, respectively, we take Δ_W to be the function given by

$$\int g(w)d\nu(w) = \int \Delta_W(w)g(w)d\mu(w)$$

(for appropriately chosen factors of proportionality for μ and ν). In

the same way, we get the modular functions Δ_A , Δ_F , and Δ_B for A , F , and B .

THEOREM 3.2. *If B is infinite, then $A \wr B$ is unimodular. If B is finite, then $A \wr B$ is unimodular iff A is unimodular. In any case, for B finite, the modular function Δ_W for $W = A \wr B$ is given by*

$$\Delta_W(fb) = \Delta_F(f) = \prod_{x \in B} \Delta_A f(x) .$$

Proof. We require, tentatively, the function δ determined as follows.

Fact 3.1 [6, (15.29)]. If α is an automorphism of a locally compact group H , and ν is a right invariant Haar measure on H , then there exists a unique real number $\delta(\alpha) > 0$ such that

$$(3.2) \quad \int g(\alpha^{-1}(x))d\nu(x) = \delta(\alpha) \int g(x)d\nu(x)$$

for each function g with compact support on H .

We now recall that each $b \in B$ determines an automorphism $\bar{b}: f \mapsto f^b$ of F . Applying [6, (15.29)], we find that

$$(3.3) \quad \Delta_W(fb) = \delta(\bar{b})\Delta_F(f)\Delta_B(b)$$

for all $fb \in A \wr B$ ($f \in F$, $b \in B$). But \bar{b} merely permutes the components of each $f \in F$, and thus, taking into account the irrelevance of the order of the da_j in (3.1) (and the corresponding fact for B infinite), we see that in our case (3.2) takes the form

$$\int g(f)df = \int g(\bar{b}^{-1}(f))df = \delta(\bar{b}) \int g(f)df .$$

That is, $\delta(\bar{b}) \equiv 1$. Furthermore, B is discrete and therefore unimodular, so that $\Delta_B(b) \equiv 1$ as well. Thus (3.3) reduces to $\Delta_W(fb) = \Delta_F(f)$. In case B is infinite, then A must be compact (Theorem 2.2(b)), whence F is compact and therefore unimodular, and so $\Delta_W(fb) = \Delta_F(f) \equiv 1$. So the question of nontrivial unimodular function arises only when B is finite, and then [6, (15.29)(a)], induction, and the special nature of direct products as semi-direct products yield

$$\Delta_W(fb) = \Delta_F(f) = \prod_{x \in B} \Delta_A f(x) .$$

Noting that if B is infinite then A must be compact and hence unimodular, we have the following corollary.

COROLLARY. *$A \wr B$ is unimodular iff A is unimodular.*

4. **Representations.** The appropriate tool for use in determining the irreducible representations of the groups $A \wr B$ is, as noted in the Introduction, the theory of induced representations. The irreducible representations of the base group can be described in terms of the irreducible representations of A . These irreducible representations of the base group can then be used to generate representations of $A \wr B$ by induction from an intermediate subgroup. There are two questions: (1) Is every representation of $A \wr B$ obtained in this way irreducible? (2) Does every irreducible representation of $A \wr B$ appear as such an induced representation? In the case of finite groups this program is completely successful: the answer to both questions is Yes. We shall see in this section that the program can be shown to be successful for a much larger class of separable locally compact $A \wr B$. For the finite case see [16], [1], [7, pp. 583-585], [8], [9].

We consider groups that are separable and locally compact; the wreath products in this class are specified by Theorem 2.2. We also assume A to be of type I: this includes the possibilities that A is compact, or Abelian, or a semi-simple Lie group, or discrete and Abelian-by-finite. We assume acquaintance with Mackey's papers on induced representations, especially [11]. By a representation of a group N we mean a continuous homomorphism $x \mapsto L_x$ of N into the group of all unitary transformations of a separable Hilbert space $H(L)$ onto itself, where continuity means that for each $\psi \in H(L)$ the function $x \mapsto L_x(\psi)$ is continuous from N into $H(L)$. If N is a closed subgroup of a group G , then $L \upharpoonright G$ will denote the representation of G induced by the representation L of N . If G is a group, then \hat{G} will denote a set of irreducible representations of G , one from each equivalence class.

We begin by considering the base group, $F = A^B$. Its irreducible representations are the tensor products of those of its component subgroups A_i ; let us make this explicit.

LEMMA 4.1. *There is a one correspondence between the set \hat{F} and the set of all maps $\mu: B \rightarrow \hat{A}$ having the property that $\mu(b)$ is the trivial representation almost always. Given μ , the corresponding representation L of F is defined by*

$$L_f = \mu(b_1)_{f(b_1)} \otimes \cdots \otimes \mu(b_n)_{f(b_n)},$$

where $\mu(b)$ is trivial for $b \neq b_1, \dots, b_n$.

Proof. If B is infinite, then A must be compact, and this is [6, II, (27.43)]. If B is finite, then A is (still) of type I, and the result follows from [11, pp. 199-200].

Consider the action of B on \hat{F} defined as follows: for $L \in \hat{F}$ and $b \in B$, $L \mapsto L^b$, where $L_f^b = L_{bfb^{-1}} = L_{fb^{-1}}$. Then, for each $L \in \hat{F}$, the *stability factor* of L in B is defined to be

$$H_L = \{b \in B: L^b \sim L\},$$

where \sim denotes equivalence. The *stability group* of L is FH_L . We can characterize the stability factors of elements of \hat{F} by using the following notion.

DEFINITION. Let H denote a subgroup of B . A function μ on B is *barely constant on the (left) cosets of H* if

$$(4.1) \quad \mu(b_i) = \mu(b_j) \quad \text{whenever} \quad b_j b_i^{-1} \in H$$

but if (4.1) does not hold with H replaced by any subgroup of B strictly containing H .

LEMMA 4.2. Assume $\mu: B \rightarrow \hat{A}$, trivial almost always. Then μ is barely constant on the cosets of H iff H is the stability factor for the representation L (of F) corresponding to μ .

Proof. Let L denote the representation corresponding to μ as in Lemma 4.1. Typical components of L_f and L_f^b will be $\mu(b_i)_{f(b_i)}$ and $\mu(b_j)_{f(b_j b)}$, respectively. For B infinite, we can argue as in [8] (or [7, pp. 583-585]) merely extending from finite to arbitrary compact groups. We see that if μ is barely constant on the cosets of H , and if $b \in H$, then $\mu(b_i) = \mu(b_j)$ when $b_i = b_j b$; thus L and L^b have equal characters and are, therefore, equivalent [6, II, (27.32)]. Assume, conversely, that b is in the stability factor for L , and choose $f \in F$ with $f(x) = e$ for $x \neq b_k$ (any fixed k). Take characters, and deduce that

$$\text{tr } \mu(b_k)_{f(b_k)} \cdot \text{deg } \mu(b_k b^{-1}) = \text{tr } \mu(b_k b^{-1})_{f(b_k)} \cdot \text{deg } \mu(b_k)$$

for all $f(b_k) \in A$. Using [6, II, (27.30)], this implies that $\mu(b_k)$ and $\mu(b_k b^{-1})$ must be equivalent. Thus μ must be constant on the cosets of $\langle b \rangle$, and the lemma is established for infinite B . If B is finite, the lemma is a consequence of [11, pp. 199-200].

We must next extend representations of F to representations of their stability groups. This can always be done [13], but it might necessitate the introduction of projective representations corresponding to nontrivial multipliers, even in the case of semi-direct products; in fact, as pointed out at the end of §3, Mackey has used a variation on a wreath product to illustrate this point. In our case, however,

projective representations are not required, as the next result shows.

LEMMA 4.3. (Specht). *Each $L \in \widehat{F}$ can be extended to $L' \in \widehat{FH}_L$. (The Mackey obstruction is trivial.)*

Proof. With L as in Lemma 4.1, each b_k -component can be represented in matrix form with entries $\mu(b_k)_{f(b_k)}(\alpha_k, \beta_k)$ [14, pp. 112-114]. Then L_f will be represented by the matrix with $\alpha_1\beta_1 \cdots \alpha_n\beta_n$ entry as indicated:

$$(u(b_1)_{f(b_1)}(\alpha_1, \beta_1) \cdots \mu(b_n)_{f(b_n)}(\alpha_n, \beta_n)) .$$

Following Specht [16, p. 330], the desired representation L' will have matrix form with the $\alpha_1\beta_1 \cdots \alpha_n\beta_n$ entry of $L'_f b$ as indicated:

$$(\mu(b_1)_{f(b_1)}(\alpha_1, \beta'_1) \cdots \mu(b_n)_{f(b_n)}(\alpha_n, \beta'_n)) ,$$

for $\beta'_i = \beta_j$ when $b_j = b_i b^{-1}$. Then $L' \downarrow F = L$, so that L' is irreducible because L is irreducible.

If M is a representation of H_L , and $\pi_L: FH_L \rightarrow FH_L/F \cong H_L$ is the natural homomorphism, then we can form the representation $L' \otimes M\pi_L$ of FH_L . We can now state our main theorem on representations. The terminology is that of [13].

THEOREM 4.1. *Assume A of type I, and $A \wr B$ separable and locally compact. Then the correspondence $M \leftrightarrow (L' \otimes M\pi_L) \uparrow (A \wr B)$ is one-one between the set of primary representations of H_L and the set of primary representations of $A \wr B$ which yield a multiple of L on restriction to F . The correspondence preserves irreducibility and type.*

Proof. The theorem is a consequence of the preceding lemmas and Theorem 8.4 of [13], if only we verify that F is regularly embedded in $A \wr B$. However, F is of type I and thus \widehat{F} is standard, and so Theorem 9.2 of [13] applies; that there is a Borel set meeting each orbit exactly once can be seen from [12, p. 64] and [13, p. 294].

COROLLARY 4.1. *The equivalence classes of irreducible representations of $A \wr B$ are in one-one correspondence with the set of triples (H, φ, M) , where H denotes a finite subgroup of B , $\varphi \rightarrow \widehat{A}$ is trivial almost always and barely constant on the cosets of H , and $M \in \widehat{H}$, provided functions $\varphi, \psi: B \rightarrow \widehat{A}$ are identified whenever the corresponding representations of F are conjugate (that is, whenever $\varphi \leftrightarrow K$ and $\psi \leftrightarrow L$ and $K^b = L$ for some $b \in B$).*

COROLLARY 4.2. *If B is torsion-free then all irreducible representations of $A \wr B$ are induced from representations of the base group.*

COROLLARY 4.3. *If A is of type I, then $A \wr B$ is of type I.*

Proof. This is a consequence of Theorem 9.3 of [13], if we note that all stability factors H_L are finite and thus of type I.

COROLLARY 4.4. *All irreducible representations of $A \wr B$ are finite dimensional iff B is finite and all irreducible representations of A are finite dimensional.*

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