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THE KRULL INTERSECTION THEOREM

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Let R be a commutative ring, I an ideal in R, and A an R-module. We always have $0 \subseteqq 0^s \subseteqq I \bigcap_{n=1}^{\infty} I^n A \subseteqq \bigcap_{n=1}^{\infty} I^n A$ where S is the multiplicatively closed set $\{1-i \mid i \in I\}$ and $0^s = 0_s \cap A = \{a \in A \mid \exists s \in S \ni sa = 0\}$. It is of interest to know when some containment can be replaced by equality. The Krull intersection theorem states that for R Noetherian and A finitely generated $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$. Since $\bigcap_{n=1}^{\infty} I^n A$ is finitely generated, $\bigcap_{n=1}^{\infty} I^n A = 0^s$. Thus if $I \subseteqq \text{rad}(R)$, the Jacobson radical of R, or R is a domain and A is torsion-free, we have $\bigcap_{n=1}^{\infty} I^n A = 0$. In this note we show that for a Prüfer domain R and a torsion-free R-module A, $I \bigcap_{i=1}^{\infty} I^n A = \bigcap_{i=1}^{\infty} I^n A$. We also consider the condition (*): $\bigcap_{n=1}^{\infty} I^n = 0$ for every ideal I in the commutative ring R. It is shown that a polynomial ring in any set of indeterminants over a Noetherian domain and the integral closure of a Noetherian domain satisfy (*).

Let R be a ring and A an R-module. If $x \in R$ and $x \notin Z(A)$, the zero divisors of A, then $(x) \bigcap_{n=1}^{\infty} (x)^n A = \bigcap_{n=1}^{\infty} (x)^n A$. Actually we can take I to be invertible and A torsion-free. However, the assumption $x \notin Z(A)$ is essential. For example, let $p \in R$ be neither a unit nor a zero divisor and let $F = Rx \oplus (\sum_{i=1}^{\infty} Ry_i)$ be the free R-module on $\{x, y_1, y_2, \cdots\}$. Let $A = F/G$ where $G = (x\text{-}py_1, x\text{-}p^2y_2, \cdots)$; it is not difficult to see that $(p) \bigcap_{n=1}^{\infty} (p)^n A \neq \bigcap_{n=1}^{\infty} (p)^n A$. Using this result, one can show that the following are equivalent: (1) $\dim R = 0$, (2) for every finitely generated (principal) ideal I and every R-module A, $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I_n A$. The first theorem gives another affirmative case.

THEOREM 1. *Let R be a reduced ring and let I be a finitely generated ideal with rank $I \leqq 1$. Then $\bigcap_{n=1}^{\infty} I^n = I \bigcap_{n=1}^{\infty} I^n$. If R is quasi-local or R is a domain, then $\bigcap_{n=1}^{\infty} I^n = 0$.*

Proof. First suppose R is a domain. By localization we can assume $\sqrt{I} = M$ the maximal ideal of R. If $B = \bigcap_{n=1}^{\infty} I \neq 0$, then $\sqrt{B} = M$, so there exists an integer m such that $I^m \subseteqq B$. Then $I^m = I^{m+1}$ which implies $I^m = 0$ by Nakayama's lemma. Next suppose R is quasi-local, by passing to R/P where P is a minimal prime we get $\bigcap_{n=1}^{\infty} I^n \subseteqq P$. Since R is reduced, we have $\bigcap_{n=1}^{\infty} I^n \subseteqq \text{nil}(R) = 0$. The general case now follows by localization.

Another affirmative case is R a Prüfer domain and A a torsion-free R-module. We first consider the quasi-local case.

LEMMA 1. *Let V be a valuation domain, I an ideal in V, and A a torsion-free V-module. Then $ib \in B = \bigcap_{n=1}^{\infty} I^n A$ where $i \in I$ and $b \in A$ implies $i \in \bigcap_{n=1}^{\infty} I^n$ or $b \in B$. In particular, $B = IB$.*

Proof. Suppose $i \notin \bigcap_{n=1}^{\infty} I^n$, then there exists an integer N such that $i \in I^{N-1} - I^N$. Now $ib \in I^m A$ for $m > N$ implies $ib = j^N j^{m-N} a$ for some $j \in I$ and $a \in A$. Now $i \notin I^N$ gives $j^N = si$ for some $s \in V$. Hence $ib = sij^{m-N}a$ so $b = sj^{m-N}a \in I^{m-N}A$ since A is torsion-free. Therefore $b \in B$.

THEOREM 2. *Let R be a Prüfer domain, I an ideal in R, A a torsion-free R-module, and $B = \bigcap_{n=1}^{\infty} I^n A$. Then $B = IB$.*

Proof. Let $y \in B$ and $J = (IB: y)$; it suffices to show $J = R$. Let M be a fixed maximal ideal; we show that $J \nsubseteq M$. Now $y \in B \subseteq B_M \subseteq \bigcap_{n=1}^{\infty} I_M^n A_M = I_M^2 \bigcap_{n=1}^{\infty} I_M^n A_M$ by Lemma 1, so $y = i^2(b/s)$ where $i \in I$, $b \in A$, $s \in R - M$ and $b/s \in \bigcap_{n=1}^{\infty} I_M^n A_M$. Let N be any maximal ideal of R, then $i^2 b = sy \in B \subseteq \bigcap_{n=1}^{\infty} I_N^n A_N$ so by Lemma 1, $i \in \bigcap_{n=1}^{\infty} I_N^n$ or $ib \in \bigcap_{n=1}^{\infty} I_N^n A$. In either case, $ib \in \bigcap_{n=1}^{\infty} I_N^n A_N$ for every maximal ideal N of R, so $ib \in B$. Therefore, $s \in J - M$.

We remark that for a Prüfer domain, $\bigcap_{n=1}^{\infty} I^n$ need not be a prime ideal, but is always a radical ideal.

Consider the condition $(*)$ on a ring. Local rings and Noetherian domains satisfy this condition. The next two propositions are straight forward and the proofs are omitted.

PROPOSITION 1. *If R satisfies $(*)$, then $Z(R) \subseteq \operatorname{rad}(R)$. Conversely, if R is Noetherian, then $Z(R) \subseteq \operatorname{rad}(R)$ implies $(*)$.*

PROPOSITION 2. *If R satisfies $(*)$, then R_M satisfies $(*)$ for every maximal ideal M. If R_M satisfies $(*)$ for every maximal ideal M, then $\bigcap_{n=1}^{\infty} I^n = I \bigcap_{n=1}^{\infty} I^n$ for every ideal I in R. If $Z(R) \subseteq \operatorname{rad}(R)$, then R satisfies $(*)$.*

The next theorem generalizes the Krull intersection theorem to rings which are locally Noetherian.

THEOREM 3. *Let R be a ring and A an R-module such that $\bigcap_{n=1}^{\infty} P_P^n A_P = 0$ for every $P \in \operatorname{spec}(R)$, then $\bigcap_{n=1}^{\infty} I^n A = 0^s$ for every ideal I in R.*

Proof. Let T be the saturation of $S = \{1 - i \mid i \in I\}$, so $T =$

$R - \bigcup_{P_\alpha \in \mathscr{S}} P_\alpha$ where $\mathscr{S} = \{P \in \operatorname{spec}(R) \mid P \cap T = \varnothing\}$. Then setting $B = \bigcap_{n=1}^{\infty} I^n A$ yields $B_P \subseteqq \bigcap_{n=1}^{\infty} I_P^n A_P = 0$ for every $P \in \mathscr{S}$. Hence $(T^{-1}B)_{T^{-1}P} = 0$ for every $P \in \mathscr{S}$, but the $T^{-1}P \in \mathscr{S}$ are precisely the prime ideals of $T^{-1}R$. Therefore $T^{-1}B = 0$, hence $B_s = 0$ and the result follows.

The next proposition will be used to prove that a polynomial ring in any number of indeterminants over a Noetherian domain satisfies $(*)$.

PROPOSITION 3. *Let R be a Noetherian ring, I an ideal in $R[X]$, and $B = \bigcap_{n=1}^{\infty} I^n$. Then $B = (B \cap R)R[X]$.*

Proof. First suppose $I \cap R = 0$, we show that $B = 0$. Suppose $0 \neq g(x) \in B$, by the Krull intersection theorem there exists a polynomial $f(x) = a_0x^n + \cdots + a_n \in I$ such that $g(x)(1 - f(x)) = 0$. Since $1 - f(x) \in Z(R[X])$, there exists $0 \neq c \in R$ such that $c(1 - f(x)) = 0$. Hence $0 = ca_0 = \cdots = ca_{n-1} = c(a_n - 1)$ so $c = ca_n$. But $ca_n = cf(x) \in I \cap R = 0$ so $c = ca_n = 0$, a contradiction. For the general case, let $J = I^m \cap R$, passing to $(R/J)[X]$ yields $B \subseteqq JR[X]$, hence $B \subseteqq \bigcap_{n=1}^{\infty} (I^n \cap R)[X] = (B \cap R)[X] \subseteqq B$.

THEOREM 4. *Let R be a Noetherian domain and $T = R[\{X_\alpha\}]$ a polynomial ring over R in any set $\{X_\alpha\}$ of indeterminants. Then T satisfies $(*)$.*

Proof. We may assume the set of indeterminants is countable and hence index it by the positive integers. By Proposition 2 we may assume that $(R, \mathscr{M})$ is local and we only need show that $\bigcap_{n=1}^{\infty} M^n = 0$ where M is a maximal ideal in T with $M \cap R = \mathscr{M}$. Let K be the algebraic closure of $k(\{z_\beta\})$ where $\{z_\beta\}$ is an uncountable set of indeterminants over $k = R/\mathscr{M}$. There exists a local ring (B, N) with $B \supseteqq R$ faithfully flat, $N = \mathscr{M}B$ and $B/N = K$ [1]. Now $B \supset R$ faithfully flat implies $MB[\{X_i\}] \neq B[\{X_i\}]$ so $MB[\{X_i\}] \subseteqq M^*$ a maximal ideal in $B[\{X_i\}]$. It is sufficient to show $\bigcap_{n=1}^{\infty} M^{*n} = 0$. Since

$$[B[\{X_i\}]/M^* \colon B/N]$$

is countable and $B/N = K$ is uncountable and algebraically closed, $B[\{X_i\}]/M^* = K$. Thus $M^* = (\mathscr{M}, X_1 - r_1, X_2 - r_2, \cdots)$ for suitable $r_i \in B$. Since a given polynomial involves only finitely many indeterminants, it suffices to show $\bigcap_{n=1}^{\infty} (\mathscr{M}, X_1 - r_1, x_m - r_m)^n = 0$ in $B[X_1, \cdots, X_m]$. Since $(\mathscr{M}, X_1 - r_1, \cdots, X_m - r_m)^n \cap B[X_1, \cdots, X_{m-1}] = (\mathscr{M}, X_1 - r_1, \cdots, X_{m-1} - r_{m-1})^n$, the result follows from Proposition 3 and induction.

The last theorem gives another class of rings where $(*)$ holds.

THEOREM 5. *Let R be Noetherian domain and R' its integral closure. Then any ring between R and R' satisfies* $(*)$.

Proof. Let $R \subseteqq T \subseteqq R'$ be a ring, since $T \subseteqq R'$ is integral, any ideal of T is contained in the contraction of an ideal of R', thus we may assume $T = R'$. It suffices to prove the result for (R, M) a local domain. Now $R \subseteqq \hat{R}/N \subseteqq \hat{R}/P_1 \oplus \cdots \oplus \hat{R}/P_n \subseteqq (\hat{R}/P_1)' \oplus \cdots \oplus (\hat{R}/P_n)'$ where $\hat{R}$ is the completion of R, $N = P_1 \cap \cdots \cap P_n$, and $P_1, \cdots, P_n$ are the minimal primes of $\hat{R}$. Now each $\hat{R}/P_i$ is a complete local domain, so each $(\hat{R}/P_i)'$ is a Noetherian domain and hence satisfies $(*)$. Every maximal ideal $\mathscr{M}$ of R' has the form $\mathscr{M} = M^* \cap R'$ for some maximal ideal M^* of $(\hat{R}/P_1)' \oplus \cdots \oplus (\hat{R}/P_n)'$ [2, p. 119]. Hence $M^* = (\hat{R}/P_1)' \oplus \cdots \oplus N \oplus \cdots \oplus (\hat{R}/P_n)'$ where N is a maximal ideal in $(\hat{R}/P_i)'$ for some i. Then $\bigcap_{n=1}^{\infty} \mathscr{M}^n = \bigcap_{n=1}^{\infty} (M^* \cap R')^n \subseteqq (\bigcap_{n=1}^{\infty} M^{*n}) \cap R' = I_i \cap R'$ where $I_i = (\hat{R}/P_1)' \oplus \cdots \oplus 0 \oplus \cdots \oplus (\hat{R}/P_n)'$. Suppose $I_i \cap R' \neq 0$, then $I_i \cap R \neq 0$ since $R \subseteqq R'$ is integral. But $0 \neq a \in I_i \cap R$ implies $a \in P_i \subseteqq Z(\hat{R})$, a contradiction.

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