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# COMPACTNESS IN ABELIAN TOPOLOGICAL GROUPS

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**If an abelian topological group $(G, t)$ satisfies Pontryagin duality, then the $t$-compact subsets and the weak compact subsets coincide. Hence if $t_1, t_2$ are group topologies on an abelian group $G$ such that $(G, t_i)$ $i = 1, 2$ satisfy Pontryagin duality and have the same set of continuous characters, then $t_1 = t_2$.**

**1. Introduction.** Our object is to prove the following two theorems:

THEOREM 1.1. *Let $(G, t)$ be an abelian topological group which satisfies Pontryagin duality. Let $w$ be the weak topology induced by the set $X$ of all its continuous characters. Then a subset $A$ of $G$ is $t$-compact iff it is $w$-compact.*

THEOREM 1.2. *Let $t_1$, $t_2$ be topologies on an abelian group $G$ such that $(G, t_i)$, $i = 1, 2$, is a topological group that satisfies Pontryagin duality. If the sets of continuous characters of $(G, t_1)$ and $(G, t_2)$ coincide, then $t_1 = t_2$.*

Glicksberg ([5] Theorem 1.2 and Corollary 2.4) proves the above two results for the special case of locally compact abelian groups. Glicksberg's work has been generalized in different manners. One such is to start with a locally compact group $(G, t)$, $G$ not necessarily abelian, consider the weak topology $w$ induced by the class of $t$-continuous irreducible unitary representations of $G$ and ask whether compactness with respect to $t$ and $w$ coincide. This corresponds to Theorem 1.1 above. Corresponding to Theorem 1.2 one can ask whether if $t_1$, $t_2$ are locally compact topologies on a not necessarily abelian group $G$ such that both $(G, t_1)$ and $(G, t_2)$ are topological groups and have the same set of continuous irreducible unitary representations, is $t_1 = t_2$? These two questions were answered in the affirmative by Hughes ([3], Theorem 1, page 18, and Corollary 1.3, page 19). (See also [4] Theorems 1 and 2.) Another direction of generalization of Glicksberg's results is to regard the set $X$ of continuous characters of $G$ as Hom $(G, T)$ where $T$ is the circle group and examine the following question. Let $G$ and $H$ be topological groups and Hom $(G, H)$, the set of all continuous homomorphisms of $G$ into $H$. Do compactness with respect to the pointwise convergent topology and the compact-open topology on Hom $(G, H)$ coincide?

Corson and Glicksberg ([1] Theorem 1) asserted that if $G$ is such that the closure of every compactly generated subgroup is of second category in itself, this is indeed so. Most recently Namioka [10] has remarked that the proof of this assertion in [1] is incomplete. Using different methods, Namioka ([10], Theorem 3.3) proves the above result in the special case when $G$ is strongly countably compact. However the assertion of Corson and Glicksberg is proved by using an independent approach by Hughes ([3] Theorem 3). (See also [4] Theorem 4.)

Our objective in this paper is to examine Glicksberg's results in [5] referred to above for the case of abelian topological groups which satisfy Pontryagin duality and which are not necessarily locally compact abelian groups. That such groups exist is well known (cf. for instance, Kaplan [7] and Venkataraman [11]). The proofs in this direction of generalization are much simpler than those in the other directions mentioned above. Our proofs avoid measure theory and Fourier transforms and use only results on the topologies on function spaces. In the case of Theorem 1.1, our approach to the proof is the same as Hughes ([3] Theorem 3) but becomes different when we invoke the use of Bohr compactifications. The proof of Theorem 1.2 depeds upon a result of mine ([11] Theorem 5.3).

2. *Proofs of Theorems* 1.1 *and* 1.2. For notation and terminology relating to topological groups we follow Hewitt and Ross [2].

Let $T$ denote the circle group. A homomorphism of a group $G$ to $T$ is called a character of $G$. The set $X$ of all continuous characters of a topological group $G$ endowed with the compact-open topology and multiplication defined pointwise is a topological group, called the character group of $G$. $G$ is said to satisfy Pontryagin duality if the evaluation map $\tau_G\colon G \rightarrow \omega$ (where $\omega$ is the character group of $X$) defined by $\tau_G(x)\chi = \chi(x)$ for every $\chi \in X$ and $x \in G$ is a topological isomorphism.

DEFINITION 2.1. Let $G$ be an abelian topological group, $X$ the character group of $G$, $W$ a limited neighbourhood of 1 in $T$, i.e. a symmetric connected neighbourhood of 1 which is contained in $\{z \in T \mid |z-1| \leqq \sqrt{2}\}$ (cf. Kaplan [7], page 650). Let $S \supset G$. By the *W-character closure of S* (in symbols $W$-chr. cl. $S$) we mean

$$\cap \{\chi^{-1}(W) \mid \chi \in X,\ \chi(S) \subset W\}\ .$$

If $S = W$-chr. cl. $S$, we say that $S$ is *W-character closed.*

PROPOSITION 2.2. *Let $G$ satisfy Pontryagin duality. Then the following properties hold*:

($P_1$): *$G$ has sufficiently many continuous characters.*

($P_2$): *Let $U$ be $W$-character closed subset of $G$ for some closed limited neighbourhood $W$ of* 1 *in $T$ such that for every compact subset $K$ of $G$ with $K$ containing the identity element $e$ of $G$, $U \cap K$ is a neighbourhood of $e$ relative to $K$. Then $U$ is a neighbourhood of $e$ in $G$.*

($P_3$): *$G$ admits a family of $W$-character closed subsets as a neighbourhood base at $e$ for some limited neighbourhood $W$ of* 1 *in $T$.*

*Proof.* Any topological group that satisfies Pontryagin duality has sufficiently many continuous characters and hence $G$ satisfies $P_1$. That such a group satisfies the property $P_2$ is proved in Theorem 5.3 of Venkataraman [11]. We use the notations in the beginning of this section. By Proposition 2.9 of Kaplan [7], $\{P(K^*, W) \mid K^* \text{ a compact subset of } X\}$ is a neighbourhood base at the identity element of $\omega$ where $W$ is (any) one limited neighbourhood of 1 in $T$ and $P(K^*, W) = \{f \in \omega \mid f(K^*) \subset W\}$. It is easy to see that for each subset $K^*$ of $X$, $P(K^*, W)$ is a $W$-character closed subset of $\omega$. Thus $\omega$ admits a neighbourhood base of $W$-character closed subsets at the identity element, i.e. $\omega$ satisfies property $P_3$. Since by hypothesis $G$ satisfies Pontryagin duality, $\tau_G\colon G \to \omega$ is a topological isomorphism. It follows that $G$ satisfies $P_3$. The proof of our proposition is complete.

For notation and terminology on topologies of function spaces we follow Kelley ([8], Chapter 7).

*Proof of Theorem* 1.1. Clearly any $t$-compact subset of $G$ is $w$-compact as $w$ is coarser than $t$. So let $A$ be a $w$-compact subset of $G$. As $G$ satisfies Pontryagin duality, the topology $t$ on $G$ is the compact open topology $\mathscr{C}$ when $G$ is regarded in the canonical manner as the character group of $X$. Also in this context, the weak topology $w$ on $G$ determined by $X$ is the topology $\mathscr{P}$ of pointwise convergence on $X$. Thus what we are required to prove is that if $A \subset G$ and $A$ is $\mathscr{P}$-compact, then $A$ is $\mathscr{C}$-compact. From the sixth of the six equivalent statements in Hughes' version of the Arzela-Ascoli Theorem (cf. Hughes [3], Proposition 15), it suffices to consider compact subset $D$ of $X$ and any sequence $\{f_n\}_n$ in $A$, and prove the following: (*) There is a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $\{f_{n_k}|_D\}_k$ is equicontinuous.

Since $A$ is by hypothesis $\mathscr{P}$-compact, a fortiori $\{f|_D \mid f \in A\}$, which we shall denote by $A|_D$, is $\mathscr{P}$-compact. So by Theorem 5, Grothendieck [6], $A|_D$ is sequentially compact. Therefore there exists a subsequence

$\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $\{f_{n_k}\}_k$ is convergent pointwise *on* $D$ to some $f$ in $A$. Now since $G$ satisfies Pontryagin duality and any compact abelian group satisfies Pontryagin duality, $G$ can be regarded as the *set* of all continuous characters of $X$ and $bX$. Here $bX$ is the so called *Bohr compactification of* $X$, i.e. $bX$ is the compact character group of $(G, d)$ where $d$ is the discrete topology on $G$. Let $\rho$ be the canonical map of $X$ to $bX$ which identifies a continuous character of $(G, t)$ with a (continuous) character of $(G, d)$. It is easily verified that in our case $\rho$ is a continuous monomorphism of $X$ onto a dense subgroup of $bX$. Furthermore the topology on $G$ of pointwise convergence on $bX$ (equivalently $\rho(X)$) is the same as $\mathscr{P}$. Thus the subset $A \subset G$ we started with can be regarded as a $\mathscr{P}$-compact subset of continuous functions from $bX$ to $T$ and the subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ is pointwise convergent on $\rho(D)$ to $f$ in $A$. Let $F$ be the closed subgroup of $bX$ generated by $D$. The topology on $A|_F$ of pointwise convergence on $bX$ is clearly compact as $A$ is $\mathscr{P}$-compact. This implies that the topology $\mathscr{P}_1$ on $A|_F$ of pointwise convergence on $\rho(D)$ is compact. Furthermore it is $T_2$. For if $g, h \in A|_F$ and $g \neq h$, as $F$ is the closed subgroup generated by $\rho(D)$, there exists $\chi$ in $\rho(D)$ such that $g(\chi) \neq h(\chi)$. So $\rho(D)$ distinguishes members of $A|_F$. So by Theorem 2 page 220 Kelley [8], the topology $\mathscr{P}_1$ is $T_2$. The topology $\mathscr{P}_2$ on $A|_F$ of pointwise convergence on $F$ is also compact as $A$ is compact with respect to pointwise convergence on $bX$, and clearly $\mathscr{P}_2$ is coarser than $\mathscr{P}_1$. So $\mathscr{P}_2 = \mathscr{P}_1$. Since the subsequence $\{f_{n_k}\}_k$ converges pointwise on $\rho(D)$ to $f$ it now follows that it converges pointwise on $F$ to $f$. As the topology on $T$ is given by an invariant metric, by Osgood's theorem (Theorem 9.5, Kelley, Namioka and co-authors [9]) the set of points of equicontinuity of $\{f_{n_k}|_F\}_k$ is residual in $F$ and, as $F$ is a closed subgroup of the compact group $bX$, hence itself is compact, this residual subset is nonvoid. As the topology on $T$ is given by an invariant metric it follows that $\{f_{n_k}|_F\}$ is equicontinuous on $F$. As $\rho$ is a continuous monomorphism of $X$ to $bX$ it follows that $\{f_{n_k}|_{\rho^{-1}(F)}\}_k$ is equicontinuous. As $\rho^{-1}(F)$ clearly contains $D$, we have that $\{f_{n_k}|_D\}_k$ is equicontinuous. Thus the statement (*) above has been proved. The proof of Theorem 1.1 is complete.

*Proof of Theorem* 1.2. Let $t_1, t_2$ be topologies on the abelian group $G$ such that $(G, t_i)$ $i = 1, 2$ are topological groups that satisfy Pontryagin duality and have the same set $X$ of continuous characters.

It follows that $(G, t_i)$ $i = 1, 2$ have the same Bohr compactification $bG$ and the canonical maps of $(G, t_i)$ $i = 1, 2$ to $bG$ are the same map $\rho$. By Theorem 1.1 it follows that $(G, t_i)$ $i = 1, 2$ have the same family of compact subsets. Now let $W$ be any closed limited neighbourhood of 1 in $T$. As $(G, t_i)$ $i = 1, 2$ have the same set $X$ of continuous characters, they have the same family of $W$-character closed subsets (cf. Definition 2.1 above). By Proposition 2.2 as $(G, t_i)$ $i = 1, 2$ satisfy Pontryagin duality, they satisfy properties $P_1$, $P_2$ and $P_3$. Thus every $W$-character closed neighbourhood of the identity of $(G, t_1)$, because of the property $P_2$ of $(G, t_2)$ and because $(G, t_1)$ have the same family of compact subsets, will be a $W$-character closed neighbourhood of identity of $(G, t_2)$ and vice versa. Because $(G, t_i)$ $i = 1, 2$ have property $P_3$, it will follow that they both have the same family of neighbourhoods of identity as neighbourhood bases. As they are both topological groups, it will follow that $t_1 = t_2$. This completes the proof of Theorem 1.2.

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