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CONCERNING σ-HOMOMORPHISMS OF RIESZ SPACES

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If L is a Riesz space (lattice ordered vector space), a Riesz homomorphism of L is an order preserving linear map which preserves the finite operations "$\vee$" and "$\wedge$". It was shown in our previous paper ["Homomorphisms of Riesz spaces," Pacific J. Math.] that there is a large class α of spaces such that if L belongs to α and φ is a Riesz homomorphism from L onto an Archimedean Riesz space, then φ preserves the order limit of sequences. In this paper the list of members of α is extended. It is further shown that there is a large class β of spaces with the property that if L belongs to α and φ is a Riesz homomorphism of L into an Archimedean Riesz space then φ preserves the order limit of sequences.

This paper is a continuation and extension of Tucker [4]. The notation and terminology of Tucker [4] will be used.

LEMMA 1. *Suppose L is a Riesz space with the principal projection property, K is an Archimedean Riesz space, and φ is a Riesz homomorphism of L into K with the property that if $\{b_1, b_2, b_3, \cdots\}$ is a countable orthogonal subset of L^+ such that $b = \bigvee b_i$, then $\varphi(b - \sum_{i=1}^{j} b_i) \to \theta$, then φ preserves the order limits of sequences.*

Proof. Suppose $f_1, f_2, f_3, \cdots$ is a sequence of points of L such that $f_1 \geqq f_2 \geqq f_3 \geqq \cdots \geqq \theta$ and $\bigwedge f_i = \theta$. Suppose, further, that n is a positive integer. For each i let $g_i = f_i - (1/2^n)f_1$, $h_i = \bigvee_p (pg_i^+ \wedge g_1)$, and $b_i = h_i - h_{i+1}$. Consider b_i and b_j where $j > i$. Now $b_j \leqq h_{i+1}$ and $b_i \leqq g_1 - h_{i+1}$, so that $\theta = h_{i+1} \wedge (g_1 - h_{i+1}) \geqq b_j \wedge b_i$. Thus $\{b_1, b_2, b_3, \cdots\}$ is a countable orthogonal set.

Since $g_1 \geqq b_i$ for each i, g_1 is an upper bound of $\{b_1, b_2, b_3, \cdots\}$. Suppose α is a point such that $g_1 - \alpha \geqq b_i$ for each i and $\alpha \geqq \theta$. Let i be a positive integer and let β be the projection of α on b_i. Then $g_1 - \beta \geqq b_i$. Now $b_i + \sum_{j=1}^{i-1} b_j + h_{i+1} = h_1 = g_1$ and $b_i \wedge (\sum_{j=1}^{i-1} b_j + h_{i+1}) = \theta$ so that $\beta \wedge (g_1 - b_i) = \theta$. Since $g_1 - \beta \geqq b_i$, $g_1 - b_i \geqq \beta$ which implies $(g_1 - b_i) \wedge \beta = \beta$, so that $\beta = \theta$. Thus for each i, $\alpha \wedge b_i = \theta$. Now $g_1 \geqq b_i$ so that $g_1 \geqq \bigvee_{j=1}^{i} b_j \vee \alpha = \sum_{j=1}^{i} b_j + \alpha = h_1 - h_{i+1} + \alpha = g_1 - h_{i+1} + \alpha$ which implies $h_{i+1} \geqq \alpha$.

Now $g_i^- \wedge g_i^+ = \theta$ which implies $g_i^- \varepsilon (g_i^+)^d$. As h_i is the projection of g_1 on g_i^+, $h_i \wedge g_i^- = \theta$.

Without loss of generality we may assume that $(1/2^n)f_1 \geqq \alpha$. Then

$h_i \wedge g_i^- = \theta$ implies $\alpha \wedge g_i^- = \theta$ for each i. The relationships $(1/2^n)f_1 \geqq \alpha$ and $(1/2^n)f_1 \geqq g_i^-$ imply that $(1/2^n)f_1 \geqq \alpha \vee g_i^- = \alpha + g_i^-$. So that, $(1/2^n)f_1 - \alpha \geqq g_i^-$. It follows that $-(1/2^n)f_1 + \alpha \leqq -g_i^-$, $-(1/2^n)f_1 + \alpha \leqq -g_i^- + g_i^+ = g_i$, and $\alpha \leqq g_i + (1/2^n)f_1 = f_i$. But $\bigwedge f_i = \theta$ so that $\alpha = \theta$. Thus $g_1 = \bigvee b_i$. By hypothesis

$$\varphi\Big(g_1 - \sum_{i=1}^{j} b_i\Big) \longrightarrow \theta \ , \ \ \varphi(g_1 - (g_1 - h_{j+1})) \longrightarrow \theta \ , \ \text{ and } \ \varphi(h_{j+1}) \longrightarrow \theta \ .$$

Since

$$h_{j+1} \geqq g_{j+1}^+ \geqq \theta \ , \quad \varphi(g_i^+) \longrightarrow \theta \ .$$

Thus if there is a point γ such that $\varphi(f_i) \geqq \gamma \geqq \theta$, then $\gamma \leqq \varphi((1/2^n)f_1) = (1/2^n)\varphi(f_1)$ which implies $\gamma = \theta$ since K is Archimedean.

We can replace Theorem 8 of [4] with a slightly stronger statement:

THEOREM 8′. *Suppose L is a Riesz space with the principal projection property, M is a uniformly closed ideal of L with the property that if $\{f_1, f_2, f_3, \cdots\}$ is a countable orthogonal subset of M^+ and there is a point h of L such that $\bigvee f_i = h$ there is an unbounded nondecreasing positive number sequence $c_1, c_2, c_3, \cdots$ such that $\{c_1f_1, c_2f_2, c_3f_3, \cdots\}$ is bounded, and π is the natural map of L onto L/M. Then L/M has the principal projection property and $\pi P_f = P_{\pi f}\pi$ for each point f of L.*

The argument given for Theorem 8 in [4] proves this statement also.

This leads to the following definition:

The statement that the Riesz space L has *Property* A means that if $\{f_1, f_2, f_3, \cdots\}$ is a countable orthogonal subset of L^+ and $\bigvee f_i$ exists, then there exists a point g of L and a nondecreasing undounded positive number sequence $c_1, c_2, c_3, \cdots$ such that $g \geqq c_if_i$ for each i.

Note that if order convergence is stable in L, then L has Property A. On the other hand Property A is more general than stability of order convergence since R^X has Property A but, as pointed out in [4], order convergence is not in general stable in R^X.

THEOREM 2. *Suppose L is a Riesz space with the principal projection property and with Property* A *and φ is a Riesz homomorphism of L into an Archimedean Riesz space K. Then φ preserves the order limits of sequences.*

Proof. Suppose $\{b_1, b_2, b_3, \cdots\}$ is a countable orthogonal subset of L^+ such that $b = \bigvee b_i$. There exists a nondecreasing unbounded positive number sequence $c_1, c_2, c_3, \cdots$ and a point g such that $c_i b_i \leqq g$. Then $\varphi(b - \sum_{i=1}^{j} b_i) = \varphi(\bigvee_{i=j+1}^{\infty} b_i) \leqq \varphi(1/c_{j+1} g) = 1/c_{j+1}\, \varphi(g)$. Since K is Archimedean $\varphi(b - \sum_{i=1}^{j} b_i) \rightarrow \theta$. By Lemma 1, φ preserves the order limits of sequences.

DEFINITION. The statement that L has *Property* B means that if $\{f_1, f_2, f_3, \cdots\}$ is a countable orthogonal subset of L^+ such that $\bigvee f_i$ exists, then there is a nondecreasing unbounded positive number sequence $c_1, c_2, c_3, \cdots$ such that $\bigvee (1/c_i) f_i$ exists.

THEOREM 3. *Suppose L is a Riesz space with the principal projection property, with Property* B, *and with the property that if φ is a Riesz homomorphism of L onto an Archimedean Riesz space and M is the kernel of φ then $P_m L$ is a subset of M for each point m of M. Then each Riesz homomorphism of L into an Archimedean Riesz space preserves the order limit of sequences.*

Proof. Suppose K is an Archimedean Riesz space and φ is a Riesz homomorphism of L into K. Now K may be embedded in a complete Riesz space in a manner which preserves the order limits of sequence (see Nakano [3], Judin [1], or Luxemburg, and Zaanen [2], p. 191) and φ may be extended to a homomorphism $\tilde{\varphi}$ from L into the completion of K. Since φ preserves the order limits of sequences if $\tilde{\varphi}$ does, K may be taken to be complete without loss of generality. For this argument it will be sufficient to assume that K has the principal projection property.

Suppose $\{b_1, b_2, b_3, \cdots\}$ is a countable orthogonal subset of L^+ such that $b = \bigvee b_i$. As L has Property B there is an unbounded nondecreasing sequence of positive numbers $c_1, c_2, c_3, \cdots$ such that $\bigvee (1/c_i) b_i$ exists. Let g denote $\bigvee (1/c_i) b_i$. Then $P_g b = b$.

Suppose α is such that $\theta \leqq \alpha \leqq \varphi(b - \sum_{i=1}^{n} b_i)$ for each n. Let $\beta = P_\alpha \varphi g$. Thus $\beta \wedge \varphi((g - (1/n)b)^+) = \theta$, $\beta \leqq (1/n)\varphi(b)$, and $\beta = \theta$ as K is Archimedean. So that $\alpha \in \{\varphi g\}^d$.

Let H be the image of φ in $\{\varphi g\}^d$ and let φ_1 be the map of L onto H. Since H is Archimedean, the kernel M of φ_1 has the property that $P_g L \subset M$. Now $P_g b = b$ so $b \in M$, i.e., $\varphi(b) \in \{\varphi g\}^{dd}$ which implies $\alpha = \theta$.

By Lemma 1, φ preserves the order limits of sequences.

COROLLARY 4. *Suppose L is a σ-complete Riesz space with the property that every Riesz homomorphism of L onto an Archimedean Riesz space preserves the order limits of sequences. Then every Riesz* *homomorphism of L into an Archimedean Riesz space preserves the order limits of sequences.*

Proof. Since L is σ-complete it has Property B. If every Riesz homomorphism of L onto an Archimedean Riesz space preserves the order limits of sequences then by Theorems A and B of [4], every uniformly closed ideal of L is a σ-ideal. By Theorem 6 of [4] every uniformly closed ideal M of L has the property that if m belongs to M then $P_m L$ is a subset of M. The result follows from Theorem 3.

Even if L were assumed to be complete and the mappings assumed to be one-to-one, Corollary 4 would not remain true if the requirement that every Riesz homomorphism of L onto an Archimedean Riesz space preserves the order limits of sequences were dropped. For instance let L be the space of bounded sequences and M be the uniformly closed ideal consisting of all sequences converging to zero. Let π be the natural map from L onto L/M. Let K be $L \times L/M$. Then L is complete and K is Archimedean. The map φ of L into K defined by $\varphi(x) = (x, \pi(x))$ is an injection which does not preserve the order limits of sequences.

If L is any Riesz space such that there exists a uniformly closed ideal M of L such that the natural map of L onto L/M does not preserve the order limits of sequences, then an Archimedean Riesz space K can be constructed so that L can be injected into K without preserving the order limits of sequences.

Clearly if L is a Riesz space with the property that if φ is a Riesz homomorphism of L into an Archimedean Riesz space then φ preserves the order limits of sequences then any Archimedean Riesz space K which is the image of L under a Riesz homomorphism has the property also. For example, let $\hat{M}$ be the space of all measurable functions defined on the interval $[0, 1]$ without identifying functions which differ only on a set of measure zero and let M be the space of all measurable functions defined on the interval $[0, 1]$ identifying functions which differ on a set of measure zero. Clearly $\hat{M}$ has a point with Property c and M is an Archimedean quotient of $\hat{M}$. Thus every Riesz homomorphism of M into an Archimedean Riesz space preserves the order limits of sequences.

So far several examples have been given of spaces with the property that every Riesz homomorphism into an Archimedean Riesz space preserves the order limits of sequences: R^X, $B[0, 1]$, the space Q of Example 2 of [4], the space of all measurable functions, L_p, $1 \leqq p < \infty$, l_p, $1 \leqq p < \infty$, c_0, the space of all functions defined on the x-axis with compact support, and the space of Example 11 of [4]. All these spaces have the principal projection property. In the following it will be shown that $B_\alpha[0,1]$, the αth Baire class of functions defined on the interval $[0, 1]$, $\alpha < \omega_1$, is uniformly complete, does not have the principal projection property, but has the property that every Riesz homomorphism into an Archimedean Riesz space preserves the order limits of sequences.

THEOREM 5. *$B_\alpha[0, 1]$ is uniformly complete.*

Proof. Suppose $f_1, f_2, f_3, \cdots$ is a sequence of functions in $B_\alpha[0, 1]$ converging relatively uniformly to f. If f is bounded, then it is in $B_\alpha[0, 1]$ as $B_\alpha[0, 1]$ is closed with respect to uniform convergence (relative uniform convergence with regulator the constant function 1). Suppose f is unbounded. Suppose further that f is in $B_\alpha^+[0, 1]$. For each positive integer n, $n \wedge f$ is in $B_\alpha[0, 1]$. Let $T(g) = g/(1 + g)$ for each g in $B_\alpha^+[0, 1]$. As T preserves continuity and pointwise convergence $T(n \wedge f) = n/(n + 1) \wedge T(f)$ is in B_α. The sequence $\{n/(n + 1) \wedge T(f)\}$ converges uniformly to $T(f)$ which is thus in $B_\alpha[0, 1]$. Therefore there exists a sequence $g_1, g_2, g_3, \cdots$ of functions in $B_{\alpha-1}[0, 1]$ converging pointwise to $T(f)$. (If α is a limit ordinal then $B_{\alpha-1}[0, 1] = \bigcup_{\gamma>\alpha} B_\gamma[0, 1]$.) These functions may be chosen so that $0 \leqq g_i \leqq 1$. Then the sequence $\{n/(n + 1)\, g_n\}$ is a sequence of functions in $B_{\alpha-1}[0, 1]$ converging pointwise to $T(f)$. Since $0 \leqq n/(n + 1)\, g_n < 1$, then $T^{-1}(n/(n + 1)g_n)$ is a sequence of functions in $B_{\alpha-1}[0, 1]$ converging pointwise to f.

Since $B_\alpha[0, 1]$ is uniformly complete it can not have the principal projection property unless it is σ-complete and $B_\alpha[0, 1]$, $\alpha < \omega_1$, is known not to be σ-complete. (If $\alpha = \omega_1$ then $B_\alpha[0, 1] = B[0, 1]$, the space of all Baire functions on $[0, 1]$, which is σ-complete.)

THEOREM 6. *If φ is a Riesz homomorphism of $B_\alpha[0, 1]$ $(\alpha > 0)$ into an Archimedean Riesz space then φ preserves the order limits of sequences.*

Proof. It will be shown that $B_\alpha[0, 1]$ has a point with Property c and the desired result will follow from Theorem 3 of [4].

Suppose $f_1, f_2, f_3, \cdots$ is a sequence of points of $B_\alpha[0, 1]$ such that $f_1 \leqq f_2 \leqq f_3 \leqq \cdots$ and $\bigvee f_i$ is the constant function 1. Denote by $USB_{\alpha-1}[0, 1]$ the set of all functions which are the greatest lower bounds of countable subsets of $B_{\alpha-1}[0, 1]$. For each f_p there is a sequence l_{ip} of functions in $USB_{\alpha-1}[0, 1]$ such that $l_{1p} \leqq l_{2p} \leqq l_{3p} \leqq \cdots \leqq f_p$ and $\bigvee l_{ip} = f_p$. Let $l_i = \bigvee_{p \leqq i} l_{ip}$.

Then $l_1, l_2, l_3, \cdots$ is a sequence of functions in $USB_{\alpha-1}[0, 1]$ such that $l_1 \leqq l_2 \leqq l_3 \leqq \cdots$, $l_i \leqq f_i$, and $\bigvee l_i$ is the constant function 1. Now $\theta \wedge l_i$ is in $USB_{\alpha-1}[0, 1]$ and $\sum_{i=1}^{n} \theta \wedge l_i$ is in $USB_{\alpha-1}[0, 1]$. For each point x there is a positive integer N such that $\theta \wedge l_n(x) = 0$ for each positive integer $n \geqq N$. (Order convergence for a bounded non-decreasing sequence in $B_\alpha[0, 1]$ is pointwise convergence.)

Thus for each x, $\sum_{i=1}^{\infty} \theta \wedge l_i(x)$ exists. For each positive integer n there exists a sequence $g_{1n}, g_{2n}, g_{3n}, \cdots$ of functions in $B_{\alpha-1}[0, 1]$ such that $g_{1n} \geqq g_{2n} \geqq g_{3n} \geqq \cdots$ and $\bigwedge g_{in} = \sum_{i=1}^{n} \theta \wedge l_i$. Then let $g_n = \bigwedge_{i \leqq n} g_{ni}$. So that $g_1, g_2, g_3, \cdots$ is a sequence of functions in $B_{\alpha-1}[0, 1]$ converging pointwise to $\sum_{i=1}^{\infty} \theta \wedge l$. Thus $\sum_{i=1}^{\infty} \theta \wedge l_i$ is in $B_\alpha[0, 1]$ and $\sum_{i=1}^{\infty} \theta \wedge l_i \leqq \sum_{i=1}^{n} f_i$ for each positive integer n.

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