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MAXIMAL CONNECTED HAUSDORFF SPACES

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A nowhere neighborhood nested space is one in which no point has a local base which is linearly ordered by set inclusion. An *MI* space is one in which every dense subset is open. In this paper we show that every Hausdorff topology without isolated points has a nowhere neighborhood nested refinement. We show that every maximal connected Hausdorff topology is *MI* and nowhere neighborhood nested, and that every connected, but not maximal connected, Hausdorff topology has a connected, but not maximal connected, nowhere neighborhood nested refinement. Every connected Hausdorff topology has a connected, *MI*, nowhere neighborhood nested refinement.

In [4] the author raised the question of the existence of nontrivial maximal connected Hausdorff spaces. The question remains open.

A topology $\mathscr{T}'$ on a set X is said to be finer than, or to be a refinement of, a topology $\mathscr{T}$ on X if $\mathscr{T} \subset \mathscr{T}'$. It is said to be strictly finer than $\mathscr{T}$, if, in addition, we have $\mathscr{T} \neq \mathscr{T}'$ We say that $(X, \mathscr{T})$ (and by abuse of language, $\mathscr{T}$) is maximal connected, if $(X, \mathscr{T})$ is connected and whenever $\mathscr{T}'$ is strictly finer than $\mathscr{T}$, $(X, \mathscr{T}')$ is not connected. An MI space (see [2]) is one is which every dense subset is open.

The following result is in the authors thesis [5].

THEOREM 1. *Every maximal connected space is an MI space.*

An irresolvable space is one which does not have a dense subset whose complement is also dense. Anderson [1] has shown that every connected Hausdorff space has a connected irresolvable refinement. If in his proof of his Theorem 1, in the fourth paragraph, we simply choose D to be an R^*-dense set which is not R^* open, we will have proved

THEOREM 2. *Let τ be an infinite cardinal number. Let R be a connected topology for X with $\Delta(R) \geqq \tau$, where $\Delta(R)$ denotes the dispersion character, or minimum cardinality of an open set of R. Then there exists a connected MI refinement R^* of R with $\Delta(R^*) \geqq \tau$.*

DEFINITION 1. Let $(X, \mathscr{T})$ be a topological space, $x \in X$. If there is a "local" base at x which is linearly ordered under set inclusion ($V \leqq W$ if $W \subset V$), we say $\mathcal{T}$ is *neighborhood nested at* x. If $\mathcal{T}$ is not neighborhood nested at any point of X, we say $\mathcal{T}$ is *nowhere neighborhood nested,* abbreviated n.n.n.

LEMMA 1. *Let* $(X\mathcal{T})$ *be a Hausdorff space,* $x \in X$. *If there is a base* $\{V_\iota\}_{\iota \in I}$ *at* x *which is linearly ordered under set inclusion, then there is a base* $\{W_k\}_{k \in K}$ *at* x *which is well ordered under set inclusion and such that if* $\delta, \sigma \in K$, $\delta < \sigma$, *then* $\text{Int}(W_\delta - W_\sigma) \neq \emptyset$.

Proof of Lemma 1. First recall that (1) every totally ordered set has a cofinal well ordered subset (2) every well ordered set has a cofinal subset which is order isomorphic with a regular cardinal A. Next, using A, one can easily construct sets with the desired properties.

COROLLARY. *If there is an ordered local base at* x, x *is adherent to a set* S *of isolated points* (isolated in S).

Proof. Choose one member of $\text{Int}(W_\iota - W_{\iota+1})$ for each ι.

THEOREM 3. *Every Hausdorff topology without isolated points has an* n.n.n. *refinement. Every connected, Hausdorff topology has a connected* n.n.n. *refinement. Every connected, Hausdorff, but not* n.n.n. *topology has a connected* n.n.n. *refinement which is not maximal connected.*

Proof of Theorem 3. For a space $(X, \mathcal{T})$, denote by $\mathcal{T}'$ the topology on X which has as a base $\mathcal{T} \cup \{D \cap T \mid T \in \mathcal{T}$ and $\text{Int}_{(X,\mathcal{T})} D$ is dense in $(X, \mathcal{T})\}$. Then one can show that $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ have the same open-and-closed subsets by appealing to the following fact: if D is a dense subset of $(X, \mathcal{T})$ and $U, V \in \mathcal{T}$ with $U \cap V \neq \emptyset$, then $U \cap V \cap D \neq \emptyset$. Using Lemma 1, $(X, \mathcal{T}')$, and, some (or all) nowhere dense subsets of $(X, \mathcal{T})$, one obtains the statements of Theorem 3.

The following theorem is an immediate corollary to Theorem 3.

THEOREM 4. *Every maximal connected Hausdorff topology is* n.n.n.

THEOREM 5. *Every Hausdorff connected topology* $\mathcal{T}_1$ *has a Hausdorff, connected, MI,* n.n.n. *refinement* $\mathcal{T}_3$.

Proof. By Theorem 2, $\mathcal{T}_1$ has a connected Hausdorff *MI* refinement $\mathcal{T}_2$. By Theorem 3, $\mathcal{T}_2$ has a connected, Hausdorff, n.n.n. refinement $\mathscr{T}_3$. It is easy to see that every refinement of an *MI* topology is *MI*. Thus, $\mathscr{T}_3$ meets the required conditions.

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