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HYPONORMAL CONTRACTIONS AND STRONG POWER CONVERGENCE

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Let T^* be a hyponormal contraction on a Hilbert space, so that $TT^* - T^*T = D \geqq 0$ and $\| T \| \leqq 1$. It is shown that if, in addition, T^* is completely hyponormal, then the sequence $\{T^n\}_{n=1,2,\cdots}$ converges strongly to 0 as $n \to \infty$. The result is obtained as a consequence of properties of the solution $w(z)$ of $(T - zI)w(z) = x$, where x is a certain vector in the range of D.

1. Let T be a bounded operator on a Hilbert space $\mathfrak{H}$ with spectrum $\sigma(T)$ and point spectrum $\Pi_0(T)$. The range and null space of T will be denoted by $R(T)$ and $N(T)$ respectively. If A is any linear manifold in $\mathfrak{H}$, its closure will be denoted by $[A]$. Also, we shall consider the set of numbers z for which $\bar{z} \in \Pi_0(T^*)$ and which will be denoted by $(\Pi_0(T^*))^*$.

Let $T_z = T - zI$ for any complex number z and let D be a nonnegative self-adjoint operator satisfying

$$(1.1) \qquad T_z T_z^* \geqq D \geqq 0 \quad \text{for all } z \text{ in } \boldsymbol{C}\,.$$

It was shown in Putnam [8] that if D has the spectral resolution

$$(1.2) \qquad D = \int_0^\infty u dF_u$$

and if x is any vector satisfying

$$(1.3) \qquad x = F((s, \infty))x\,, \quad s > 0\,,$$

then $T_z^{-1}x$ is bounded and weakly continuous on $\boldsymbol{C} - P$, where $P = \{z: z \in \Pi_0(T) \text{ or } \bar{z} \in \Pi_0(T^*)\}$. (Actually, the set P occurring in [8] was defined differently but should have been defined as above.) This result will be strengthened below to the following

THEOREM 1. *Suppose* (1.1), (1.2) *and that* $x \in \mathfrak{H}$ *satisfies*

$$(1.4) \qquad k_x \equiv \int_{+0}^\infty u^{-1} d \,\| F_u x \|^2 < \infty\ .$$

Then there exists a vector-valued function $w(z)$ *on* $\boldsymbol{C}$ *satisfying*

$$(1.5) \qquad T_z w(z) = x \text{ and } \| w(z) \| \leqq k_x^{1/2}\,, \quad z \in \boldsymbol{C}\,,$$

and having the following properties. At every point $z_0 \notin \Pi_0(T)$, $w(z)$ *is weakly continuous, that is, for every* f *in* $\mathfrak{H}$, $(w(z), f)$ *is continuous at* z_0. *Further, if* $\mathfrak{H}$ *is separable then, for every* f *in* $\mathfrak{H}$, *the function* $(w(z), f)$ *is Lebesgue planar measurable on the set* $C - (\Pi_0(T^*))^*$. *In addition, if* α *is any rectifiable curve in* C *with arc length measure* m_α *and if* $m_\alpha(\alpha \cap (\Pi_0(T^*))^* = 0$ *then* $(w(z), f)$ *is* m_α*-measurable as well as* $dz\,(= dx + idy)$*-measurable on* α.

REMARKS. Note that if $z \notin \Pi_0(T)$ then necessarily $w(z) = T_z^{-1}x$, and that, for any f in $\mathfrak{H}$, $(w(z), f)$ is analytic in $C - \sigma(T)$. Further, it is clear that all vectors x of (1.3) satisfy (1.4) and hence that the set of vectors x satisfying (1.4) is dense in $R(D)$.

That the set $\Pi_0(T^*)$ occurring in the statement of Theorem 1 and, more generally, the point spectrum of any bounded operator on a separable Hilbert space, is Lebesgue planar measurable follows from a result of Dixmier and Foiaş [3] as Nikolskaya [7]. We are indebted to K. F. Clancey for informing us of these facts.

Recall that a bounded operator S is said to be hyponormal if $S^*S - SS^* \geqq 0$ and completely hyponormal if, in addition, there does not exist any non-trivial reducing subspace of S on which its restriction is normal. If $S_z = S - zI$, then $S_z{}^*S_z - S_zS_z{}^* = S^*S - SS^*$. Clearly, if S is hyponormal then $\Pi_0(S) \subset (\Pi_0(S^*))^*$ and any eigenvector of S belonging to z is also an eigenvector of S^* belonging to $\bar{z}$. In particular, $\Pi_0(S)$ must be empty whenever S is completely hyponormal. Further, it is easy to see that if T^* is hyponormal then (1.1) holds with $D = TT^* - T^*T$. Consequently, in view of Theorem 1, we have the following

THEOREM 2. *Let* T^* *be completely hyponormal on* $\mathfrak{H}$ *and let* $D = TT^* - T^*T\,(\geqq 0)$ *have the spectral resolution* (1.2). *If* $x \in \mathfrak{H}$ *satisfies* (1.4) *then there exists a vector-valued function* $w(z)$ *on* C *satisfying the conditions of Theorem* 1. *Thus, relation* (1.5) *holds and* $w(z)$ *is weakly continuous at all points* $z_0 \notin \Pi_0(T)$. *If* $\mathfrak{H}$ *is separable, then, since* $\Pi_0(T^*)$ *is now empty,* $(w(z), f)$ *is Lebesgue planar measurable in* C *and is measurable with respect to arc length measure and to the* $dz = dx + idy$ *measure on all rectifiable curves in* C.

As a consequence of Theorem 2 there will be proved

THEOREM 3. *Let* T^* *be completely hyponormal on* $\mathfrak{H}$ *and suppose that* T *is a contraction, that is,* $\|T\| \leqq 1$. *Then* $\{T^n\}_{n=1,2,\ldots}$ *converges* *strongly to* 0 *as* $n \to \infty$, *that is*, $\| T^n f \| \to 0$ *as* $n \to \infty$ *for every* f *in* $\mathfrak{H}$.

REMARKS. It follows from Theorem 3 that if T^* is any hyponormal contraction then T can be written as the direct sum $T = T_1 \oplus N$, where T_1^* is completely hyponormal, $T_1^n \to 0$ strongly as $n \to \infty$, and N is normal. Clearly, $N = \int z dK_z$ can be further decomposed as $N = \int_{|z|<1} z dK_z + \int_{|z|=1} z dK_z = N_1 \oplus N_2$, where $N_1^n \to 0$ strongly as $n \to \infty$ and N_2 is unitary. Hence, one has the following

COROLLARY 1 OF THEOREM 3. *Let* T^* *be any hyponormal contraction on a Hilbert space. Then* $T = T_2 \oplus U$ *where* $T_2^n \to 0$ *strongly as* $n \to \infty$ *and* U *is unitary, where it is understood that either component of the direct sum may be missing.*

Thus, if T^* is any completely nonunitary (cf. Sz.-Nagy and Foiaş [11], p. 72) hyponormal contraction, then $T^n \to 0$ strongly as $n \to \infty$, so that T is of class C_0. (cf. [11], p. 72). In was shown in [8], p. 167, that if T^* is a hyponormal contraction for which $T^n \not\to 0$ then T has a nontrivial invariant subspace. The above Corollary yields the stronger result that T^* (hence T) even has a unitary part. Also, it follows from the Corollary that if T^* is a hyponormal contraction for which $T^n f \not\to 0$ as $n \to \infty$ whenever $f \neq 0$, then T must be unitary. In case T^* is also subnormal, this last result was obtained by Stampfli [10].

COROLLARY 2 OF THEOREM 3. *Let T be a completely hyponormal contraction on a Hilbert space. Then* T^* *is* (unitarily equivalent to) *the restriction of the adjoint of a unilateral shift to an invariant subspace.*

Proof. Actually, every contraction S satisfying $S^n \to 0$ strongly as $n \to \infty$ is unitarily equivalent to the restriction of the adjoint of a unilateral shift to an invariant subspace (Foiaş [4], de Branges and Rovnyak [1, 2]. See also Halmos [5], problem 121, and Sz.-Nagy and Foiaş [11] p. 95. Note that the unilateral shift in question is, in general, not the simple unilateral shift.

2. *Proof of Theorem* 1. The proof will be an extension and refinement of the argument given in [8]. Let z be fixed and let $T_z T_z^*$ have the spectral resolution

$$T_z T_z^* = \int_0^\infty u dE_u^{(z)} . \tag{2.1}$$

Then, by an argument like that on pp. 165-166 of [8],

$$\int_0^\infty \lim_{t\to 0+} (u+t)^{+1} d\, \| E_u^{(z)} x \|^2 \leqq k_x \,,$$

where k_x is defined by (1.4). It follows that

$$E^{(z)}(\{0\})x = 0 \quad \text{and} \quad \int_{+0}^\infty u^{-1} d\, \| E_u^{(z)} x \|^2 \leqq k_x \,, \tag{2.2}$$

and hence, for any z in $\boldsymbol{C}$,

$$y(z) = \int_{+0}^\infty u^{-1/2} dE_u^{(z)} x \text{ is defined and } \| y(z) \|^2 \leqq k_x \,. \tag{2.3}$$

Next, let $T_z = T - zI$ have the polar factorization (see Kato [6], pp. 334-335)

$$T_z = U(z)G(z) \,, \tag{2.4}$$

where $G(z) = (T_z^* T_z)^{1/2}$ and $U(z)$ is partially isometric with initial set $[R(G(z))]$ and final set $[R(T_z)]$. Then $T_z U^*(z) y(z) = (U(z)G(z)U^*(z))y(z) = (T_z T_z^*)^{1/2} y(z) = x$. On putting

$$w(z) = U^*(z)y(z) \,, \tag{2.5}$$

one sees that (1.5) follows from (2.3).

Next, it will be shown that the above defined bounded vector-valued function $w(z)$ on $\boldsymbol{C}$ is weakly continuous at every point z_0 not in $\Pi_0(T)$. It must be shown that $w(z)$ converges weakly to $w(z_0)$, that is, for any f in $\mathfrak{H}$, $(w(z), f) \to (w(z_0), f)$ as $z \to z_0$. If this limit relation did not hold however, then, since $w(z)$ is bounded, there would exist a z_0 and a sequence $\{z_n\}$ such that $w(z_n) \to p$ (weakly) as $z_n \to z_0$ with $p \neq w(z_0)$. It follows from the relation $T_z w(z) = x$, on letting $z = z_n$ and noting that $\| T - T_n \| \to 0$, that $T_{z_0} p = x$ and, since $T_{z_0} w(z_0) = x$, that

$$T_{z_0}(p - w(z_0)) = 0 \,. \tag{2.6}$$

Since $z_0 \notin \Pi_0(T)$, then $p = w(z_0)$, a contradiction.

There remains then to establish the measurability of $w(z)$ in the sense described in Theorem 1, at least if $\mathfrak{H}$ is separable. To this end, we first shall show that, whether or not $\mathfrak{H}$ is separable, if T is any operator with the polar factorization of (2.4), then

$$U(z) \to U(z_0) \text{ strongly as } z \to z_0 \text{ whenever } z_0 \notin \Pi_0(T) \,. \tag{2.7}$$

Assume then that $z_0 \notin \Pi_0(T)$. Note (cf. [6], pp. 334-335) that

$U(z)$ is defined for vectors in $R(G(z))$ by $U(z)$: $G(z)u \to T_z u$ and that $U(z)$ is then extended by continuity to be isometric on $[R(G(z))]$. For y in $N(G(z))(=N(T_z))$, $U(z)y = 0$. Since $z_0 \notin \Pi_0(T)$, then $N(G(z_0)) = 0$ and so $U(z_0)$ is isometric.

Since $R(G(z_0))$ is dense in $\mathfrak{H}$, relation (2.7) will follow if it is shown that

$$(2.8) \quad U(z)v \longrightarrow U(z_0)v \text{ (strongly) as } z \longrightarrow z_0\,, \text{ whenever } v \in R(G(z_0))\,.$$

Suppose then that $v \in R(G(z_0))$, so that $v = G(z_0)u$ for some vector u. In view of $U(z)G(z)u = T_z u$ and $U(z_0)G(z_0)u = T_{z_0}u$, we have $U(z)v - U(z_0)v = (T_z - T_{z_0})u - U(z)(G(z) - G(z_0))u$. Since $\| T_z - T_{z_0} \| \to 0$, hence also $\| G(z) - G(z_0) \| \to 0$, as $z \to z_0$, relation (2.8), hence also (2.7), follows. By symmetry, we have also

$$(2.9) \quad U^*(z) \longrightarrow U^*(z_0) \text{ strongly as } z \longrightarrow z_0 \text{ whenever } \bar{z}_0 \notin \Pi_0(T^*)\,.$$

Henceforth, it will be supposed that T is the operator occurring in the statement of Theorem 1. By (2.2) and (2.3), $y(z) \in [R(T_z T_z^*)^{1/2}] = [R(T_z)]$, and this set is the initial set of $U^*(z)$. Since $w(z) = U^*(z)y(z)$, it follows that $U(z)w(z) = U(z)U^*(z)y(z) = y(z)$. We shall show that

$$(2.10) \qquad y(z) \longrightarrow y(z_0) \text{ weakly as } z \longrightarrow z_0,\ \bar{z}_0 \notin \Pi_0(T^*)\,.$$

If (2.10) did not hold then, since $y(z)$ is (uniformly) bounded in C, there would exist a sequence $\{z_n\}$ for which $z_n \to z_0$ and $y(z_n) \to q$ (weakly) as $n \to \infty$ with $q \neq y(z_0)$. Since $w(z)$ is also bounded, we may choose a subsequence of $\{z_n\}$, which will also be denoted by $\{z_n\}$, such that $w(z_n) \to p$ (weakly).

Let f be arbitrary in $\mathfrak{H}$. Then $(y(z_n)f) \to (q, f)$ and also $(y(z_n), f) = (U(z_n)w(z_n), f) = (w(z_n), U^*(z_n)f)$. In view of (2.9), we have $(w(z_n), U^*(z_n)f) \to (p, U^*(z_0)f) = (U(z_0)p, f)$, and hence $q = U(z_0)p$. Since $y(z_0) = U(z_0)w(z_0)$, we see that $q - y(z_0) = U(z_0)(p - w(z_0))$. But, as noted earlier, $T_{z_0}(p - w(z_0)) = 0$ (cf. (2.6)), so that $p - w(z_0) \in N(G(z_0))$ and hence $U(z_0)(p - w(z_0)) = 0$. Thus $q = y(z_0)$, a contradiction, and so (2.10) is proved.

In summary, we see that the vector-valued function $w(z)$ on C is weakly continuous at $z_0 \notin \Pi_0(T)$. The vector-valued function $y(z)$ is weakly continuous at z_0 if $\bar{z}_0 \notin \Pi_0(T^*)$. Also, the operator-valued function $U^*(z)$ on C is strongly continuous and, hence, $U(z)$ is weakly continuous at z_0 if $\bar{z}_0 \notin \Pi(T^*)$.

Suppose now that $\mathfrak{H}$ is separable. Then, as noted earlier, $\Pi_0(T^*)$

(hence also $(\Pi_0(T^*))^*$) is Lebesgue planar measurable. It will be shown that for any f in $\mathfrak{H}$, the function $(w(z), f)$ is Lebesgue planar measurable on $C - (\Pi_0(T^*))^*$. For, let $\{\phi_n\}(n = 1, 2, \cdots)$ be any complete orthonormal system for $\mathfrak{H}$. Then $(w(z), f) = (y(z), U(z)f) = \sum_{n=1}^{\infty} (y(z), \phi_n)(\phi_n, U(z)f)$. But each term of the summation is a function continuous at all points z for which $\bar{z}$ is not in $\Pi_0(T^*)$. In particular, each such term, and hence the sum, is (planar) measurable on $C - \Pi_0(T^*))^*$. (The argument is similar to that used in [9], p. 384, in connection with the proof of Stone's theorem on unitary groups.)

Finally, a similar argument establishes the assertion of the last part of Theorem 1 and the proof is now complete.

3. *Proof of Theorem* 3. Without loss of generality it may be supposed that $\mathfrak{H}$ is separable. It follows from Theorem 2 that if $w(z)$ is defined by (2.5) and if f is arbitrary in $\mathfrak{H}$, then $(w(z), f)$ is (bounded and) measurable with respect to arc length and to the measure $dz = dx + idy$ on every circle $C_r = \{z: |z| = r\}$, $0 < r < \infty$. Let

$$(3.1) \qquad y(r) = -(2\pi i)^{-1} \int_{C_r} w(z)dz \left(= -r(2\pi i)^{-1} \int_C w(rt)dt \right),$$

where $C = C_1$ and all circles are oriented positively. It is understood, of course, that $y(r)$ is defined in terms of the relation $(y(r), f) = -(2\pi i)^{-1} \int_{C_r} (w(z), f)dz$ for any f in $\mathfrak{H}$ and that the latter integral is a Lebesgue integral. A similar remark applies to the other integrals of this section.

The set $\Pi_0(T) \cap \{z: |z| = 1\}$ is empty; otherwise, T would have a normal part. (In fact, if T is any contraction and if z is an eigenvalue of T satisfying $|z| = 1$ with eigenvector f then $\bar{z}$ is an eigenvalue of T^* with the same eigenvector f; cf. [11], p. 8.) If z is fixed and $|z| = 1$, then, by Theorem 2, $w(rz) \to w(z)$ (weakly) as $r \to 1 - 0$. For any fixed f in $\mathfrak{H}$, it follows from (3.1) and the uniform boundedness convergence theorem that

$$(y(r), f) = -(2\pi i)^{-1} \int_{C_r} (w(z), f)dz$$
$$\longrightarrow -(2\pi i)^{-1} \int_C (w(z), f)dz \text{ as } r \longrightarrow 1 - 0,$$

Thus, $y(r) \to y(1)$ (weakly) as $r \to 1 - 0$. Similarly, $y(r) \to y(1)$ (weakly) as $r \to 1 + 0$. But, if $r > 1$, $-(2\pi i)^{-1} \int_{C_r} T_z^{-1} v dz = v$, for arbitrary v, so that, if x is any vector satisfying (1.4), $y(r) = x$ for $r > 1$ and hence $y(1) = x$. Hence, we have for such vectors x, (3.2) $$y(r) \longrightarrow x \text{ (weakly) as } r \longrightarrow 1 - 0 .$$

In view of (1.5),

$$\begin{aligned} Ty(r) &= -(2\pi i)^{-1} \int_{C_r} Tw(z)dz \\ &= -(2\pi i)^{-1} \int_{C_r} (T - z + z)w(z)dz , \\ &= -(2\pi i)^{-1} \int_{C_r} zw(z)dz . \end{aligned}$$

Similarly, one sees that $T^n y(r) = -(2\pi i)^{-1} \int_{C_r} z^n w(z)dz$ for $n = 1, 2, \cdots$, and hence

(3.3) $$T^n y(r) \longrightarrow 0 \text{ (strongly) as } n \longrightarrow \infty, \text{ for } r < 1 .$$

Next, let $\mathfrak{M} = \{v: T^n v \to 0$ (strongly) as $n \to \infty\}$. Since T is a constraction, $\mathfrak{M}$ is a subspace invariant under T. Also, by (3.3), each $y(r)$, $r < 1$, belongs to $\mathfrak{M}$. Hence, by (3.2), if u is any vector in $\mathfrak{M}^\perp$, $0 = (y(r), u) \to (x, u)$ as $r \to 1 - 0$, and so $x \in \mathfrak{M}$, where x is any vector satisfying (1.4). Since such vectors are dense in $R(D)$, $R(D) \subset \mathfrak{M}$.

Let now $\mathfrak{L}$ denote the least subspace containing $R(D)$ and reducing T. It will be shown that

(3.4) $$\mathfrak{L} \subset \mathfrak{M} .$$

To see this, note that if $u \in \mathfrak{M}$ then $Tu \in \mathfrak{M}$. Also, $TT^* - TT^* = D$ and hence $T^n T^* u = T^{n-1} T^* Tu + T^{n-1} Du$. Since $Du \in \mathfrak{M}$, then $T^{n-1} Du \to 0$ as $n \to \infty$ and hence $\limsup_{n\to\infty} \| T^n T^* u \| \leqq \| Tu \|$. Repetition of this argument shows that $\limsup_{n\to\infty} \| T^n T^* u \| \leqq \| T^k u \|$ for $k = 1, 2, \cdots$, and hence $T^n Tu \to 0$ as $n \to \infty$, so that $T^* u \in \mathfrak{M}$. Thus, whenever u is in $\mathfrak{M}$ so also are Tu and $T^* u$. Since $R(D) \subset \mathfrak{M}$, the desired relation (3.4) follows.

It is clear that $\mathfrak{L}^\perp$ also reduces T and that $T \mid \mathfrak{L}^\perp$ is normal. Since T^* is completely hyponormal then $\mathfrak{L}^\perp = 0$, and so by (3.4), $\mathfrak{M} = \mathfrak{H}$. This completes the proof of Theorem 3.

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