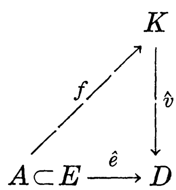
PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

METRIC FAMILIES

J. F. McClendon

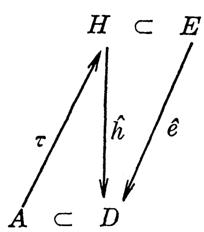
A (continuous) metric family is a disjoint collection of metric spaces whose metrics are compatible with a given topology on the disjoint union. The purpose of this paper is to give some examples of these objects and to develop some of their basic properties. Most theorems about metric spaces can at least be formulated for metric families — some are true, some are true only with extra hypotheses, and some are false. Examples of each kind will be given. The main positive results are a version of Dugundji's extension theorems, a cross section theorem, a generalization of one of Michael's selection theorems, and a generalization of one of Coban's selection theorems.

Here a rough description of the results described above will be given. First consider the following diagram:



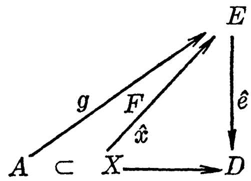
Here $\hat{e}: E \to D$ is a metric family and $\hat{v}: K \to D$ is a sub-family of a vector family. It is shown (Theorem 3.3) that under certain conditions f extends to a continuous $F: E \to K$ with $\hat{v}F = \hat{e}$. If D is a point then the result coincides with the extension theorem of Dugundji [5, 6].

Now consider the following diagram.



Here $\hat{h}: H \to D$ is a sub-family of a vector family $E \to D$. τ is a partial cross section of $\hat{h}$. Theorem 4.2 is a fairly general theorem giving hypotheses on E, H, and τ which guarantee that $\hat{h}$ has a cross section which extends τ. The result is sufficiently general to include the Tietze extension theorem. Subsequent results specialize Theorem 4.2 and give conditions on $\hat{e}$ and H which guarantee a cross-section-extension for any τ.

Finally, consider the following diagram.



Here $\hat{e}: E \rightarrow D$ is a vector family and $F: X \rightarrow E$ is a multivalued function such that $\hat{e}F = \hat{x}$. g is a partial selection for F. Theorem 5.1 gives conditions on $\hat{e}$ and F which permit g to be extended to a continuous selection for F. If D is a point then this becomes one of Michael's selection theorems [9]. In general the problem presented by the diagram can be thought of as a lifting problem with extra condition. The lifting problem (take A empty for simplicity) is to find $f: X \rightarrow E$ with $\hat{e}f = \hat{x}$ and is the type of problem often studied in algebraic topology when $\hat{e}$ is fibration. The extra condition is to make f a selection for F. Of course the problem can be reformulated as a pure lifting problem (not for fibrations) or a pure selection problem (not metric valued). One can also let $X_d = \hat{x}^{-1}(d)$ and $F_d = F \mid X_d$. Then F can be thought of as a family of multifunctions with parameter space D. In Theorem 5.5 $\hat{e}: E \rightarrow D$ is simply a metric family but the resulting selection-extension is not necessarily continuous. If E is a single metric space the theorem coincides with one of Coban [3].

I wish to thank C. Himmelberg for some helpful lectures and conversations.

1. Definitions and examples. We will use the following terminology. A space is a topological space. A map is a continuous function. If $p: E \rightarrow D$ is a map then a section of p is a map $s: D \rightarrow E$ with $ps = id_D: D \rightarrow D$. A local section of p is a map $s: V \rightarrow E$, V open in D, with $ps = id_V: V \rightarrow V$. Now suppose $p: E \rightarrow D$ is a given map.

$$E_d = p^{-1}(d)$$
$$E_A = p^{-1}(A)$$
$$E^* = E \times_D E = \{(e, e') \in E \times E \mid pe = pe'\} \subset E \times E$$
$$= \bigcup_{d \in D} E_d \times E_d \text{ (a disjoint union)}$$
$$S \subset E,\ S_d = S \cap E_d,\ S_A = S \cap E_A$$

DEFINITION[1] 1.1. (E, p, D, ρ) is a metric family if $p: E \to D$ is a map, $\rho: E^* \to R$ is a map, and $\rho \mid E_d \times E_d$ is a metric for each E_d, $d \in D$.

Usually we will simply say that E is a metric family or that E is a metric family over D. ρ can also be called a fiber metric and E can be called a fiber metric space. Later it will be convenient to use $\hat{e}: E \to D$ rather than $p: E \to D$ when more than one family is involved.

EXAMPLE 1.2. Let E be a metric space with metric d and $p: E \to D$ any map. Then $d \mid E^* = \rho$ makes $E \to D$ a metric family.

EXAMPLE 1.2′. Let $E \to D$ be a metric family and $E \to D = E \to B \to D$. Then $E \to B$ is a metric family by $E \times_B E \subset E \times_D E \to R$. If D is a point then this is 1.2.

EXAMPLE 1.3. Let D be any space and (M, α) any metric space. Let $E = D \times M$ and $p: E \to D$ be the natural projection. Here $E^* = \{(d, m, d, m') \mid d \in D,\ m, m' \in M)$ and E^* is homeomorphic to $D \times M \times M$. Define $\rho: E^* \to R$ by $\rho(d, m, d, m') = \alpha(m, m')$. That is,

$$E^* \xrightarrow{\text{proj.}} M \times M \xrightarrow{\alpha} R\,.$$

Then E is a metric family. Call E a product family.

EXAMPLE 1.4. Let $E \to D$ be a metric family and $S \subset E$. Then $S \to D$ is a metric family with metric obtained by restricting that of E.

DEFINITION 1.5. Let $p: E \to D$ and $p': E' \to D$ be metric families. A map $f: E \to E'$ such that $p'f = p$ is an isometry if each $f_d: E_d \to E'_d$ is an isometry (not necessarily onto) and f is an embedding (=homeomorphism onto its image).

The question arises as to which metric families are isometrically embeddable in a product family.

EXAMPLE 1.6. Let $D = R$. Let E as a set be $R \times R$ with the topology generated by usual opens and the following set

$$S = R \times R - \{(s, t) \mid t = 0,\ -\infty < s < 0 \text{ or } 0 < s < +\infty\}$$

[1] This should be compared to corresponding notions of J. Dauns and K. H. Hofmann, Representation of rings by sections, Memoirs of the A. M. S. no. 83, 1968, and J. M. G. Fell, An extension of Mackey's method to Banach *-algebraic bundles, Memoirs of the A. M. S. no. 90, 1969.

$p: E \to D$ is $p(s, t) = s$. $\rho: E^* \to R$ is $(s, t, s, t') = |t - t'|$. Then (E, p, D, ρ) is a metric family. E is not regular since the closed set $A = \{(s, t) \mid t = 0, 0 < s < \infty\}$ and the point $(0, 0)$ cannot be put into disjoint open sets. If E were embeddable in a product family we would have

$$\begin{array}{c} E \subset R \times M \\ \searrow \; \swarrow \\ R \end{array}$$

with M metric—but this would imply that E is regular,

EXAMPLE 1.7. Let $\mathscr{B} = (p: E \to B, G, F, \mathscr{A} = \{(V, h_V)\})$ be a fiber bundle in the sense of [Steenrod, 11]. That is, p is F-locally trivial, G a topological group, F is an effective left G-space and $\mathscr{A}$ is a G atlas (really, $\mathscr{B}$ is a coordinate bundle but it determines a unique fiber bundle). Suppose now that α is a metric for F and that the G action preserves the metric, i.e., $\alpha(gf, gf') = \alpha(f, f')$. Then call $\mathscr{B}$ a metric bundle. Then $E \to B$ is a metric family since $E_V \leftrightarrow V \times F$ is (Example 1.3) and the equivariance assumption permits us to piece these together to form a metric family.

Recall [e.g. Husemoller, 7] a space X is a G-space, G a topological group, if there is a continuous function $G \times X \to X$, $(g, x) \to gx$, such that $1x = x$ and $g(g'x) = (gg')x$ all $g, g' \in G$, $x \in X$. It is called a free G-space (or a fixed-point-free G-space) if $gx = x$ for some x implies $g = 1$. It is calld an effective G-space if $gx = x$ for all x implies $g = 1$. A free G-space is a Cartan principal G-space if the function $u: X^* \to G$ uniquely defined by $u(x, x')x' = x$ is continuous. Here $p: X \to X/G = D$ is the quotient map and $X^* = X \times_D X$.

THEOREM 1.8. *If G is a metrizable topological group and $p: X \to X/G$ a Cartan principal G-space then Y is a metric family.*

Proof. Recall [e.g., Montgomery-Zippen, 10] that G has a left invariant metric α, so $\alpha(gg_1, gg_2) = \alpha(g_1, g_2)$ all $g_1, g_2 \in G$. Define $\rho: X^* \to R$ by $\rho(x, x') = \alpha(u(g, g'), 1)$. Then ρ is continuous and we need only show that it gives a metric on each fiber. The function u always satisfies the following conditions: $u(x, x') = 1 \Leftrightarrow x = x'$, $u(x, x')\, u(x', x'') = u(x, x'')$, $u(x', x) = u(x, x')^{-1}$. The first shows that $\rho(x, x') = 0 \Leftrightarrow x = x'$. Also $\rho(x, x') = \alpha(u(x, x'), 1) = \alpha(1, u(x, x')^{-1}) = \alpha(1, u(x', x)) = \alpha(u(x', x), 1) = \rho(x', x)$. The triangle inequality is also easy to check.

COROLLARY 1.9. *G a topological group and H a metrizable subgroup. Then* $G \to G/H$ *is a metric family.*

Proof. If H is any subgroup of G then $G \to G/H$ is a Cartan principle fiber space so the result follows from 1.8.

Let $X \to X/G$ be a Cartan principle G-space and F a G-space. Then $X \times F$ is a G-space (diagonal action) and there is a natural map $(X \times F)/G \to X/G$ called a Cartan fiber space.

THEOREM 1.10. *Suppose F has a G-invariant metric. Then a Cartan fiber space is a metric family.*

Proof. Similar to 1.8 (which is a special case of 1.10).

Note that 1.10 actually includes Example 1.7 since a Steenrod fiber bundle is homeomorphic cover the base to a (locally trivial) Cartan fiber space. In general a Cartan fiber space need not have a local cross section as the case $G \to G/H$ shows.

Recall [Atiyah, 1] that $p: E \to D$ is a vector family if each E_d is a vector space (over R here, but the general case is similar) and the functions

$$E \times_D E \longrightarrow E \qquad (e, e') \longrightarrow e - e'$$
$$R \times E \longrightarrow E \qquad (t, e) \longrightarrow te$$

are continuous. $p: E \to D$ need not have a continuous section. However, if it does then the zero section $d \to 0_d$ will then be continuous.

DEFINITION 1.11. Let $E \to D$ be a vector family. Suppose $|\cdot|: E \to R$ $(e \to |e|)$ is a function such that

(a) it is a norm on each E_d $(|e| \geqq 0, = 0 \Leftrightarrow e = 0, |e + e'| \leqq |e| + |e'|, |\lambda e| = |\lambda||e|)$

(b) the function $|\cdot|: E \to R$ is continuous.

Then $E \to D$ will be called a *normed vector family.*

Thus $\rho(e, e') = |e - e'|$ makes the normed vector family E into a metric family. Any normed vector bundle or vector bundle with metric [Atiyah, 1, p. 13] will be a normed vector family and hence a metric family.

2. Elementary properties.

THEOREM 2.1. *Let* $p: E \to D$ *be a map and* $\rho: E \times_D E \to R$ *a function. Then the following are equivalent.*

(a) ρ *is continuous at* (e, e')

(b) *Given* $\varepsilon > 0$ *there are opens* V, V' *of* E, $e \in V$, $e' \in V'$ *such that* $|\rho(\bar{e}, \bar{e}') - \rho(e, e')| < \varepsilon$ *for* $\bar{e} \in V$, $\bar{e}' \in V'$, $pe = p\bar{e}'$.

COROLLARY 2.2. *Let* $p: E \to D$ *be a metric family,* $e \in E$, $\varepsilon > 0$. *Then there is an open* W *in* E *with* $e \in W$ *such that* $\bar{e}, \bar{e}' \in W$, $p\bar{e} = p\bar{e}'$ *imply* $\rho(\bar{e}, \bar{e}') < \varepsilon$.

Theorem 2.1 gives the following verbal description of a metric family: each E_d is a metric space and if a pair of points in E_d is topologically close to a pair of points in E_d, then the distance between the points of the first pair is close to the distance between the points of the second pair. It is possible to have each E_d metric but $E \to D$ not a metric family (see example below). Corollary 2.2 shows that there is an ε-*strip* around each point of E, where V is an ε-*strip* if diam $(V_d) < \varepsilon$ for all $d \in D$. If there is a local section through $e \in E$ then there is something better than an ε-strip.

DEFINITION 2.3. Let $p: E \to D$ be a metric family.

(1) $\varepsilon > 0$, $A \subset E$, $B_\varepsilon(A) = B(A; \varepsilon) = \{e \in E \mid A_{p(e)} \neq \phi$ and $\rho(e, A_{p(e)}) < \varepsilon\}$

(2) $\sigma: V \to E$ a local section of p, $B_\varepsilon(\sigma) = B(\sigma; \varepsilon) = \{e \in E \mid pe \in V, \rho(e, \sigma pe) < \varepsilon\} = B_\varepsilon(\sigma(V))$.

Usually $B_\varepsilon(A)$ is not open in E, as the case A = a point shows. However we do have the following facts.

THEOREM 2.4. *Let* $p: E \to D$ *be a metric family and* $\sigma: V \to E$ *a local section of* p *and* $\varepsilon > 0$.

(1) $B(\sigma; \varepsilon)$ *is open in* E.

(2) *Let* $e_0 \in B(\sigma; \varepsilon)$. *Suppose* $\tau': W' \to E$ *is a local section of* E *with* $e_0 \in \tau'(W')$. *Then there is a* $\delta > 0$ *and an open neighborhood* W *of* pe_0 *such that* $e_0 \in B(\tau; \delta) \subset B(\sigma; \varepsilon)$ *where* $\tau = \tau' \mid W$.

Proof. (1) Let $s: E_V \to E_V^*$ be defined by $s(e) = (e, \sigma pe)$. Then s is continuous since σ is and $B(\sigma; \varepsilon) = (\rho s)^{-1}[0, \varepsilon)$.

(2) Let $p(e_0) = d_0$ and $\rho(e_0, \sigma(d_0)) = \varepsilon_1 < \varepsilon_2 < \varepsilon$. Let $W = \{d \mid \rho(\tau'(d), \sigma(d)) < \varepsilon_2\}$. Then $W = \{d \mid \tau'(d) \in B(\sigma; \varepsilon_2)\} = \tau'^{-1}(B(\sigma; \varepsilon_2))$ so W is open in D and $d_0 \in W$. Set $\delta = \varepsilon - \varepsilon_2 > 0$. Then $e \in B(\tau; \delta)$, $pe \in W$ implies $\rho(e, \tau pe) < \delta$ so $\rho(e, \sigma pe) < \rho(e, \tau pe) + \rho(\tau pe, \sigma pe) < \delta + \varepsilon_2 = \varepsilon$. Thus $e_0 \in B(\tau; \delta) \cap p^{-1}(W) \subset B(\sigma; \varepsilon)$ proving the result.

Part (2) of the theorem shows that is $\mathscr{S}$ if a family of local sections of p and there is at least one section through each $e \in E$ then the family $\{B(\sigma \mid W, \varepsilon) \mid \sigma \in \mathscr{S}, \varepsilon > 0, W \text{ open}, W \subset \operatorname{dom} \sigma\}$ is a basis for a topology on E. Call $\mathscr{S}$ a *full* family of local sections of p if it contains at least one local section through each point of E.

DEFINITION 2.5. Let $p: E \to D$ be a metric family and $\mathscr{S}$ a family of local sections of p. The coarse $\mathscr{S}$-topology on E is the topology sub-generated by $\mathscr{S}' = \{B(\sigma \mid W, \varepsilon) \mid \varepsilon > 0, \sigma \in \mathscr{S}, W \text{ open}, W \subset \operatorname{dom} \sigma\}$. If $\mathscr{S}$ consists of all local cross sections of p then call the resulting topology the coarse topology of E. A function $h: E \to Z$ is coarsely continuous (open, closed, etc.) if it is continuous (open, closed, etc.) for the coarse topology on E. If the topology on E is the same as the coarse topology then E will be called a coarse metric family.

Example 1.6 shows that a metric family may have many local sections and still not be a coarse metric family. A metric family may not have any sections (or even local sections) but when it does the set of sections has a natural metric (possibly infinite valued). More generally, consider

$$\begin{array}{ccc} & & E \\ & & \downarrow \hat{e} \\ X & \xrightarrow{\hat{x}} & D \end{array}$$

$\hat{e}: E \to D$ a metric family

DEFINITION 2.6. $C_D(X, E) = \{f: X \to E \mid f \text{ continuous}, \hat{e}f = \hat{x}\}$. If $f, g \in C_D(X, E)$ then $d(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\} \in [0, +\infty]$. If $\hat{x} = id: D \to D$ write Sect $\hat{e}$ for $C_D(D, E)$.

THEOREM 2.7. *Suppose E has the coarse $\mathscr{S}$-topology for some full family of local sections $\mathscr{S}$. If each E_d is complete then so is $C_D(X, E)$.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C_D(X, E)$. Then each $\{f_n(x)\}$ is Cauchy so $f_n(x) \to f(x)$. Now let $f(x_0) \in B(\sigma; \varepsilon)$ where $\sigma: V \to E$ is a local section from $\mathscr{S}$ and $\hat{x}(x_0) = d_0$, $\sigma(d_0) = f(x_0)$ The usual argument shows that there is an N such that $n > N$ implies $\rho(f_n(x), f(x)) < \varepsilon/2$. Select such an n so $f_n(x_0) \in B(\sigma, \varepsilon/2)$ and there is an open set W in X with $x_0 \in X$ and $f_n(W) \subset B(\sigma; \varepsilon/2)$. If $x \in W$ then $\rho(f(x), \sigma\hat{x}x) \leqq \rho(f(x), f_n(x)) + \rho(f_n(x), \sigma\hat{x}(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ so $f(W) \subset B(\sigma; \varepsilon)$. Thus f is continuous and by the proof $f_n \to f$ in $C_D(X, E)$.

We now consider briefly separation properties and some examples. Let us say that a map $p: E \to D$ is a Hausdorff [normal] family if for any two distinct points $x, y \in E_D$ [disjoint closed sets A, B of E_d] there are disjoint open sets V, V' of E with $x \in V$, $y \in V'$ [$A \subset V$, $B \subset V'$].

Note that if E is a Hausdorff family and D a Hausdorff space then E is a Hausdorff space. However if D and Y are normal spaces then $D \times Y$ is a normal family but not necessarily a normal space. It is easy to see that a metric family is a Hausdorff family.

EXAMPLE 2.9. A metric family need not be a normal family. Note first the following general fact. Suppose X a Hausdorff space and A a closed subspace with metric $d: A \times A \to R$. Define $D = X/A$ and $\rho: X \times_D X \to R$ by $\rho \mid A \times A = d$ and $\rho \mid \Delta(x) \equiv 0$. Then $(X \to X/A, \rho)$ is a metric family.

Now take $X = I \times I$ as a set with $A = I \times 0$, $X' = I \times (0, 1]$. Give X' the R^2 usual topology and take as a basis at $(x, 0)$ all $U_\varepsilon = [B_\varepsilon(x, 0) \cap X'] \cup \{(x, 0)\}$. Let $B = [0, 1/2] \times 0$, $B' = (1/2, 1] \times 0$. Then B and B' are closed in A but can not be separated by disjoint opens in X. Thus $X \to X/A$ is a metric family which is not a normal family (not even a regular family).

EXAMPLE 2.10. It is possible to have each E_d metric but $E \to D$ not a metric family. Let $E = \{a_1, a_2, b_1, b_2\} \to D = \{a, b\}$. The opens of E are $\{a_1, b_2\}$, $\{a_2, b_2\}$, $\{b_1\}$, $\{b_2\}$, $\{a_1, b_1, b_2\}$, $\{b_1, b_2\}$, $\{a_2, b_1, b_2\}$ and E, ϕ. The opens of D are D, ϕ, and $\{b\}$. $p(a_i) = a$, $p(b_i) = b$, $i = 1, 2$. $E \to D$ has each E_d metric (since discrete) but E is not a Hausdorff family since a_1 and a_2 can't be globally separated.

Call $p: E \to D$ is a completely regular family if for each $d \in D$ and each $A \subset E_d$, A closed, $b \in E_d$, $b \notin A$, there is a function $f: E \to R$, continuous on E, with $f(A) = 0$, $f(b) = 1$.

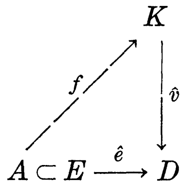
EXAMPLE 2.11. Let W be a Hausdorff regular space such that all coninuous functions on W are constant [I wish to thank J. Porter for pointing out to me the relevance of these spaces]. Let w_0, w_0' be distinct points of W and set $D = W/\{w_0, w_0'\}$ (i.e., identify w_0 and w_0'). First note that $W \to D$ is a normal family since only $A = \{w_0\}$ and $B = \{w_0'\}$ must be separated in W but W is Hausdorff. Also $W \to D$ is a metric family since ρ: $W^* \to R$, $\rho(w_0, w_0') = 1$, $\rho(w_0', w_0) = 1$, $\rho(w, w) = 0$ all w, is a metric. Thus $W \to D$ shows that a normal family need not be a completely regular family and a metric family need not be a completely regular family.

3. Extensions. In this section we will prove a version of Dugundji's extension theorem [Dugundji, 5, 6] for metric families.

Let $V \to D$ be a vector family. A subset K of V is *convex* if each K_d is convex ($k, k' \in K_d$ implies $tk + (1 - t)k' \in K_d$ for $0 \leqq t \leqq 1$). V is *locally convex* if it has a basis of convex subsets. For the theorem below we can get by with somewhat less than local convexity.

DEFINITION 3.1. (cf. 5, p. 417, 418). A vector family $\hat{v}: V \to D$ is of type m if for every metric family $\hat{e}: E \to D$ and continuous $f: E \to V$, $\hat{v}f = \hat{e}$, the following is true: for each $e \in E$ and neighborhood W of $f(e)$ there is a neighborhood U of e and a convex set C of U such that $f(U) \subset C \subset W$.

We study the following commutative diagram of spaces and maps.



$K \subset V \to D$, K convex, V a vector family of type m, $E \to D$ a metric family.

DEFINITION 3.2. A subset A of E is *smooth* if $\hat{e}A = D$, $e \to \rho(e, A)$ is a continuous function $(E - A) \to R$, and the sets $B(\sigma; \varepsilon)$ for local sections $\sigma: V \to E$, $\sigma(V) \subset A$, $\varepsilon > 0$, form a basis at each point of A.

THEOREM 3.3. *Suppose A is a smooth closed subset of E and $E - A$ paracompact. Then f extends to a map $F: E \to K$ with $\hat{v}F = \hat{e}$.*

Proof. Let $e \in E - A$. Select $a_e \in A$ such that $\rho(e, a_e) < 2\rho(e, A)$. Let $\sigma = \sigma_e: v \to E$ be a local section such that $a_e \in \sigma(v) \subset A$. Define

$$N_e = \{e' \in E \mid \rho(e', \sigma\hat{e}(e')) < 2\rho(e', A)\}$$

Then N_e is an open neighborhood of e since both $e' \to \rho(e', \sigma\hat{e}(e'))$ and $e' \to \rho(e', A)$ are continuous. The family of all N_e's is an open cover of $E - A$ so we get a locally finite subcover $\mathscr{U} = \{V_\lambda \mid \lambda \in \Lambda\}$ and partition of unity $\{\Pi_\lambda: (E - A) \to R\}$ subordinate to it. For each λ select e with $N_e \supset V_\lambda$ and define $\sigma_\lambda = \sigma_e$. Define $F: E \to K$ by

$$F(e) = \begin{cases} f(e) & e \in A \\ \Sigma_\lambda \Pi_\lambda(e) f\sigma\hat{e}(e) & e \in (E - A) \end{cases}$$

F is continuous on $(E - A)$ by the usual argument. Let $a \in A$ and Suppose W a neighborhood of $F(a) = f(a)$ in K. Let U be a neighborhood of a and C a convex subset of K such that $f(U \cap A) \subset C \subset W$. We may assume that $U = B(\tau; \varepsilon)$ for a local section τ of $A \to D$

which passes through a. Let $U' = B(\tau; \varepsilon/3)$. Then to prove F continuous at a we need only show $F(U') \subset W$. Let $y \in U'$. If $y \in A$ then $F(y) = f(y) \in W$. Suppose that $y \in U' - A$. If $y \in U_\lambda$ then $\rho(\sigma_\lambda \hat{e}(y), \tau\hat{e}(y) \leqq \rho(\sigma_\lambda \hat{e}(y), y) + \rho(y, \tau\hat{e}(y)) \leqq 2\rho(y, A) + \rho(y, \tau\hat{e}(y))$ (since $y \in U_\lambda \subset$ some $N_e) \leqq 3\rho(y, \tau\hat{e}y) < \varepsilon$. Thus $\sigma_\lambda \hat{e}y \in U$ and $f\sigma_\lambda \hat{e}y \in C$. Since C is convex, $F(y) \in C \subset W$ showing $F(U') \subset W$.

Dugundji's extension theorem is the case $D =$ point since in this case every A is smooth and E metric so $E - A$ is paracompact. Now let L be a topological vector space of type m (as in [5] or the case $D =$ point in Definition 3.1) and T a convex subset. Consider

$$\begin{array}{ccc} E & & T \\ \cup & \nearrow_{g} & \\ A & & \end{array}$$

$E \to D$ a metric family.

COROLLARY 3.4. *If A is a closed smooth subset of E and $E - A$ is paracompact then g extends to a continuous function $G: E \to T$.*

Proof. Use $V = D \times L \to D$ and $K = D \times T \to D$. Let $f = (\hat{e}, g): A \to K$. The theorem gives an extension $F = (\hat{e}, G): E \to K$ and G is the desired extension of g.

Now consider

$$\begin{array}{ccc} D \times M & & T \\ \cup & \nearrow_{g} & \\ A & & \end{array}$$

$T =$ convex subset of $L =$ vector space of type m. $M =$ metric space.

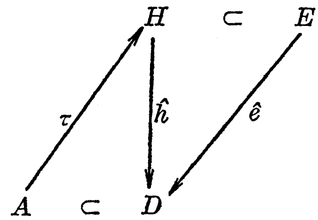
COROLLARY 3.5. *Let A be a closed smooth subset of $D \times M$ and suppose $(D \times M) - A$ is paracompact. Then g extends to a continuous $G: D \times M \to T$.*

Just as in the case $D =$ point we can say something about linearity. Suppose $S \to D$ is any map and $V \to D$ a vector family. Let $C_D(\mathrm{S}, V)$ be the set of map's $h: S \to V$ with $\hat{v}h = \hat{s}$. Then if $C_D(S, V)$ is not empty it has a natural vector space structure (add values). The proof of Theorem 3 gives a linear function

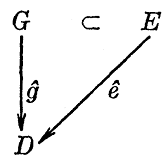
$$\alpha: C_D(A, V) \longrightarrow C_D(E, V) \qquad \alpha(f) = F$$

and if $\beta: C_D(E, V) \to C_D(A, V)$ is the function $\beta G = G \mid A$ then we have $\beta\alpha =$ identity.

4. **Cross sections.** In this section we will study the following commutative diagram $E \to D$ a normed vector family.



The objective is to find conditions under which $\hat{h}: H \to D$ has a cross section which extends τ. If $A = \phi$ this is a pure cross section problem and by introducing an auxiliary space we can reduce the problem to a cross section problem. Define $G = G(\tau) \subset E$ by $G \cap E_d = H_d$ if $d \notin A$ and $G \cap E_d = \{\tau(d)\}$ if $d \in A$. Thus



and $\hat{h}$ has a cross section extending τ iff $\hat{g}$ has a cross section.

Recall [Dold, 4] that a cover $\mathscr{U}$ of a space is *numerable* if there is a family $\{\Pi_\lambda: D \to [0, 1]\}$ of continuous functions such that $\{\Pi_\lambda^{-1}(0, 1]\}$ is locally finite and a refinement of $\mathscr{U}$ and $\Sigma_\lambda \Pi_\lambda(d) = 1$ all $d \in D$.

DEFINITION 4.1. (H, τ) is *numerably sectioned in* E if for every $\varepsilon > 0$ and every open convex E' of E with $\hat{e}(E' \cap G) = D$ there is a family $\mathscr{S} = \mathscr{S}(\varepsilon, E')$ of local sections of $\hat{e}$ such that

(a) $E' \cap G \subset \bigcup \{B_\varepsilon(\sigma) \mid \sigma \in \mathscr{S}\}$

(b) $\mathscr{U} = \mathscr{U}(\varepsilon, E') = \{\hat{e}(B_\varepsilon(\sigma) \cap G \cap E') \mid \sigma \in \mathscr{S}\}$ is a numerable cover of D.

The dependence on τ of the above condition appears to be unavoidable in the general case. However, if D is paracompact it can often be eliminated (see 4.9 below).

THEOREM[2] 4.2. *Suppose that* $\hat{e}: E \to D$ *is a normed vector family*

[2] This theorem and its corollaries are valid, with the same proof, under the weaker hypotheses: $|\cdot|: E \to R$ is upper semi-continuous and a pseudo-norm on each E_d.

and Sect $(\hat{e})$ *is complete. Suppose* H_d *is closed in* E_d *and convex for* $d \in D - A$ *and that* (H, τ) *is numerably sectioned in* E. *Then* $\hat{h}$ *has a cross section extending* τ.

LEMMA 4.3. *Suppose* $\hat{e}: E \to D$ *is a normed vector family, each* H_d *is convex, and* (H, τ) *is numerably sectioned in* E. *Suppose* $\varepsilon > 0$ *and* E' *open convex with* $\hat{e}(E' \cap G) = D$. *Then* $B(G \cap E'; \varepsilon) \to D$ *has a cross section.*

Proof. Let $\mathscr{U} = \mathscr{U}(\varepsilon, E')$, $\{\Pi_\lambda: D \to [0, 1]\}$ be as in Definition 4.1. For each $\lambda \in \Lambda$, $\Pi_\lambda^{-1}(0, 1] \subset V \in \mathscr{U}$ for some V. Pick such a V and its section σ and let $\sigma_\lambda = \sigma \mid \Pi_\lambda^{-1}(0, 1]$. Define $\sigma = \Sigma \Pi_\lambda \sigma_\lambda: D \to E$. Then σ is continuous by the usual argument. Each σ_λ is a local section of $B(G \cap E'; \varepsilon) \to D$ and $B(G_d, \varepsilon)$ is convex so σ is a section of $B(G \cap E'; \varepsilon) \to D$.

Proof of 4.2. We will construct a sequence $\sigma_1, \sigma_2, \cdots$ of sections of $e: E \to D$ such that

$$(A_n)(n > 1)\rho(\sigma_n, \sigma_{n-1}) < 1/2^{n-2}$$
$$(B_n)\rho(\sigma_n, G) < 1/2^n .$$

By (A), $\{\sigma_n\}$ is a Cauchy sequence in Sect $\hat{e}$ and by completeness $\sigma_n \to \sigma$ and σ is a continuous section of $\hat{e}$. By (B) and the fact that each G_d is closed in E_d we see that $\sigma(D) \subset G$ so that σ is the desired cross section.

Lemma 4.4 gives σ_1. Suppose that $\sigma_1, \sigma_2, \cdots, \sigma_n$ are defined with properties (A) and (B). The lemma gives a cross section

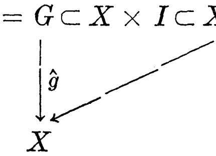
$$\sigma_{n+1}: D \longrightarrow B(G \cap B(\sigma_n, 1/2^n), 1/2^{n+1})$$

since (B_n) shows $\hat{e}(G \cap B(\sigma_n, 1/2^n)) = D$. Thus (B_{n+1}) $\rho(\sigma_{n+1}, \sigma_n) < 1/2^{n+1}$ and (A_{n+1}) $\rho(\sigma_{n+1}, \sigma_n) < 1/2^{n-1}$ are clear (since $1/2^{n+1} + 1/2^n < 1/2^{n-1}$).

It's interesting to note that the above theorem includes the Tietze extension theorem. To be precise it includes a deduction of the Tietze extension theorem from the Uryrohn theorem on the existence of certain real valued functions on a normal space.

COROLLARY 4.4. (Tietze 12) *X normal $\supset A$ closed. $f: A \to [0, 1]$ a map. Then f extends to a map $F: X \to [0, 1]$.*

Proof. Consider $[((X - A) \times I) \cup G_f] = G \subset X \times I \subset X \times R$



where G_f is the graph of f. For $t \in I$ define $\sigma_t(x) = (x, t)$. These and the compactness of I allow $\mathscr{S}_\varepsilon$ to be finite in Definition 4.1. Also $\hat{g}$ is an open map so $\mathscr{U}$ of 4.1 is a finite open cover. Urysohn's theorem permits the construction of a subordinate partition of unity [Bourbaki, 2] and the result now follows from 4.2.

DEFINITION 4.5. $\hat{e}: E \to D$ is a *Banach family* if it is a normed vector family (see 1.11), has the coarse topology for a full family of local sections (see 2.5), and each E_d is complete.

COROLLARY 4.6. *Suppose $\hat{e}: E \to D$ is a Banach family. Suppose that each H_d is closed in E_d and convex and that (H, τ) is numerably sectioned in E. Then h has a cross section extending τ.*

Proof. By 2.7 Sect $\hat{e}$ is complete.

DEFINITION 4.7. H is sectioned in E there exists H' dense in H such that for every $h \in H'$ there is a local section $\sigma: V \to E$ of $\hat{e}$ with $h \in \sigma(V)$.

COROLLARY 4.8. *Suppose D paracompact, $\hat{e}: E \to D$ a normed vector family, Sect $\hat{e}$ complete, $\hat{h}$ coarsely open and onto. Suppose H is sectioned in E and each H_d is closed and convex. Then $\hat{h}$ has a cross section extending τ for any closed A and partial section τ.*

Proof. For any $\varepsilon > 0$, E' (as in 4.1) take $\mathscr{S}$ = all local sections of E. Then (a) is clear. If we can show that $\hat{g}$ is coarsely open then $\mathscr{U}$ of Definition 4.1 will be an open cover and will be numerable by the paracompactness of D. Let $d \in \hat{e}(B(\sigma; \varepsilon) \cap G)$. If $d \in D - A$ then $d \in \hat{h}(B(\sigma; \varepsilon)) \cap (D - A) \subset \hat{e}(B(\sigma; \varepsilon) \cap G)$. If $d \in A$ select W open in D such that $\tau(W \cap A) \subset B(\sigma; \varepsilon)$ and check that $d \in W \cap \hat{h}(B(\sigma; \varepsilon) \cap H) \subset \hat{e}(B(\sigma; \varepsilon) \cap G)$. So $\hat{g}$ is coarsely open because $\hat{h}$ is. Thus 4.2 applies.

COROLLARY 4.9. *Suppose D paracompact $\hat{e}: E \to D$ a Banach family, $\hat{h}$ open and onto, each H_d closed and convex. Then $\hat{h}$ has a cross section extending τ.*

Proof. This follows from 4.8. H is sectioned since $\hat{e}$ has a full family of local cross sections. Sect $\hat{e}$ is complete by 2.7.

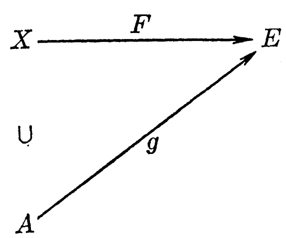
COROLLARY 4.10. *D paracompact, $\hat{e}: E \to D$ a Banach family, $\hat{e}$ open and onto. If A is any closed subset of D and τ is a cross section of $\hat{e}$ over A then τ extends to a cross section of $\hat{e}$.*

Note that in 4.8–4.10 the hypotheses are independent of A. Recall [4] that a map $\hat{t}\colon T \to D$ has the section extension property if each cross section over a set A which extends to a halo extends to a cross section over D. $V \supset A$ is a halo if there is a continuous function $\alpha\colon D \to [0, 1]$ with $A \subset \alpha^{-1}(1)$ and $(D - V) \subset \alpha^{-1}(0)$.

COROLLARY 4.11. *Suppose $T \to D$ satisfies the hypotheses of 4.8 or 4.9 or 4.10, over each set of a numerable covering of D. Then $T \to D$ has the section extension property.*

Proof. Since Dold showed that the section extension property is local [4, p. 229].

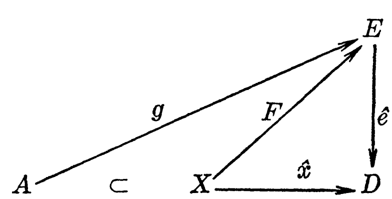
5. **Selections.** In this section some of the results of § 4 are used to prove selection theorems. Call $F\colon X \to E$ a *multifunction* if it is a relation which assigns to each $x \in X$ a nonempty subset of E. The graph of $F = G(F) = \{(x, e) \mid e \in F(x)\} \subset X \times E$. A *selection* for F is a single valued function $f\colon X \to E$ such that $f(x) \in F(x)$ for all $x \in X$. Consider



F a multifunction

g a continuous selection for $F \mid A$

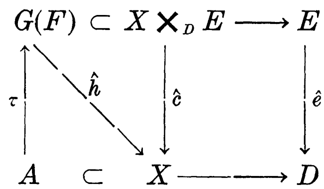
The selection-extension problem [Michael, 9] is: can we find a continuous selection for F which extends g? Here we shall study the following version



F is a multifunction, g, $\hat{x}$, $\hat{e}$ are maps, $\hat{e}F = \hat{x}$, $\hat{e}g = \hat{x} \mid A$ and g is a selection for $F \mid A$. The object is to find a continuous selection for F extending g. Note that $\hat{e}f = \hat{x}$ will follow. Recall that a multifunction F is *lower semi-continuous* if F^{-1} of an open set is open (recall $F^{-1}(S) = \{x \in X \mid F(x) \cap S \neq \emptyset\}$).

THEOREM 5.1. *Suppose X paracompact, A closed, $\hat{e}\colon E \to D$ a Banach family. Suppose F is lower semi-continuous and has closed, convex values. Then F has a continuous selection extending g.*

Proof. Consider the following diagram.

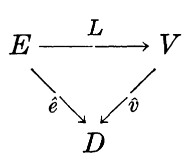


Here $X \times_D E = \{(x, e) \in X \times E \mid \hat{x}(x) = \hat{e}(e)\} = C$. $C \to X$ gets the structure of a Banach family from $\hat{e}$. Specifically $(x, e) + (x, e') = (x, e + e')$ is the addition, $\lambda(x, e) = (x, \lambda e)$ the scalar multiplication, C_x is isometric with $E_{\hat{x}x}$ so is complete. If (x, e) is given with $\hat{x}x = d = \hat{e}e$ then there is an open V of D and local section σ over V through e. $\sigma' = (id, \sigma\hat{x})$ is a local section of $\hat{c}$ over $\hat{x}^{-1}(V)$ through (x, e). It is not hard to check that $\hat{h}$ is an open map since F is lower semi-continuous (this is Prop. 1.2 of [3]). The hypotheses show that each $G(F)_d$ is closed and convex. Define $\tau(a) = (a, g(a))$. Now Corollary 4.9 gives a cross section $\sigma\colon X \to G(F)$ of $\hat{h}$ which extends τ and the composition $X \to G(F) \subset X \times_D E \to E$ is the desired selection-extension.

COROLLARY 5.2. [Michael, 9] *E a Banach space, $F\colon X \to E$ a lower semi-continuous multifunction with closed convex values. X paracompact $\supset A$ closed and $g\colon A \to E$ a continuous selection for $F \mid A$. Then F has a continuous selection which extends g.*

Proof. This is the case $D =$ point of Theorem 5.1.

In [9] Michael proves several variants of 5.2. Each of these can be generalized to the metric family setting. Theorem 5.1 is a consequence of Theorem 4.9. Similarly each of the theorems of § 4 gives rise to a selection theorem. Now consider the following situation.

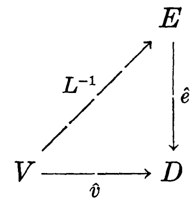


$E \to D$ a Banach family

$V \to D$ a vector family

COROLLARY 5.3. *Suppose L is linear* (i.e. each L_d is continuous linear $-L$ is a family of linear operators), *open, and onto, and V is paracompact. Then L has a continuous right inverse* ($M\colon V \to E$ such that $LM = id$). *In fact, any partial right inverse on a closed subset A of V can be extended to a right inverse.*

Proof. Consider

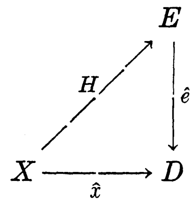


Since L is open and onto, $F = L^{-1}$ is a lower semi-continuous multifunction. It has closed convex values since L is linear and V is Hausdorff. Theorem 5.1 gives a continuous selection for L^{-1} which is a right inverse for L.

Until now we have looked for continuous cross-sections and selections but for our final result we will relax this restriction. Henceforth sections and selections are simply single valued functions and all continuity conditions will be stated explicitly. In [3] Coban proves a selection theorem for a multifunction from a σ-space to a complete metric space. We will generalize this to the metric family setting. Several of Coban's other results can be generalized in the same manner. A space X is a σ-space if for every family $\mathscr{U}$ of open sets there is a σ-discrete refinement $\mathscr{W}$ of closed sets such that $\bigcup \mathscr{U} = \bigcup \mathscr{W}$. Coban points out [3, p. 275] that each of the following is a sufficient condition for a space X to be a σ-space: (a) X is weakly paracompact and completely normal, (b) X has a σ-discrete filter, (c) X is a symmetric space satisfying the first axiom of countability, (d) X has a uniform structure, (e) X has a refining sequence of coverings. Recall that a subset of a topological space is an F_σ set if it is a countable union of closed sets. In a σ-space every difference of closed sets is an F_σ set. In particular open sets are F_σ sets.

DEFINITION. 5.4. $E \to D$ a metric family, V open in E, $V_n = \{e \in E \mid \rho(e, E - V) > 1/n\}$. V is *approachable* if V_n is open for all sufficiently large n.

Consider the following situation



Assume that $\hat{e}$ is a metric family with the coarse topology defined by a full family of continuous local sections and E_d complete. Suppose that X is a σ-space and H is a lower semi-continuous multifunction with $\hat{e}H = \hat{x}$ and closed values.

THEOREM 5.5. *Under the hypotheses above H has a selection h such that $h^{-1}(V)$ is an F_σ set for every approachable open V of E.*

Proof. For $n = 1, 2, \cdots$, we will find a function $h_n\colon X \to E$ with $\hat{e}h_n = \hat{x}$ and subsets $X_{n,1}, X_{n,2}, \cdots$ such that

(A_n) $(n > 1)\rho(h_n, h_{n-1}) < 1/2^{n+1}$

(B_n) $\rho(h_n, H) < 1/2^{n-1}$

(C_n) $h_n \mid X_{n,k}$ is continuous

(D_n) $\{X_{n,k} \mid k = 1, 2, \cdots\}$ is a disjoint cover of X by F_σ sets.

If this has been done then define $h\colon X \to E$ by $h = \lim h_n$. h is well defined by (A) and the completeness of each E_d and it is a selection for H by (B). Consider

$$h^{-1}(V) = \bigcup_{n=1}^{\infty} h_n^{-1}(V_n) \qquad V_n \text{ as in Definition 5.4.}$$

By [8, p. 398] this equation is true in each X_d so it is true in X. There is no loss in generality in assuming each V_n open so by (C) and (D) each $h_n^{-1}(V_n)$ is an F_σ and hence $h^{-1}(V)$ is an F_σ set.

Proof of $A_{n+1} - D_{n+1}$ assuming $A_n - D_n$ {with modifications for $n = 0$}. Let $h_n, X_{n,1}, X_{n,2}, \cdots$, satisfy $A_n - D_n$ {ignore this if $n = 0$}. Fix k, for now, and define $H_k\colon X_{n,k} \to E$ by $H_k(x) = H(x) \cap B(h_n(x), 1/2^{n+1})$ {if $n = 0$, use $X_{0,1} = X$, $X_{0,k} = \phi$ for $k > 1$, and $H_1 = H$}. By (B_n) $H_k(x)$ is nonempty and H_k is lower semi-continuous on $X_{n,k}$ because h_n is continuous there. Define $\mathscr{U} = \{H_k^{-1}(B(\sigma, 1/2^{n+2}) \mid \sigma$ a local section of $\hat{e}\}$. Then $\mathscr{U}$ is an open cover of $X_{n,k}$ so there are discrete families of closed sets $\mathscr{W}_p = \{A_{p,\alpha} \mid \alpha \in I_p\}$ $p = 1, 2, \cdots$ such that $\mathscr{W} = \bigcup \mathscr{W}_p$ refines $\mathscr{U}$ and covers $X_{n,k}$. If $\alpha \in I_p$ select σ_α a local section with $A_{p,\alpha} \subset H_k^{-1}B(\sigma_\alpha, 1/2^{n+2})$. Thus $\rho(H(x), \sigma_\alpha(x)) < 1/2^{n+2}$ for $x \in A_{p,\alpha}$. Now define

$$W'_{i,k} = \bigcup \mathscr{W}_i$$

$$W_{i,k} = W'_{i,k} \setminus \bigcup_{j=1}^{i-1} W'_{j,k} = W'_{i,k} \setminus \bigcup_{j=1}^{i-1} W_{j,k}$$

Then $W_{i,k}$ is a difference of closed sets so an F_σ subset of $X_{n,k}$, so an F_σ subset of X. The $W_{i,k}$'s form a disjoint cover of $X_{n,k}$ by F_σ subsets of X. Define

$$g'_{ik}: W'_{ik} \longrightarrow E \qquad g'_{ik} = \sigma_\alpha \text{ on } A_{i\alpha}$$

$$g_{ik}: W_{ik} \longrightarrow E \qquad g_{ik} = g'_{ik} \mid W_{ik} .$$

Thus g'_{ik} is continuous on each $A_{i,\alpha}$ and $\mathscr{W}_i$ discrete so g'_{ik} continuous on W'_{ik} and g_{ik} continuous on W_{ik}.

Now consider the family of all W_{ik}, $i, k = 1, 2, \cdots$. This is a countable family so it can be reordered and renamed as $X_{(n+1),1)}$ $X_{(n+1),2)} \cdots$. Hence the $X_{(n+1),k}$ form a disjoint cover of X by F_σ subsets of X which proves (D_{n+1}). Define $h_{n+1}: X \to E$ by $h_{n+1} \mid W_{ik} = g_{ik}$. Then (C_{n+1}) and (B_{n+1}) are easily checked. For $n > 1$, condition (A_{n+1}) is a consequence of the definition of H_k and $\mathscr{U}$ since $1/2^{n+1} + 1/2^{n+2} < 1/2^n$. This completes the proof.

DEFINITION 5.6. $E \to D$ is a locally embedable metric family if for every $d \in D$ there is an open $W = W(d)$ neighborhood of d and a metric space $M = M(d)$ and an isometric embedding of metric families

$$\begin{array}{ccc} N_W & \hookrightarrow & W \times M \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

COROLLARY 5.7. *Suppose in* 5.5 *that* E *is a locally embedable metric family. Then* H *has a selection* h *such that* $h^{-1}(V)$ *is an* F_σ *set for all* V *in some basis of* E.

Proof. In $W \times M$ if $V = W' \times B$ $B = B_\varepsilon(m)$ then $V_n = W' \times B_n$ is open so V is approachable and sets $V \cap E_W$ are a basis of E.

DEFINITION 5.8. $E \to D$ is a sub-euclidean metric family if for every $d \in D$ there is a neighborhood $W = W(d)$ of d and an integer $n = n(d)$ and a closed isometric embedding of metric families.

$$\begin{array}{ccc} E_W & \hookrightarrow & W \times \mathbf{R}^n \\ & \searrow \quad \swarrow & \end{array}$$

COROLLARY 5.9. *Suppose in* 5.5 *that* $E \to D$ *is a sub-euclidean metric family. Then* H *has a selection* h *such that* $h^{-1}(V)$ *is an* F_σ *for every open* V *of* E.

Proof. Let A be closed in E and define $g_A: E \to [0, +\infty]$ by $g(e) = \rho(e, A)$. Then $V_n = g_{E-V}^{-1}(1/n, +\infty]$ so we need only show that each g_A is lower semi-continuous. We may assume $E = D \times \boldsymbol{R}^n$. Let $\varepsilon > 0$ and $g(e) > \varepsilon$. Suppose $e = (d_0, u_0)$ and $g(e) = \alpha > \beta_2 > \beta_1 > \varepsilon$. Set $\gamma = \beta_2 - \beta_1$. We will prove that there is a neighborhood W of d_0 such that $e = (d, u) \in W \times B(u_0, \gamma)$ implies $g(d, u) > \varepsilon$. Suppose this is false. Then for each $W \ni d_0$ get $d \in W$, $u \in B(u_0, \gamma)$ and $g(d, u) \leqq \varepsilon$. So we can find $u' \in R^n$ with $(d, u') \in A$, $d(u, u') \leqq \beta_1$. Then $\{(d_W, u'_W)\}$ is a net on A, $\{u'_W\}$ is a net on $\overline{B(u_0, \beta_1 + \gamma)}$ so has a convergent subnet converging to $\bar{u}$. But this gives a net on A converging to $(d_0, \bar{u})$ and A is closed so $(d_0, \bar{u}) \in A$. Hence $g(e) \leqq \rho(e, (d_0, \bar{\mu})) = d(u_0, \bar{\mu}) \leqq \beta_1 + \gamma = \beta_2 < \alpha$, a contradiction.

COROLLARY. *Let* $p: E \to X$ *be a metric fiber bundle with fiber* F *a closed subset of* $\boldsymbol{R}^n$ *and* X *a* σ*-space. Then* p *has a cross section* σ *such that* $\sigma^{-1}(V)$ *is an* F_σ *set for all opens* V *of* E.

Proof. Apply 5.9 with $X = D$ and $H = p^{-1}$.

Note that we cannot expect a continuous cross section in the setting of Corollary 5.10 as is shown by the Hopf bundle $S^3 \to S^2$ with fiber S^1.

REFERENCES

1. M. F. Atiyah, *K-Theory*, Harvard lecture notes, 1964.
2. N. Bourbaki, *General Topology II*, Addison-Wesley, 1966.
3. M. M. Coban, *Many-valued mappings and Borel sets I*. Trans. Moscow Math. Soc., **22** (1970), 258–280 (Amer. Math. Soc. translations).
4. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math., **78** (1963), 223–255.
5. J. Dugundji, *Topology*, Allyn and Bacon, Boston (1966).
6. ———, *An extension of Tietze's theorem*, Pacific J. Math., **1** (1951) 353–367.
7. D. Husemoller, *Fiber Bundles*, McGraw-Hill, New York (1966).
8. K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Ser. Sec. Math. Astron. Phys., **13** (1965), 397–403.
9. E. A. Michael, *Continuous selections I*, Ann. of Math., (2) **63** (1956), 361–382.
10. D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience, New York, 1955.
11. N. Steenrod, *The topology of fiber bundles*, Princeton, 1951.
12. H. Tietze, *Uber Funktionen, die auf einer abgeschlossen Menge stetig sind*, J. für die R. und Ang. Math., **145**, (1915), 9–14.

Received February 8, 1974 and in revised form February 3, 1975.

UNIVERSITY OF KANSAS