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AN ASYMPTOTIC ANALYSIS OF AN ODD ORDER LINEAR DIFFERENTIAL EQUATION

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Let q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and let n be a positive integer. With respect to the equation $u^{(2n+1)} + qu = 0$, we study the relationship between the existence of oscillatory solutions and the asymptotic behavior of nonoscillatory solutions.

There is no additional hypothesis on q which will ensure that every solution of

$$u^{(2n+1)} + qu = 0 \tag{1}$$

is oscillatory. In particular, it follows from a result of P. Hartman and A. Wintner [5] that there is a solution u of (1) such that

$$(-1)^k u^{(k)}(t) > 0 \tag{2}$$

whenever $t \geqq 0$ and $0 \leqq k \leqq 2n$. We shall call a solution of u of (1) *strongly decreasing* if and only if there is $c \geqq 0$ such that (2) is true whenever $t \geqq c$ and $0 \leqq k \leqq 2n$. Since we know that (1) has a strongly decreasing solution, the best result one can hope for in an oscillation theorem is that every eventually positive solution of (1) is strongly decreasing. G. V. Anan'eva and V. I. Balaganskii [1] (see also C. A. Swanson [7, p. 175]) have shown that if

$$\int_0^\infty t^{2n-1} q(t)dt = \infty \ , \tag{3}$$

then every eventually positive solution of (1) is strongly decreasing. Our first result extends this.

THEOREM 1. *If* (3) *fails and the second order equation*

$$w''(t) + \frac{1}{(2n-2)!}\left(\int_t^\infty (s-t)^{2n-2} q(s)ds\right) w(t) = 0 \tag{4}$$

is oscillatory, then every eventually positive solution of (1) *is strongly decreasing.*

Although the conclusion of Theorem 1 limits the asymptotic behavior of nonoscillatory solutions of (1) (if u is nonoscillatory then either u or $-u$ is eventually positive), it does not in fact ensure the existence of oscillatory solutions.

THEOREM 2. *Suppose that every eventually positive solution of* (1) *is strongly decreasing. Then if u is a solution of* (1), *and if any of $u, u', u'', \cdots, u^{(2n)}$ has a zero in $[0, \infty]$, u is oscillatory.*

COROLLARY 1. *With the hypotheses of Theorem* 2, *the solution space Q of* (1) *has a basis each member of which is oscillatory, and Q has a $2n$-dimensional subspace each member of which is oscillatory.*

COROLLARY 2. *With the hypotheses of Theorem* 1, *the conclusions of Corollary* 1 *hold.*

Finally, we offer a comparison theorem.

THEOREM 3. *Suppose that p is a continuous function from $[0, \infty)$ to $(0, \infty)$ with $p(t) \geqq q(t)$ whenever $t \geqq 0$. Suppose also that every eventually positive solution of* (1) *is strongly decreasing. Then every eventually positive solution of*

$$u^{(2n+1)} + pu = 0 \tag{5}$$

is strongly decreasing.

A. C. Lazer has shown [6, Theorem 1.2] that in the third order case the existence of a nontrivial oscillatory solution of (1) implies that every eventually positive solution of (1) is strongly decreasing. The following example shows that this is not true in general.

EXAMPLE. Suppose $1 < r < 2$. Now

$$r(r-1)(r-2)(r-3)(r-4) > (r+2)(r+1)r(r-1)(r-2)\ ,$$

so $\alpha > \gamma$, where

$$\alpha = \min\{r(r-1)(r-2)(r-3)(r-4) : 1 \leqq r \leqq 2\}$$

and

$$\begin{aligned}\gamma &= \min\{(r+2)(r+1)r(r-1)(r-2) : 1 \leqq r \leqq 2\} \\ &= \min\{r(r-1)(r-2)(r-3)(r-4) : 3 \leqq r \leqq 4\}\ .\end{aligned}$$

Let $n = 2$, and, noting that α and γ are negative, let β be a positive number such that $\alpha > -\beta < \gamma$. Let q be given by $q(t) = \beta(t+1)^{-5}$. The polynomial equation

$$r(r-1)(r-2)(r-3)(r-4) + \beta = 0 \tag{6}$$

has two complex roots, so (1) has nontrivial oscillatory solutions. On the other hand, (6) has a solution r in the interval $(3, 4)$, and u, given by $u(t) = (t + 1)^r$, satisfies (1) and is not strongly decreasing. The example is complete.

LEMMA. *Suppose u is a solution of* (1), $c \geqq 0$, *and* $(-1)^k u^{(k)}(c) > 0$ *for* $k = 0, \cdots, 2n$. *Then* (2) *is true for* $0 \leqq t \leqq c$ *and* $k = 0, \cdots, 2n$.

Proof. Let v be given on $[-c, 0]$ by $v(t) = u(-t)$. If $k = 0, \cdots, 2n + 1$ then $v^{(k)}(t) = (-1)^k u^{(k)}(-t)$, so

$$v^{(2n+1)}(t) - q(-t)v(t) = 0$$

and $v^{(k)}(-c) > 0$ for $-c \leqq t \leqq 0$ and $k = 0, \cdots, 2n$. Thus

$$v(t) = v(-c) + \sum_{m=1}^{2n} \frac{(t + c)^m}{m!} v^{(m)}(-c) + \int_{-c}^{t} \frac{(t - s)^{2n}}{(2n)!} q(-s)v(s)ds \tag{7}$$

if $-c \leqq t \leqq 0$. But clearly the solution of (7) is positive, so $v(t) > 0$ if $-c \leqq t \leqq 0$. Now, if $k = 0, \cdots, 2n$ and $-c \leqq t \leqq 0$,

$$\begin{aligned} v^{(k)}(t) = v^{(k)}(-c) &+ \sum_{m=k+1}^{2n} \frac{(t + c)^{m-k}}{(m - k)!} v^{(m)}(-c) \\ &+ \int_{-c}^{t} \frac{(t - s)^{2n-k}}{(2n - k)!} q(-s)v(s)ds \,, \end{aligned}$$

so $v^{(k)}(t) > 0$, i.e., $(-1)^k u^{(k)}(-t) > 0$. The proof is complete.

Proof of Theorem 1. Assume that (3) fails. We shall show that if there is an eventually positive solution of (1) which is not strongly decreasing then (4) is nonoscillatory. Let u be an eventually positive solution of (1) which is not strongly decreasing. Find $a \geqq 0$ such that $u(t) > 0$ if $t \geqq a$. Now $u^{(2n+1)} < 0$ on $[a, \infty)$, so $u^{(2n)}$ is eventually one-signed. Since $u^{(2n)}$ is eventually one-signed, $u^{(2n-1)}$ is eventually one-signed. Continuing this, we see that there is $c \geqq a$ such that none of $u, u', \cdots, u^{(2n)}$ has a zero in $[c, \infty)$. Let j be the largest integer such that $u^{(i)} > 0$ on $[c, \infty)$ if $i \leqq j$ (we write $u = u^{(0)}$). Note that $j \neq 2n + 1$. Now $u^{(j+1)} < 0$ on $[c, \infty)$, so $u^{(j)}$ is bounded. Thus, if $j \leqq k \leqq 2n$, $u^{(k)}u^{(k+1)} < 0$ on $[c, \infty)$. But $u^{(2n+1)} < 0$, so if $j \leqq k \leqq 2n$ then $u^{(k)} > 0$ on $[c, \infty)$ if k is even and $u^{(k)} < 0$ on $[c, \infty)$ if k is odd. Since $u^{(j+1)} < 0$ (recall how j was chosen), this says $j + 1$ is odd and j is even. By hypothesis, $j \neq 0$. Suppose $j < 2n$. Now

$$-u^{(j+1)}(t) = \frac{1}{(2n - j - 1)!} \int_{t}^{\infty} (s - t)^{2n-j-1} q(s)u(s)ds$$

if $t \geqq c$. Also, $u^{(j-1)}$ is increasing on $[c, \infty)$ since $u^{(j)} > 0$, so, if $s \geqq t \geqq c$,

$$\begin{aligned}u(s) &\geqq \frac{1}{(j-2)!}\int_c^s (s-\xi)^{j-2}u^{(j-1)}(\xi)d\xi \\ &\geqq \frac{1}{(j-2)!}\int_t^s (s-\xi)^{j-2}u^{(j-1)}(\xi)d\xi \\ &\geqq \frac{u^{(j-1)}(t)}{(j-2)!}\int_t^s (s-\xi)^{j-2}d\xi = \frac{u^{(j-1)}(t)}{(j-1)!}(s-t)^{j-1}\,.\end{aligned}$$

Since $(2n-j-1)!\,(j-1)! \leqq (2n-2)!$, this says

$$\begin{aligned}-u^{(j+1)}(t) &\geqq \frac{u^{(j-1)}(t)}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds\,, \\ u^{(j+1)}(t)/u^{(j-1)}(t) &\leqq -\frac{1}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds\end{aligned} \tag{8}$$

if $t \geqq c$. Let v be given on $[c, \infty)$ by $v = u^{(j)}/u^{(j-1)}$, and note that $v > 0$ on $[c, \infty)$. Now if $t > c$,

$$v'(t) = u^{(j+1)}(t)/u^{(j-1)}(t) - v(t)^2\,,$$

so (8) says

$$v'(t) + v(t)^2 \leqq -\frac{1}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds\,. \tag{9}$$

But a classical result of M. Bôcher [2], [3] (see also C. de la Vallée Poussin [8], A. Wintner [9], C. A. Swanson [7, Theorem 2.15, p. 63], and W. A. Coppel [4, Theorem 4, p. 6]) says that the existence of a positive solution of (9) implies that (4) is nonoscillatory. The proof is complete, if $j < 2n$.

Suppose $j = 2n$. Now

$$\begin{aligned}u^{(2n)}(t) &\geqq \int_t^\infty q(s)u(s)ds \\ &\geqq \frac{1}{(2n-2)!}\int_t^\infty q(s)\left(\int_c^s (s-\xi)^{2n-2}u^{(2n-1)}(\xi)d\xi\right)ds \\ &\geqq \frac{1}{(2n-2)!}\int_t^\infty q(s)\left(\int_t^s (s-\xi)^{2n-2}u^{(2n-1)}(\xi)d\xi\right)ds\end{aligned}$$

if $t \geqq c$. But this and standard iteration methods say that there is a continuously differentiable function w from $[c, \infty)$ to $[u^{(2n-1)}(c), \infty)$ such that $w(c) = u^{(2n-1)}(c)$ and

$$w'(t) = \frac{1}{(2n-2)!}\int_t^\infty q(s)\left(\int_t^s (s-\xi)^{2n-2}w(\xi)d\xi\right)ds$$

if $t \geqq c$. But w clearly satisfies (4) on $[c, \infty)$, and can be extended to a nonoscillatory solution of (4) on $[0, \infty)$, so the proof is complete.

Proof of Theorem 2. Let u be a nonoscillatory solution of (1). If u is eventually negative, we may replace u by $-u$, so we assume that u is eventually positive. Now there is $c \geqq 0$ such that (2) holds whenever $t \geqq c$ and $k = 0, \cdots, 2n$. Now our lemma says that if $k = 0, \cdots, 2n$ then $u^{(k)}$ has no zeroes in $[0, c]$ and thus has no zeros at all. The proof is complete.

Proof of Corollary 1. If k is an integer in $[1, 2n+1]$, let z_k be the solution of (1) such that $z_k^{(j)}(0) = 0$ if $j \neq k-1$ and $z_k^{(k-1)}(0) = 1$. Clearly $\{z_1, \cdots, z_{2n+1}\}$ is a basis for Q, and Theorem 3 says that each z_k is oscillatory. Also, if u is in the $2n$-dimensional subspace spanned by $\{z_2, \cdots, z_{2n+1}\}$, then $u(0) = 0$ so u is oscillatory. The proof is complete.

Corollary 2 is now immediate from Theorem 2 and Corollary 1.

Proof of Theorem 3. We shall assume the existence of an eventually positive solution of (5) which is not strongly decreasing, and show the existence of an eventually positive solution of (1) which is not strongly decreasing. Let v be an eventually positive solution of (5) which is not strongly decreasing. Let $c \geqq 0$ be such that none of $v, v', \cdots, v^{(2n)}$ has a zero in $[c, \infty)$, and let j be the largest integer such that $v^{(i)} > 0$ on $[c, \infty)$ if $i \leqq j$. By hypothesis, $j \neq 0$, and we know that j is even. Now

$$v^{(j)}(t) \geqq \frac{1}{(2n-j)!}\int_t^\infty (s-t)^{2n-j}p(s)v(s)ds$$

if $t \geqq c$, and

$$v(t) \geqq v(c) + \frac{1}{(j-1)!}\int_c^t (t-s)^{j-1}v^{(j)}(s)ds$$

if $t \geqq c$, so

$$\begin{aligned}(10)\qquad v(t) &\geqq v(c) + \frac{1}{(j-1)!(2n-j)}\int_c^t (t-s)^{j-1}\left(\int_s^\infty (\xi-s)^{2n-j}p(\xi)v(\xi)d\xi\right)ds \\ &\geqq v(c) + \frac{1}{(j-1)!(2n-j)!}\int_c^t (t-s)^{j-1}\left(\int_s^\infty (\xi-s)^{2n-j}q(\xi)v(\xi)d\xi\right)ds\end{aligned}$$

if $t \geqq c$. Now (10) and standard iteration techniques say that there is a continuous function u from $[c, \infty)$ to $[0, \infty)$ such that $u(t) \leqq v(t)$ whenever $t \geqq c$ and such that

$$\begin{aligned}(11)\qquad u(t) &= v(c) \\ &\quad + \frac{1}{(j-1)!(2n-j)!}\int_c^t (t-s)^{j-1}\left(\int_s^\infty (\xi-s)^{2n-j}q(\xi)u(\xi)d\xi\right)ds\end{aligned}$$

if $t \geqq c$. The fact that u has only nonnegative values, together with (11), says $u(t) \geqq v(c)$ whenever $t \geqq c$; in particular, u has no zeros in $[c, \infty)$. Differentiation of (11) yields that u satisfies (1) on $[c, \infty)$, and $u' > 0$ on (c, ∞), i.e., (2) is not true for $k = 1$ and $t > c$. Clearly u can be extended to a solution of (1) on $[0, \infty)$, and this solution is eventually positive but not strongly decreasing. The proof is complete.

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