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REARRANGING FOURIER TRANSFORMS ON GROUPS

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Let G denote an infinite locally compact abelian group and X its character group. Let θ be a suitable Haar measure on X, and $1 < p < 2$. For a θ-measurable function ϕ on X, we define $\theta_\phi(t) = (\{\chi \in X: |\phi(\chi)| > t\})$ and $\phi^*(x) = \inf\{t > 0: \theta_\phi(t) \leqq x\}$ for $x > 0$. ϕ^* is called the nonincreasing rearrangement of ϕ. Note that even though ϕ is defined on X, the domain of ϕ^* is $(0, \infty)$. A nonnegative function g defined on $(0, \infty)$ is called admissible if g is nonincreasing and $\lim_{x\to\infty} g(x) = 0$.
Theorems:

1. Let G be nondiscrete with a compact open subgroup and g admissible. Then $g|_N = \hat{f}^*|_N$, where N is the set of positive integers, for some $f \in L^p(G)$ if $\sum_{k=1}^\infty g(k)^p k^{p-2} < \infty$.

2. Let G be nondiscrete with no compact open subgroup and g admissible. Then $g = \hat{f}^* m$ a.e. for some $f \in L^p(G)$ if $\int_0^\infty g(x)^p x^{p-2} dx < \infty$.

3. Let G be an infinite discrete abelian group which contains $Z, Z(r^\infty)$ or $Z(r)^{\aleph_0}$ as a subgroup, g admissible. Then $g|_{(0,1)} = \hat{f}^*|_{(0,1)} m$ a.e. for some $f \in L^p(G)$ if $\int_0^1 g(x)^p x^{p-2} dx < \infty$.

I. Introduction. As usual the Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined on X such that $\hat{f}(\chi) = \int_G f\chi d\lambda$, where λ is a fixed but arbitrary Haar measure on G. For $1 < p < 2, \hat{f} \in L^{p'}(G)$ and p' is the conjugate exponent of p. The set of real numbers, n-dimensional Euclidean space, the circle group, the integers, the r-adic integers, the countable product of the group of integers modula r and the subgroup of the circle whose elements have order a power of r are denoted by R, R^n, T, Z, Δ_r, $\Pi Z(r)$ and $Z(r^\infty)$, respectively. Also p will denote any number such that $1 < p < 2$. Let m be $1/\sqrt{2\pi}$. Lebesque measure on R.

Hardy and Littlewood [1], [2] characterized functions on Z such that every rearrangement is the Fourier transform of a function in $L^p(T)$, $2 < p < \infty$. They also characterized functions on Z such that some rearrangement is the Fourier transform of afunction in $L^p(T)$, $1 < p < 2$. Hewitt and Ross [4] generalized these results to arbitrary compact infinite abelian groups. We are interested in the case of LCA (locally compact abelian) groups. Here are our results.

THEOREM 1. *Let G be nondiscrete with a compact open subgroup,* *and g an admissible function. Then* $g|_N = \hat{f}^*|_N$ *for some* $f \in L^p(G)$ *if and only if* $\sum_{k=1}^{\infty} g(k)^p k^{p-2} < \infty$. *Moreover, there exists a constant* A_p *that depends on p only such that*

$$\left(\sum_{k=1}^{\infty} g(k)^p k^{p-2}\right)^{1/p} \leqq A_p \|f\|_p$$

for evrey such f.

THEOREM 2. *Let G be a nondiscrete* LCA *group with no compact open subgroup and g an admissible function. Then* $g = \hat{f}^*$ *for some* $f \in L^p(G)$ *if and only if* $\int_0^{\infty} g(x)^p x^{p-2} dx < \infty$. *Moreover, there exists* A_p *that depends only on p such that*

$$\left(\int_0^{\infty} g(x)^p x^{p-2} dx\right)^{1/p} \leqq A_p \|f\|_p$$

for every such f.

THEOREM 3. *Let G be an infinite discrete abelian group containing Z,* $Z(r^\infty)$ *or* $Z(r)^{\aleph_0}$ *as a subgroup and g an admissible function. Then* $g|_{(0,1)} = \hat{f}^*|_{(0,1)}$ *for some* $f \in L^p(G)$ *if and only if* $\int_0^1 g(x)^p x^{p-2} dx < \infty$. *Moreover there exists* A_p *that depends only p such that*

$$\left(\int_0^1 g(x)^p x^{p-2} dx\right)^{1/p} \leqq A_p \|f\|_p$$

for every such f.

Theorems 1 and 2 give us a complete solution for all nondiscrete LCA groups. Theorem 3 holds for "almost all" discrete abelian groups, but I am not able to settle the case where G contains $\prod_{n=1}^{*\infty} Z(r_n)$ as a subgroup, with $r_n \to \infty$.

The forward implications "$\Rightarrow$" of all three theorems and the existence of the constants A_p are due to Hunt [5]; see Stein and Weiss [6], Chapter V, Corollary 3.16.

II. A few lemmas.

LEMMA 1. *Let G be a LCA group and H an open subgroup of G. Let* $H^\perp = \{\chi \in X : \chi = 1 \text{ on } H\}$. *Then for each* $f_0 \in L^p(H)$, *there exists* $f \in L^p(G)$ *such that* $\hat{f}^* = \hat{f}_0^* m$ a.e. (where we use suitable Haar measures on X and $X/H^\perp$ for the definitions of $\hat{f}^*$ and $\hat{f}_0^*$).

Proof. Let $f_0 \in L^p(H)$ and define $f(x) = f_0(x)$ if $x \in H$ and $f(x) = 0$ otherwise. Since H is open, f is still λ-measurable in G

and $f \in L^p(G)$. Choose Haar measure λ_H on H to be the restriction of λ to H. Choose $\theta_{H^\perp}$ to be the normalized Haar measure on $H^\perp$, and θ_X to be an arbitrary Haar measure on X. Then a Haar measure θ_1 on $X/H^\perp$ exists so that Weil's theorem applies [3; Vol. II, 28.54]. $\hat{f}$ is clearly constant on each coset of $H^\perp$. That is, $\hat{f}(\chi) = \hat{f}_0(\chi H^\perp)$ for all $\chi \in X$. A calculation, using Weil's theorem shows that $\hat{f}^* = \hat{f}_0^* m$ a.e.

For the rest of this paper, we let g be a fixed admissible function on $(0, \infty)$, $1 < p < 2$ and $\int_0^\infty g(x)^p x^{p-2} dx$ is finite.

LEMMA 2. (i) $\int_0^1 g(ct)dm(t) < \infty$ *for all* $c > 0$.
(ii) $0 \leqq \int_0^\infty g(ct) \sin xt\, dm(t) \leqq \int_0^{\pi/x} g(ct) \sin xt\, dm(t) < \infty$ *for all* $x > 0$, $c > 0$.

Proof. (i) Since

$$\int_0^1 g(ct)^p dm(t) \leqq \int_0^1 g(ct)^p t^{p-2} dm(t) \leqq \int_0^\infty g(ct)^p t^{p-2} dm(t)$$
$$= \frac{1}{c^{p-1}} \int_0^\infty g(t)^p t^{p-2} dm(t) < \infty ,$$

we see that $\int_0^1 g(ct)^p dm(t)$ is finite and hence $\int_0^1 g(ct)dm(t)$ is finite.
(ii) For $k = 1, 2, \cdots$, let

$$\nu_k = (-1)^{k+1} \int_{(k-1)\pi/x}^{k\pi/x} g(ct) \sin xt\, dm(t) .$$

It is clear that $\nu_1 \geqq \nu_2 \geqq \nu_3 \geqq \cdots \geqq 0$ and $\nu_k \to 0$.
It follows that

$$\int_0^\infty g(ct) \sin xt\, dt = \sum_{k=1}^\infty (-1)^{k+1} \nu_k$$

and hence

$$0 \leqq \int_0^\infty g(ct) \sin xt\, dt \leqq \nu_1 = \int_0^{\pi/x} g(ct) \sin xt\, dm(t) < \infty .$$

This completes the proof of Lemma 2.

Define $G_c(x) = \int_0^{|x|} g(ct)dm(t)$ for $x \in R$. This is well-defined because $\int_0^1 g(ct)dm(t) < \infty$ by (i) of Lemma 2 and g is bounded in between 1 and $|x|$.

LEMMA 3. (i) $G_c(x) = o(x^{1/p})$ *as* $x \to 0$ *and as* $x \to \infty$.

(ii) $\int_0^\infty G_c(x)^p x^{-2} dm(x) < \infty$ *for all* $c > 0$.

Proof. See [7], Vol. I, Ch. I, §9.16.

LEMMA 4. *There exists* $f \in L^p(R)$ *such that* $\hat{f}^* = gm$ a.e.

Proof. Define, for $x \in R$

$$\varphi(x) = \int_0^\infty g(2t) \sin xt \, dm(t) .$$

Then, by part (ii) of Lemma 2 $0 \leqq \varphi(x) \leqq G_2(\pi/x)$, for $x > 0$, because $0 \leqq \varphi(x) \leqq \int_0^{\pi/x} g(2t) \sin xt \, dm(t) \leqq \int_0^{\pi/x} g(2t) \, dm(t) = G_2(\pi/x)$. Since G_2 is an even function, we have that $|\varphi(x)| \leqq G_2(\pi/x)$ for all $x \in R\backslash\{0\}$. Part (ii) of Lemma 3 says that $G_2(\pi/x) \in L^p(R)$. If follows then that $\varphi \in L^p(R)$. Define, for $n \in N$,

$$\varphi_n(x) = \int_0^n g(2t) \sin xt \, dm(t) \quad (x \in R) .$$

Let $x > 0$. For each n, choose $m \in N$ such that $|2m\pi/x - n| \leqq \pi/x$. Then

$$\begin{aligned} |\varphi_n(x)| &\leqq \int_0^{2m\pi/x} g(2t) \sin xt \, dm(t) + \left| \int_{2m\pi/x}^n g(2t) \sin xt \, dm(t) \right| \\ &\leqq \int_0^\infty g(2t) \sin xt \, dm(t) + g\left(\frac{2(2m-1)\pi}{x}\right) \left| \frac{2m\pi}{x} - n \right| \\ &\leqq \varphi(x) + g\left(\frac{2\pi}{x}\right)\frac{\pi}{x} \leqq \varphi(x) + \int_0^{\pi/x} g(2t) dm(t) \\ &= \varphi(x) + G_2\left(\frac{\pi}{x}\right) . \end{aligned}$$

This shows that $|\varphi_n(x)| \leqq |\varphi(x)| + |G_2(\pi/x)|$ for all $x \in R\backslash\{0\}$. Since $\varphi_n(x) \longrightarrow \varphi(x)$ pointwise and $\varphi(x), G_2(\pi/x) \in L^p(R)$, we must have $\|\varphi_n - \varphi\|_p \longrightarrow 0$ be the dominated convergence theorem. So we can obtain φ by approximating φ_n. Let us compute φ_n:

$$\begin{aligned} 2i\varphi_n(x) &= 2i\int_0^n g(2t) \sin xt \, dm(t) = \int_0^n g(2t) (e^{-ixt} - e^{ixt}) dm(t) \\ &= \int_R g(-2t) I_{[-n,0]}(t) e^{-ixt} dm(t) \\ &\quad - \int_R g(2t) I_{[0,n]}(t) e^{-ixt} dm(t) . \end{aligned}$$

Recall that the Haar measure m on R is chosen so that the inversion theorem holds. We know that $g(2t)I_{[0,n]}(t)$ and $g(-2t)I_{[-n,0]}(t) \in$

$L^1(R)$ and $\varphi_n \in L^p(R)$. Hence, by [3; Vol. II, 31.44 (b)], we have

$$2i\varphi(x) = \begin{cases} -g(2x) & \text{if } x \geqq 0 \\ & m \text{ a.e.} \\ g(-2x) & \text{if } x < 0 \end{cases}$$

Now define $f = 2i\varphi$ so that $|\hat{f}(x)| = g(|2x|)$ m a.e. It is then easy to check that $\hat{f}^* = gm$ a.e., which is what we needed to prove.

LEMMA 5. *For each $n \in N$, there exists $f \in L^p(R^n)$ such that* $\hat{f}^* = gm$ a.e.

Proof. By Lemma 4, we may assume that $n > 1$. Define, for $k \in N$,

$$\varphi(x) = \int_0^\infty g(2^n t) \sin xt \, dm(t)$$

$$\varphi_k(x) = \int_0^k g(2^n t) \sin xt \, dm(t)$$

$$f(x_1, \cdots, x_n) = 2^n i\varphi(x_i)\frac{\sin x_2}{x_2} \cdots \frac{\sin x_n}{x_n}$$

$$f_k(x_1, \cdots, x_n) = 2^n i\varphi_k(x_1)\frac{\sin x_2}{x_2} \cdots \frac{\sin x_n}{x_n}$$

Let $m_n = m \times m \times \cdots \times m$ on R^n, $x = (x_1, \cdots, x_n)$. Then

$$\varphi(x),\ \varphi_k(x),\ \frac{\sin x}{x} \in L^p(R) .$$

Therefore

$$\begin{aligned} &\int_{R^n} |f_k - f|^p dm_n \\ &= 2^{np}\int_{R^n}\left|\varphi_k(x_1) - \varphi(x_1)\right|^p \left|\frac{\sin x_2}{x_2}\right|^p \cdots \left|\frac{\sin x_n}{x_n}\right|^p dm_n \\ &= 2^{np}\int_R \left|\varphi_k - \varphi\right|^p dm \left(\int_R \left|\frac{\sin x}{x}\right|^p dm\right)^{n-1} . \end{aligned}$$

As in the proof of Lemma 4, we have $\|\varphi_k - \varphi\|_p \to 0$, and so $\|f_k - f\|_p \to 0$ in $L^p(R^n)$. Straight forward calculations show that

$$\hat{f}_k(x_1, \cdots, x_n) = \begin{cases} g(-2^n x_1) \text{ if } -k \leqq x_1 < 0 \text{ and } x_j \in [-1, 1] \\ \quad \text{for } 2 \leqq j \leqq n \\ -g(2^n x_1) \text{ if } 0 \leqq x_1 \leqq k \text{ and } x_j \in [-1, 1] \\ \quad \text{for } 2 \leqq j \leqq n \\ 0 \quad \text{otherwise} \end{cases}$$

m_n a.e. and hence

$$\hat{f}(x_1, \cdots, x_n) = \begin{cases} g(-2^n x_1) & \text{if} \quad x_1 < 0, |x_j| \leqq 1, 2 \leqq j \leqq n \\ -g(2^n x_1) & \text{if} \quad x_1 > 0, |x_j| \leqq 1, 2 \leqq j \leqq n \\ 0 & \text{otherwise} \end{cases}$$

m_n a.e. It follows that

$$\mathrm{m}_n\{x \in R^n: |\hat{f}(x)| > t\} = 2^n m\{x_1 > 0: g(2^n x_1) > t\} .$$

This in turn shows that for $x > 0$

$$\hat{f}^*(x) = \inf\{t > 0: 2^n m\{x_1 > 0: g(2^n x_1) > t\} \leqq x\} = g(x)$$

m a.e., which completes the proof of Lemma 5.

III. Proof for the nondiscrete case. Let G be an infinite LCA group. To prove Theorem 1 and Theorem 2, Lemma 1 and the structure theorem [3, Vol. I, 24.30] shows that we may assume $G = K \times R^n$, where K is a compact abelian group.

Proof of Theorem 1. In this $n = 0$, so that $G = K$. Then there exists $f_0 \in L^p(K)$, by [4], such that $\hat{f}_0^*|_N = g|_N$.

Proof of Theorem 2. In this case $n > 0$. By Lemma 5, there exists $f_0 \in L^p(R^n)$ such that $\hat{f}_0^* = gm$ a.e. Define $f(x, y) = f_0(y)$ for $x \in K$ and $y \in R^n$. Let $m_n = m \times \cdots \times m$ be the Haar measure on R^n, λ_K be the normalized Haar measure on K and $\lambda_{K\times R^n}$ the Haar measure on $K \times R^n$ so that Weil's theorem holds. It follows that f is in $L^p(K \times R^n)$ and $||f||_p = ||f_0||_p$. Moreover, for $\chi_1 \in \hat{K}$, $\chi_2 \in R^n$, we have

$$f(\chi_1\chi_2) = \begin{cases} f_0(\chi_2) & \text{if} \quad \chi_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\theta_{\hat{K}\times R}n$, $\theta_{\hat{K}}$ and θ_{R^n} the Haar measures on $\hat{K} \times R^n$, $\hat{K}$ and R^n respectively, so that Planchevel's theorem holds. Then Weil's theorem holds for these measures by [3, 31.46(c)]. Clearly $\theta_{\hat{K}}$ is the discrete measure on $\hat{K}$. Then for $t > 0$

$$\begin{aligned} (\theta_{\hat{K}\times R^n})_{\hat{f}}(t) &= \int_{\hat{K}\times R^n} I_{\{\chi: |\hat{f}(\chi)| > t\}} d\theta_{\hat{K}\times R^n} \\ &= \int_{R^n}\int_{\hat{K}} I_{\{\chi: |\hat{f}(\chi)| > t\}} d\theta_{\hat{K}} d\theta_{R^n} \\ &= \int_{R^n} I_{\{x: |\hat{f}_0(x)| > t\}} d\theta_{R^n} = (\theta_{R^n})_{\hat{f}_0}(t) , \end{aligned}$$

and it follows that for $x > 0$,

$$\begin{aligned}\hat{f}^*(x) &= \inf\{t > 0: (\theta_{\hat{K}\times R^n})_{\hat{f}}(t) \leqq x\} = \inf\{t > 0: (\theta_{R^n})_{\hat{f}_0}(t) \leqq x\} \\ &= \hat{f}_0^*(x) = g(x)m \text{ a.e.}\end{aligned}$$

Note that Theorem 1 is essentially the theorem in [4].

IV. *Proof of Theorem* 3. For each $n = 1, 2, \cdots$, let r_n be an integer $\geqq 2$. Denote by θ the normalized Haar measure on $X = \prod_{n=1}^{\infty} Z(r_n)$ and λ the usual restriction of Lebsque measure to $[0, 1]$. Define a function $\varphi: X \to [0, 1]$ via

$$\varphi(\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{p_1 p_2 \cdots p_n} \qquad \varepsilon = (\varepsilon_1, \cdots, \cdots) \in X .$$

Then g is measure preserving; in fact, the following is well known.

LEMMA 6. *E is measurable in X if and only if $\varphi(E)$ is measurable in $[0, 1]$, and $\theta(E) = \lambda(\varphi(E))$. φ is an onto map and φ is one-to-one on X except for a countable set. Moreover,*

$$\int_x h \circ \varphi d\theta = \int_0^1 h d\lambda$$

for all bounded λ measurable functions h on $[0, 1]$.

LEMMA 7. *Theorem 3 is true if $G \supset Z$.*

Proof. By Lemma 1, we may assume $G = Z$. Define

$$a_0(n) = \frac{1}{2\pi}\int_0^{2\pi} g(t)\sin nt\, dt \quad \text{for } n \in Z .$$

The values of the integrals involved are finite, by (i) of Lemma 2. Also $a_0 \in l^p(Z)$ because

$$\begin{aligned}(2\pi)^p \sum_{n\in Z} |a_0(n)|^p &= \sum_{n\in Z}\left|\int_0^{2\pi} g(t)\sin nt\, dt\right|^p \leqq \sum_{\substack{n\in Z\\ n\neq 0}}\left|\int_0^{\pi/n} g(t)dt\right|^p \\ &= \sum_{\substack{n\in Z\\ n\neq 0}} G_1\Big(\frac{\pi}{n}\Big)^p \leqq \int_R G_1^p\Big(\frac{\pi}{x}\Big)dx = \pi\int_R G_1^p(y)y^{-2}dy .\end{aligned}$$

The last integral is finite by (ii) of Lemma 3. Similarly, if we define

$$b_0(n) = \frac{1}{2\pi}\int_0^{2\pi} g(t)\cos nt\, dt \quad \text{for } n \in Z$$

then $b_0 \in l^p(Z)$. So if we set $c(n) = b_0(n) - ia_0(n) = 1/2\pi\int_0^{2\pi} g(t)e^{-int}dt$ for $n \in Z$, then $c \in l^p(Z)$ and $\hat{c}(t) = g(t)$ a.e. [3, 31.44, (b)]. Since g is nonincreasing in $[0, 2\pi]$, we then have $\hat{c}^* = g\theta$ a.e.

LEMMA 8. *Theorem* 3 *is true is* $G \supset \Pi^* Z(r)$, *where* $r \in N$, $r \geqq 2$.

Proof. We may assume that $G = \Pi^* Z(r)$, by Lemma 1.

Let $X = Z(r)^{\aleph_0}$, the character group of G. Define $\varphi(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon_n / r_n$ for $\varepsilon = (\varepsilon_n) \in X$, and note that Lemma 6 applies to φ. For a real number t, denote $[t]$ by the greatest integer which is not greater than t. For $m \in N$, define

$$\chi_m(t) = e^{i2\pi[r^m t]/r}$$

for $t \in [0, 1]$. Then $\chi_m \circ \varphi(\varepsilon) = e^{i(2\pi/r)\varepsilon_m}$ where $\varepsilon \in X$ and ε_m is the mth component of ε. It follows that G is isomorphic to the group of finite products of elements in $\{\chi_m \circ \varphi\}_{m=1}^{\infty}$. In this proof we write $I_{m,\nu}$ for the characteristic function of the interval $[v/r^m, (v+1)/r^m]$

$$\chi_m(t) = \sum_{u=1}^{r^{m-1}} \left(\sum_{j=0}^{r-1} w^j I_{m,(u-1)r+j}(t) \right)$$

for θ a.e. t, where $w = e^{i(2\pi/r)}$. And hence

$$\chi_{m_1}^{l_1}(t) \cdots \chi_{m_k}^{l_k}(t) = \sum_{u=1}^{r^{m_1-1}} a_u \left(\sum_{j=0}^{r-1} w^{l_1 j} Im_1, (u-1)r + j(t) \right)$$

where $a_u^r = 1$ for all $u = 1, \cdots, r^{m_1-1}$; $m_1 > m_2 > \cdots > m_k$ and $0 \leqq l_1, l_2, \cdots, l_k \leqq r-1$, $l_1 > 0$.

Define a function f on G via

$$f(\chi_{m_1}^{l_1} \circ \varphi, \cdots, \chi_{m_k}^{l_k} \circ \varphi) = \int_X g \circ \varphi(\varepsilon) \chi_{m_1}^{l_1} \circ \varphi(\varepsilon) \cdots \chi_{m_k}^{l_k} \circ \varphi(\varepsilon) d\varepsilon .$$

Define, for $u = 1, 2, \cdots, r^{m_1-1}$ and $j = 0, \cdots, r-1$,

$$k_{(u-1)r+j} = \int I_{m_1,(n-1)r+j}(t) g(t) dt, \; b_{(u-1)r+j} = a_u w^{jl_1} .$$

Then $\{k_0, k_1, \cdots, k_{r^{m_1}-1}\}$ is a positive nonincreasing sequence, and

$$\left| \sum_{l=0}^{s} b_l \right| \leqq r \quad \text{for all} \quad s = 0, 1, 2, \cdots, r^{m_1} - 1$$

In fact,

$$\sum_{j=0}^{r-1} b_{(u-1)r+j} = \sum_{j=0}^{r-1} a_u w^{jl_1} = a_u \sum_{j=0}^{r-1} w^{jl_1} = 0 .$$

It follows that

$$|f(\chi_{m_1}^{l_1} \circ \varphi, \cdots, \chi_{m_k}^{l_k} \circ \varphi)| = \left| \int_0^1 g(t) \chi_{m_1}^{l_1}(t), \cdots, \chi_{m_k}^{l_k}(t) dt \right|$$

$$= \sum_{u=1}^{r^{m_1-1}} \left(\sum_{j=0}^{r-1} a_u w^{jl_1} \int I_{m_1,(u-1)r+j}(t)\, g(t)dt \right) = \left| \sum_{l=0}^{r^{m_1-1}} b_l k_l \right|$$
$$\leqq k_0 \max_{0 \leqq s \leqq r^{m_1-1}} \left| \sum_{l=0}^{s} b_l \right| \leqq k_0 r = r \int_0^{1/r^{m_1}} g(t)dt = rG\left(\frac{1}{r^{m_1}}\right) .$$

Writing Σ' for a sum over all $(m_1, \cdots, m_k, l_1, \cdots, l_k)$ satisfying $k \in N$, $m_1 > m_2 > \cdots > m_k \geqq 0$, $0 < l_1 \leqq r-1$, $0 \leqq l_j \leqq r-1$ for $j = 2, \cdots, k$, we obtain

$$\| f \|_p^p = \Sigma' | f(\chi_{m_1}^{l_1}\varphi, \cdots, \chi_{m_k}^{l_k}\varphi) |^p \leqq \Sigma' r^p G^p\left(\frac{1}{r^{m_1}}\right)$$
$$\leqq \sum_{m_1=0}^{\infty} r^{m_1} r^p G^p\left(\frac{1}{r^{m_1}}\right) = r^{p+1} \sum_{m_1=0}^{\infty} r^{m_1-1} G^p\left(\frac{1}{r^{m_1}}\right)$$
$$\leqq r^{p+1} \sum_{m_1=0}^{\infty} (r^{m_1} - r^{m_1-1}) G^p\left(\frac{1}{r^{m_1}}\right) \leqq r^{p+1} \int_0^{\infty} G^p\left(\frac{1}{x}\right) dx < \infty .$$

So $f \in L^p(G)$ and hence $\hat{f} = g \circ \varphi$. It follows that $\hat{f}^* = gI_{[0,1]}$ m a.e.

LEMMA 9. *Theorem 3 is true if G contains $Z(r^\infty)$, $(r \geqq 2)$.*

Proof. We may assume that $G = Z(r^\infty)$ by Lemma 1. Let Δ_r be the group of r-adic integers; then $Z(r^\infty)$ is a discrete group with $Z(r^\infty)^\wedge = \Delta_r$. Define

$$\varphi(\varepsilon) = \sum_{n-1}^{\infty} \frac{\varepsilon_n}{r^n} \varepsilon = (\varepsilon_n) \in \Delta_r .$$

As in Lemma 6, φ is a measure preserving map from Δ_r onto [0, 1], and

$$\int_{\Delta_r} h \circ \varphi d^\theta = \int_0^1 h dt \tag{2}$$

for all bounded measurable functions h on [0, 1], where θ is the normalized Haar measures on Δ_r. We write $I_{m,s_1,\dots,s_m}$ for the characteristic function of the interval

$$\left[\frac{r^{m-1}s_1 + r^{m-2}s_2 + \cdots + s_m}{r^m}, \frac{r^{m-1}s_1 + r^{m-2}s_2 + \cdots + s_m + 1}{r^m} \right] .$$

For $m \in N$, define

$$\chi_m(t) = \sum_{s_1 \cdots s_m = 0}^{r-1} w_m^{s_1 + rs_2 + \cdots + r^{m-1}s_m} I_{m,s_1,\cdots,s_m}(t) \tag{3}$$

where $w_m = e^{i(2\pi/r^m)}$. Then $\chi_m \circ \varphi(\varepsilon) = w_m^{\varepsilon_1 + r\varepsilon_2 + \cdots + r^{m-1}\varepsilon_m}$ θ a.e. where $(\varepsilon) \in \Delta_r$ and $\varepsilon_1, \cdots, \varepsilon_m$ are the first m coordinates of (ε). It follows that G is isomorphic to the group generated by $\{\chi_m \circ \varphi\}_{m=1}^\infty$. Define for

$m, l \in N$ and $(l, r) = 1$

$$f(\chi_m^l) = \int_{\Delta_r} g \circ \varphi(\varepsilon)\chi_m^l \circ \varphi(\varepsilon) d\theta$$

Then f is a function on G, and by (2) and (3),

$$\begin{aligned} f(\chi_m^l) &= \int_0^1 g(t)\chi_m^l(t)dt \\ &= \sum_{s_1,\cdots,s_m=0}^{r-1} (w_m^l)^{s_1+rs_2+\cdots+r^{m-1}s_m} \int I_{m,s_1,\cdots,s_m}(t)g(t)dt \,. \end{aligned}$$

Let $k_{r^{m-1}s_1+\cdots+s_m} = \int I_{m,s_1,\cdots,s_m}(t)g(t)dt$. Then $\{k_0, k_1, \cdots, k_{r^m-1}\}$ is a positive, nonincreasing sequence. Let $b_{r^{m-1}s_1+\cdots+s_m} = (w_w^l)^{s_1+rs_2+\cdots+r^{m-1}s_m}$. For any $0 \leqq s \leqq r^m - 1$, we write $s = r^{m-1}s_1 + \cdots + s_m$ with $0 \leqq s_1, \cdots, s_m < r$. Then

$$\begin{aligned} \sum_{n=0}^{s} b_n &= \sum_{n=1}^{r^{m-1}s_1+\cdots+s_m} b_n \\ &= \left(\sum_{u=1}^{r^{m-2}s_1+\cdots+s_{m-1}} \sum_{h=0}^{r-1} b_{(u-1)r+h} \right) + \left(\sum_{j=0}^{s_m} b_{r^{m-1}s_1+\cdots+rs_{m-1}+j} \right) \end{aligned}$$

For each $u = 1, \cdots, r^{m-2}s_1 + \cdots + s_{m-1}$. Choose $0 \leqq u_1, \cdots, u_{m-1} < r$ such that $(u-1)r = r^{m-1}u_1 + \cdots + ru_{m-1}$, and hence

$$\begin{aligned} \sum_{h=0}^{r-1} b_{(u-1)r+h} &= \sum_{h=0}^{r-1} b_{r^{m-1}u_1+\cdots+ru_{m-1}+h} \\ &= \sum_{h=0}^{r-1} (w_m^l)^{u_1+ru_2+\cdots+r^{m-2}u_{m-1}+r^{m-1}h} \\ &= (w_m^l)^{u_1+ru_2+\cdots+r^{m-2}u_{m-1}} \sum_{h=0}^{r-1} (w_m^l)r^{m-1}h \\ &= (w_m^l)^{u_1+ru_2+\cdots+r^{m-2}u_{m-1}} \sum_{h=0}^{r-1} (e^{i(2\pi l)/r})^h = 0 \,. \end{aligned}$$

The last equality holds because $(l, r) = 1$. This shows that

$$\left| \sum_{n=0}^{s} b_n \right| = \left| \sum_{j=0}^{s_m} b_{r^{m-1}s_1+\cdots+rs_{m-1}+j} \right| \leqq s_m + 1 \leqq r$$

and hence

$$\begin{aligned} |f(\chi_m^l)| &= \left| \sum_{n=0}^{r^m-1} b_n k_n \right| \leqq k_0 \max_{0 \leqq s \leqq r^m-1} \left| \sum_{n=0}^{s} b_n \right| \leqq rk_0 \\ &= r\int_0^{1/r^m} g(t)dt = rG_1\left(\frac{1}{r^m}\right) \end{aligned}$$

for all $m, l \in N$ and $(l, r) = 1$. Denote by Σ' the sum over $(m, l) \in N$, $(l, r) = 1$ and $0 \leqq l < r^m$. Then we have

$$||f||_p^p = \Sigma' |f(\chi_m^l)|^p \leqq \Sigma' r^p G_1^p\left(\frac{1}{r^m}\right) \leqq \sum_{m=0}^{\infty} r^m r^p G_1^p\left(\frac{1}{r^m}\right).$$

As in Lemma 8, we conclude that $f \in L^p(G)$ and $\hat{f}^* = gI_{[0,1]}m$ a.e.

Patching Lemmas 7, 8 and 9 together gives the proof of Theorem 3.

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The remaining open question is whether Theorem 3 holds if $G = \prod_{n=1}^{\infty *} Z(r_n)$ where $r_n \in N$, $r_n \geqq 2$ for all n and $\lim_{n\to\infty} r_n = \infty$.

References

1. G. H. Hardy and J. E. Littlewood, *some new properties of Fourier constants*, Math. Annalen, **97** (1926), 159–209.

2. ———, *some new propeties of Fourier constants*, J. London Math. Soc., **6** (1931), 3–9.

3. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, 2 vols. (Springer-Verlag, Inc. Berlin-Heidelberg-New York 1963, 1970).

4. ———, *Rearrangement of L^r Fourier Series on compact abeliangroups*, Proc. London Math. Soc., (3) **29** (1974), 317–330.

5. R. A. Hunt, *On $L(p, g)$ spaces, L'* Enseignement Math., **12** (1966), 249–275.

6. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces.* (Princeton, New Jersey, Princeton University Press, 1971).

7. A. Zygmund, *Trigonometric Series, 2nd edition,* 2 vols. (Cambridge Press 1959, reprinted 1968).

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