PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

PROJECTIVE QUASI-COHERENT SHEAVES OF MODULES

RICHARD A. LEVARO

Let R be a commutative ring and $\tilde{R}$ the structure sheaf over the prime spectrum of R.

THEOREM: Suppose R has only finitely many minimal primes. Then $\tilde{R}$ is a projective $\tilde{R}$-Module if and only if R is a finite direct product of local rings.

Let R be a nonzero commutative ring with identity, and let $x = \text{Spec}(R)$, the prime spectrum of R endowed with the Zariski topology. Let $\tilde{R}$ be the structure sheaf of R on X. We shall use the terminology and notation of [5] in describing the category of $\tilde{R}$-Modules, $\text{Mod}(\tilde{R})$.

There is a functor $T: \text{mod}(R) \to \text{Mod}(\tilde{R})$ given $T(M) = \tilde{M}$ and $T(f) = \tilde{f}$, where $\tilde{M}$ is the $\tilde{R}$-Module associated to M, and $\tilde{f}$ is defined at each stalk of $\tilde{M}$ to be the localization of f. The functor T is full, faithful and exact; moreover T preserves direct sums [5, Corollaire I.1.3.8 and I.1.3.9.]. In addition, T determines an equivalence between $\text{mod}(R)$ and the category of quasi-coherent $\tilde{R}$-Modules. In § 1, we shall show that if $\tilde{R}$ is a generator, then $\text{Mod}(\tilde{R})$ is equivalent to $\text{mod}(R)$. In § 2 necessary and sufficient conditions are given for $\tilde{R}$ to be a projective $\tilde{R}$-Module.

The author wishes to thank J. J. Rotman and the Referee for their suggestions regarding the preparation of this paper.

1. The equivalence of $\text{Mod}(\tilde{R})$ and $\text{mod}(R)$.

C. J. Mulvey [8] has given a necessary and sufficient condition for $\tilde{R}$ to be a generator in $\text{Mod}(\tilde{R})$. For the case of the affine scheme $(X = \text{Spec}(R), \tilde{R})$, we can state Mulvey's condition as follows:

PROPOSITION 1.1 (Mulvey, [8]). *A necessary and sufficient condition that $\tilde{R}$ be a generator in* $\text{Mod}(\tilde{R})$ *is that the stalks of $\tilde{R}$ may be generated by global sections of $\tilde{R}$ of arbitrarily small support. If this condition holds, then $X = \text{Spec}(R)$ is necessarily a regular topological space.*

THEOREM 1.2. *The following are equivalent:*

(i) $T: \text{mod}(R) \to \text{Mod}(\tilde{R})$ *is an equivalence of categories,* i.e., *every $\tilde{R}$-Module is quasi-coherent;*

(ii) *$\tilde{R}$ is a generator for the category* $\text{Mod}(\tilde{R})$;

(iii) $X = \text{Spec}(R)$ *is* T_1;

(iv) *$R/N(R)$ is von Neumann regular, where $N(R)$ is the nil-*

radical of R. If $\tilde{R}$ is a flabby (flasque) $\tilde{R}$-Module, then the equivalent conditions (i)–(iv) *are satisfied.*

Proof. (i) implies (ii). Since R is a generator of mod(R), this implication is clear.

(ii) implies (i). Since $\tilde{R}$ is a generator, it is immediate that every $\tilde{R}$-Module is of the form $\tilde{M}$.

(ii) implies (iii). Because $\tilde{R}$ is a generator, by Proposition 1.1, $X = \operatorname{Spec}(R)$ is a regular topological space. But X is always T_0, so it is also T_1.

(iii) implies (iv). This is well-known and appears as a exercise in [2, page 143].

(iv) implies (ii). Since $R/N(R)$ is von Neumann regular and $X = \operatorname{Spec}(R)$ is homeomorphic to $\operatorname{Spec}(R/N(R))$, X has a basis of closed and open sets. We shall use the criterion of Proposition 1.1 to show $\tilde{R}$ is a generator. Let $x \in X$, and let U be an open set in X with $x \in U$. Let V be an open and closed (basic) set such that $x \in V \subseteqq U$. Define sections $s_1 \in \tilde{R}(V)$ and $s_0 \in \tilde{R}(X - V)$ by $s_1(z) = 1_z \in R_{p_z}$ for all $z \in V$, and $s_0(z) = 0_z \in R_{p_z}$ for all $z \in X - V$. Since V partitions X, we can collate s_1 and s_0 to obtain a global section s of $\tilde{R}$ with $s(z) = 1_z$ if $z \in V$ and $s(z) = 0_z$ if $z \notin V$. Clearly s generates $\tilde{R}_x$, and the support of s is $V \subseteqq U$. Therefore, by the Proposition, $\tilde{R}$ is a generator.

For the last statement, suppose $\tilde{R}$ is flabby and $s \in R$. Then the restriction map $\tilde{R}(X) \rightarrow \tilde{R}(D(s))$ is onto, and hence the localization map $R \rightarrow R_s$ is onto. Now $D(s) \approx \operatorname{Spec}(R_s)$, and because $R \rightarrow R_s$ is onto, $\operatorname{Spec}(R_s)$ is a homeomorphic to a closed set of X. Hence the usual basis is both open and closed; therefore points in X are closed and X is T_1.

R. Wiegand has shown, using different techniques, that a reduced prescheme $(X, \mathscr{R})$ is regular (i.e., X can be covered by open sets U_i such that $(U_i, \mathscr{R} \mid U_i)$ is the affine scheme of a von Nuemann regular ring) if and only if every $\tilde{R}$-Module is quasi-coherent [9].

The Theorem provides examples of rings for which there are projectives in Mod$(\tilde{R})$.

COROLLARY 1.3. *Suppose $R/N(R)$ is von Neumann regular where $N(R)$ is the nilradical of R. The $\tilde{R}$-Module $\boldsymbol{F}$ is projective if and only if $\boldsymbol{F}(X)$ is a projective R-module. In particular, P is a projective R-module if and only if $\tilde{P}$ is a projective $\tilde{R}$-Module.*

2. **Projective quasi-coherent $\tilde{R}$-Modules.** Suppose $\tilde{R}$ is a projective $\tilde{R}$-Module. If P is a projective R-module, then there is an R-module Q such that $P \oplus Q \cong \sum R$; hence $\tilde{P} \oplus \tilde{Q} \cong \sum \tilde{R}$ since T

preserves direct sums. Therefore, $\tilde{P}$ is a projective $\tilde{R}$-Module. Thus, to discover when projective R-module yield projective $\tilde{R}$-Modules, it is enough to determine when $\tilde{R}$ is projective.

PROPOSITION 2.1. *If R is a local* (not necessarily Noetherian) *ring, then $\tilde{R}$ is a projective $\tilde{R}$-Module.*

Proof. Since $\operatorname{Hom}_{\tilde{R}}(\tilde{R}, \boldsymbol{F})$ is naturally isomorphic to $\boldsymbol{F}(X)$ for every $\tilde{R}$-Module $\boldsymbol{F}$, we need only show the global section functor is exact. Let p_x be the unique maximal ideal of R. For any $\tilde{R}$-Module $\boldsymbol{F}$, $\boldsymbol{F}_x = \varinjlim \boldsymbol{F}(U)$ where the direct limit is taken over all open sets contaiinng x. Because $X = \operatorname{Spec}(R)$ is the only open set containing x, $\boldsymbol{F}_x = \boldsymbol{F}(X)$. Now, the formation of stalks is exact, so $\operatorname{Hom}_{\tilde{R}}(\tilde{R},\ )$ is exact, i.e., $\tilde{R}$ is projective.

R. Bkouche [1] introduced the notion of soft rings.

DEFINITION. The ring R is *soft* (mou) if $\operatorname{Max}(R)$, the maximal spectrum of R, is Hausdorff and $J(R) = 0$, where $J(R)$ is the Jacobson radical of R.

For our purposes, we need a notion a bit more general.

DEFINITION. The ring R is *quasi-soft* if for every $x \in \operatorname{Max}(R)$, the localization map $\alpha_x: R \longrightarrow R_{p_x}$ is onto.

Every local ring is quasi-soft, but not necessarily soft. Every von Neumann regular ring is quasi-soft. The relation between soft and quasi-soft rings is given by the following.

PROPOSITION 2.2. *If R is quasi-soft, then $R/J(R)$ is soft, where $J(R)$ is the Jacobson radical of R. Every soft ring is quasi-soft.*

Proof. If R is quasi-soft, then $\operatorname{Max}(R)$ is regular as can be seen by imitating the proof for soft rings [1, Proposition 1.6.1 and 1.6.2]. But $\operatorname{Max}(R)$ is always T_1; hence $\operatorname{Max}(R)$ is Hausdorff. Since $\operatorname{Max}(R) \approx \operatorname{Max}(R/J(R))$ and $J(R/J(R)) = 0$, $R/J(R)$ is soft.

Now suppose R is soft, $x \in \operatorname{Max}(R)$, and let $\alpha_x: R \longrightarrow R_{p_x}$ be the localization map. Because $J(R) = 0$ and $\operatorname{Max}(R)$ is Hausdorff, $V_M(\ker(\alpha_x)) = \{x\}$, where $V_M(I) = \operatorname{Max}(R) \cap V(I)$ for an ideal I of R. Therefore, $R/\ker(\alpha_x)$ is a local ring with maximal ideal p_x, and so every element outside p_x is invertible. By the universal mapping property of localization, $R/\ker(\alpha_x) \cong R_{p_x}$; hence R is quasi-soft.

Quasi-softness is the condition we must investigate to find necessary conditions for $\tilde{R}$ to be a projective $\tilde{R}$-Module in view of the following result.

PROPOSITION 2.3. *If $\tilde{R}$ is a projective $\tilde{R}$-Module, then R is quasi-soft.*

Proof. Let $x \in \operatorname{Max}(R)$ and set $A = \{x\}$. Then $A \subseteqq X$ is closed, and we have the exact sequence

$$0 \longrightarrow \tilde{R}_{X-A} \longrightarrow \tilde{R} \xrightarrow{\alpha} \tilde{R}_A \longrightarrow 0$$

of $\tilde{R}$-Modules [4, Théorème 2.9.3.]. Since $\tilde{R}$ is projective, $\operatorname{Hom}_{\tilde{R}}(\tilde{R}, \ )$ is exact, and hence $\operatorname{Hom}_{\tilde{R}}(\tilde{R}, \tilde{R}) \xrightarrow{\alpha_*} \operatorname{Hom}_{\tilde{R}}(\tilde{R}, \tilde{R}_A)$ is onto. Now $\operatorname{Hom}_{\tilde{R}}(\tilde{R}, \tilde{R}) \cong R$ and $\operatorname{Hom}_{\tilde{R}}(\tilde{R}, \tilde{R}_A) \cong R_{p_x}$, and it is routine to check that α_* may be identified with the localization map $\alpha_x: R \rightarrow R_{p_x}$ (i.e., the obvious diagram commutes). Therefore R is quasi-soft.

We can now state and prove the

MAIN THEOREM. *Suppose R has only finitely many minimal primes. Then $\tilde{R}$ is a projective $\tilde{R}$-Module if and only if R is finite direct product of local rings.*

Proof. Since R has only finitely many minimal primes, R is the finite direct product of connected rings, say $R = R_1 \times R_2 \times \cdots \times R_n$ each having only finitely many minimal primes. If $\tilde{R}$ is a projective $\tilde{R}$-Module, $\tilde{R}_i$ is a projective $\tilde{R}_i$-module for each i. By Proposition 2.3 R_i is quasi-soft. Hence $\operatorname{Max}(R_i)$ is finite, since each prime ideal of a quasi-soft, ring is contained in a unique maximal ideal [1, Proposition 1.6.1]. Also, since R_i is quasi-soft, $\operatorname{Max}(R_i)$ is the continuous image of $\operatorname{Spec}(R_i)$ [1, Proposition 1.6.2]. (See also [3]). Thus, $\operatorname{Max}(R_i)$ is finite and discrete, but also connected being the continuous image of $\operatorname{Spec}(R_i)$. Therefore $\operatorname{Max}(R_i)$ consists of a single point, and hence R_i is local.

Conversely, if $R = R_1 \times \cdots \times R_n$ where each R_i is local, then $\tilde{R}_i$ is a projective $\tilde{R}_i$-Module by Proposition 2.1. Hence, $\tilde{R}$ is a projective $\tilde{R}$-Module.

The Main Theorem resolves the problem of determining the projectivity of $\tilde{R}$ for rings with only finitely many minimal primes; in particular, for Noetherian rings and integral domains.

Let R be a discrete valuation domain. In this case; $X = \operatorname{Spec}(R) = \{(0), p\}$, where p is the unique maximal ideal of R. Since R is local,

$\tilde{R}$ is a projective $\tilde{R}$-Module. Since $U = \{(0)\}$ is smallest open set containing (0), $\tilde{R}_U$ is also a projective $\tilde{R}$-Module. Thus, there are examples of projective $\tilde{R}$-Modules which are not quasi-coherent. Furthermore, since $\tilde{R} \oplus \tilde{R}_U$ is a generator for Mod$(\tilde{R})$ [6, Proposition 3.1.1], in this case Mod$(\tilde{R})$ has a small projective generator. Hence Mod$(\tilde{R})$ is equivalent to a category of modules [7, Theorem 4.1, page 104], but the functor T is not the equivalence since $X =$ Spec(R) is not T_1.

REFERENCES

1. R. Bkouche, *Couples spectraux et faisceaux associés. Applications aux anneaux fonctions*, Bull. Soc. Math. France, **98** (1970), 253-295.

2. N. Bourbaki, *Elements of Mathematics, Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1972.

3. G. De Marco and A. Orsatti, *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc., **30** (1971), 459-466.

4. R. Godement, *Topological Albèbrique et Théorie des Faisceaux*, Hermann, Paris, 1958.

5. A. Grothendieck and J. A. Dieudonné, *Eléments de Géométrie Algébrique* I, Springer-*Verlag*, Berlin, 1972.

6. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J., **9** (1957), 119-221.

7. B. Mitchell, *Theory of Categories*, Academic Press, New York and London, 1965.

8. C. J. Mulvey, *A condition for a ringed space to be a generator in its category of modules*, J. Algebra, **15** (1970), 312-313.

9. R. Wiegand, *Generators of modules over commutative rings*, J. Algebra, **27** (1973), 454-461.

Received November 11, 1974 and in revised form March 10, 1975.

WASHINGTON UNIVERSITY