PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

BEHAVIOR OF Φ-BOUNDED HARMONIC FUNCTIONS AT THE WIENER BOUDARY

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For a strongly convex $\Phi(t)$, denote by $H\Phi$ the class of Φ-bounded harmonic functions, and by $C\Phi$ the class of continuous functions f on the Wiener harmonic boundary such that the composite $\Phi(|f|)$ is integrable with respect to a harmonic measure. Theorem: $u \in H\Phi$ if and only if u is a solution of the Dirichlet problem with boundary values $f \in C\Phi$ on the Wiener harmonic boundary.

1. For a strongly convex $\Phi(t)$ Naim [4] developed an integral representation of Φ-bounded harmonic functions in terms of the Martin minimal boundary and fine topology, and for $\Phi(t) = t^p$ $(p > 1)$ Schiff [7] extended the results in the framework of the Wiener compactification. In view of the fact that the Wiener compactification is "smaller" than the Martin, the latter sharpens the former.

The purpose of the present paper is to show that Naim's theory of Φ-bounded harmonic functions does carry over in its full generality to the Wiener compactification setting.

2. The set-up is, as in Naim [4], a locally compact, noncompact, connected and locally connected Hausdorff space Ω with a Brelot harmonic sheaf H such that 1 is superharmonic (cf. Brelot [1]). An increasing nonnegative function $\Phi(t)$ on $[0, \infty)$ is said to be strongly convex if $\Phi(t)$ is convex and $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$. Following Parreau [5], a harmonic function u is said to be Φ-bounded if the function $\Phi(|u|)$ has a harmonic majorant, and u is said to be quasibounded if $u = u_1 - u_2$ where u_1, u_2 are limits of nondecreasing sequences of nonnegative bounded harmonic functions.

Throughout this paper we base our arguments on the Wiener compactification (cf. Sario and Nakai [6], and Loeb and Walsh [3]). Let Δ be the Wiener harmonic boundary, $P(x, t)$ the harmonic kernel, and μ the harmonic measure (centered at $x_0 \in \Omega$, say). It is now classic that u is quasibounded if and only if $u(x) = \int_\Delta P(x, t)f(t)d\mu(t)$ for some μ-integrable function f on Δ. In this case u has a continuous extension to Δ and $u = f$ μ-a.e.

3. Denote by $H\Phi(\Omega)$ the class of Φ-bounded harmonic functions on Ω, and by $C\Phi(\Delta)$ the class of extended real-valued continuous functions f on Δ such that $\Phi(|f|)$ is μ-integrable.

THEOREM 1. *A harmonic function u belongs to the class $H\Phi(\Omega)$ if and only if*

$$u(x) = \int_{\Delta} P(x, t) f(t) d\mu(t) \tag{1}$$

for some $f \in C\Phi(\Delta)$. In this case $u = f$ on Δ.

Proof. First assume $f \in C\Phi(\Delta)$. Since $|f| \leqq \Phi(|f|) + M$ for some constant M, f is μ-integrable and therefore the function (1) is a quasibounded harmonic function. By Jensen's inequality

$$\begin{aligned}\Phi(|u(x)|) &\leqq \Phi\Big(\int_{\Delta} P(x, t)\,|f(t)|\,d\mu(t)\Big)\\ &\leqq \int_{\Delta} P(x, t)\Phi(|f(t)|)d\mu(t)\end{aligned}$$

and the last function is a harmonic function since $\Phi(|f|)$ is μ-integrable.

Conversely let $u \in H\Phi(\Omega)$. In view of the fact that every Φ-bounded harmonic function is quasibounded for a strongly convex Φ (Parreau [5]), it suffices to show that the function $\Phi(|u|)$ is μ-integrable on Δ. Choose a positive harmonic function h on Ω such that $h \geqq \Phi(|u|)$. Set

$$u_n(x) = \int_{\Delta} P(x, t)[(u(t) \cap n) \cup (-n)]d\mu(t)$$

for $n \geqq 1$. Here $\cup$ and $\cap$ stand for pointwise maximum and minimum operations on functions. It is easy to see that each u_n is bounded and harmonic, and that the subharmonic $\Phi(|u_n|)$ monotonically increase to $\Phi(|u|)$ as $n \to \infty$.

Take an exhaustion $\{\Omega_i\}_{i \in I}$ of Ω by regular inner regions (Loeb [2]), and denote by v_i the harmonic function on Ω_i with the boundary value $\Phi(|u_n|) \mid \partial\Omega_i$. Clearly $v_i \leqq h$ for all $i \in I$, and taking the supremum over $i \in I$ we conclude that

$$\int_{\Delta} P(x, t)\Phi(|u_n(t)|)d\mu(t) \leqq h(x)$$

on Ω. In particular

$$\int_{\Delta} \Phi(|u_n(t)|)d\mu(t) = \int_{\Delta} P(x_0, t)\Phi(|u_n(t)|)d\mu(t) \leqq h(x_0) < \infty$$

and by monotone convergence theorem the function $\Phi(|u|)$ is μ-integrable on Δ, as desired.

4. A function $u \in H\Phi(\Omega)$ is said to be $H\Phi$-minimal if $u \geqq 0$ and every $v \in H\Phi(\Omega)$ with $0 \leqq v \leqq u$ is proportional to u.

THEOREM 2. *u is $H\Phi$-minimal if and only if u is HB-minimal.*

Proof. Clearly every HB-minimal function is $H\Phi$-minimal. Conversely assume that u is $H\Phi$-minimal. It suffices to show that u is bounded. In fact the bounded harmonic function

$$\int_{\Delta} P(x, t)(u(t) \cap n)d\mu(t)$$

is in the class $H\Phi(\Omega)$ and has u as a harmonic majorant. Therefore u is bounded as a constant multiple of a bounded function.

COROLLARY. *If the space $HB(\Omega)$ of bounded harmonic functions is of finite dimension, then $H\Phi(\Omega) = HB(\Omega)$.*

5. In general the class $H\Phi(\Omega)$ is not a linear space (it is linear if, for instance, $\Phi(t)\cdot[\Phi(1/2t)]^{-1}$ is bounded for all large t). But it always forms a lattice with respect to the lattice operations $\vee$, $\wedge$ induced from the usual function ordering in the class of all harmonic functions.

THEOREM 3. *Let u, $v \in H\Phi(\Omega)$. Then the least harmonic majorant $u \vee v$ and the greatest harmonic minorant $u \wedge v$ belong to the class $H\Phi(\Omega)$. Moreover*

$$(u \vee v)(x) = \int_{\Delta} P(x, t)(u(t) \cup v(t))d\mu(t) ,$$

$$(u \wedge v)(x) = \int_{\Delta} P(x, t)(u(t) \cap v(t))d\mu(t)$$

on Ω.

REFERENCES

1. M. Brelot, *Lectures on potential theory,* Tata Inst. Fund. Res., Bombay, 1960.
2. P. Loeb, *An axiomatic treatment of pairs of elliptic differential equations,* Ann. Inst. Fourier, **16** (1966), 167-208.
3. P. Loeb and B. Walsh, *An axiomatic approach to the boundary theories of Wiener and Royden,* Bull. Amer. Math. Soc., **74** (1968), 1004-1007.
4. L. Lumer-Naim, *H^p spaces of harmonic functions,* Ann. Inst. Fourier, **17** (1967), 425-469.
5. M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann,* Ann. Inst. Fourier, **3** (1951), 103-197.
6. L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces,* Springer, 1970.
7. J. Schiff, *H^p-spaces of harmonic functions and the Wiener compactification,* Math. Z., **132** (1973), 135-140.

Received February 13, 1975 and in revised form March 18, 1975.

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