PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

GENERALISATION OF A "SQUARE" FUNCTIONAL EQUATION

A. K. GUPTA

Recently the difference equation defining the triangular array of binomial coefficients, known as Pascal's triangle, has been extended to a square functional equation which generates a tableau of numbers. In the present paper these results have been generalised and the generating function for this new set of numbers has been obtained. Several relations among these numbers, which help construct the tableau, are studied. Some further generalisations of these numbers are also given.

1. Introduction. Let I, I^+ and R denote the set of integers, the set of nonnegative integers and the set of real numbers, respectively. The function f defined on the lattice $I^+ \times I^+$ which satisfies the difference equation

$$f(n + 1, r) = f(n, r) + f(n, r - 1) \tag{1}$$

and is uniquely determined by initial values on $I^+ \times \{0\}$ and $\{0\} \times I^+$ describes the well known triangular array of numbers. This has been generalised by many authors (see Gupta [4], [5], Cadogan [1], Stanton and Cowan [6]). In [5] Gupta has studied the square functional equation

$$g(n + 1, r + 1) = g(n, r + 1) + g(n + 1, r) + g(n, r) \tag{2}$$

which together with the boundary conditions $g(n, 0) = g(0, r) = 1, \forall n$, $r \in I^+$ uniquely determines a tableau. However, here we obtain a more general class of functions defined by

$$g\colon I^+ \times I^+ \longrightarrow R$$

satisfying the general square functional equation

$$\begin{aligned} g(n, r) = p_1 g(n - 1, r) &+ p_2 g(n, r - 1) \\ &+ p_3 g(n - 1, r - 1) p_i \in R\,, \qquad i = 1, 2, 3\,, \end{aligned} \tag{3}$$

subject to certain initial conditions $g(n, 0) = p_1^n$. It may be noted that $g(n, r)$ is not symmetric in n and r. Also notice that if $p_i = 1$, $i = 1, 2, 3$ this reduces to the case studied earlier (Gupta [5]). In the next section we will give some results for this generalised function $g(n, r)$. However details will be skipped since they are similar to the results in Gupta [5] or Stanton and Cowan [6].

2. Properties of $g(n, r)$. Let us define the generating function $A(x, y)$ by

$$A(x, y) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} g(n, r) x^n y^r \tag{4}$$

and invoke the recurrence relation (3), along with the initial conditions, we get

$$\frac{1}{1 - p_2 x - p_1 y - p_3 xy} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} g(n, r) x^n y^r \tag{5}$$

which expresses $A(x, y)$ as a rational function, and may be used to evaluate $g(n, r)$ either explicity or asymptotically. However if we let $g(n, 0) = d$, $\forall n$ the generating function is given by

$$A(x, y) = \frac{d(1 - p_1 y)}{(1 - y)(1 - p_1 y - p_2 x - p_3 xy)} .$$

LEMMA 1. *$g(n, r)$ is the coefficient of x^n in the expansion of $(p_2 + p_3 x)^r/(1 - p_1 x)^{r+1}$ and is given by*

$$g(n, r) = \left(\frac{p_3}{p_2}\right)^n p_2^r \sum_k \binom{r}{n-k} \binom{r+k}{k} \left(\frac{p_1 p_2}{p_3}\right)^k . \tag{6}$$

Proof. Define $A_r(x) = \sum_{n=0}^{\infty} g(n, r) x^n$. Then it is easy to show, by using (3) and the initial conditions that

$$A_r(x) = \frac{(p_2 + p_3 x)^r}{(1 - p_1 x)^{r+1}} . \tag{7}$$

By expanding the right-hand side of (7) we get the result (6).

Again if we let $g(n, 0) = d$, then $g(n, r)$ is the coefficient of x^n in the expansion of

$$\frac{d}{1 - x}\left[\frac{p_2 + p_3 x}{1 - p_1 x}\right]^r .$$

LEMMA 2. *We have,*

$$\begin{aligned} g(n, r) &= \left(\frac{p_3}{p_2}\right)^n \sum_\alpha \binom{n}{\alpha} \binom{r+\alpha}{n} \left(\frac{p_1 p_2}{p_3}\right)\alpha , \\ &= \left(\frac{p_3}{p_2}\right)^n \sum_\beta \binom{r}{\beta} \binom{n}{\beta} \left(1 + \frac{p_1 p_2}{p_3}\right)^\beta . \end{aligned} \tag{8}$$

Proof. Similar to the proof of Lemma 2 of [6].

LEMMA 3. *If $d_m = \sum_{n+r-m}^{\infty} g(n, r)$, then*

$$d_{m+2} = (p_1 + p_2)d_{m+1} + p_3 d_m \; . \tag{9}$$

If we let $p_i = 1$, $i = 1, 2, 3$ *we get Lemma* 4 *of* [6].

The following relation can easily be verified and corresponds to Lemma 3 of [5].

$$g(n, r + s) = \sum_k g(k, r)[g(n - k, s) - p_1 g(n - k - 1, s)] \; . \tag{10}$$

The relationship of $g(n, r)$ with the hypergeometric function ${}_2F_1$ is given by

$$g(n, r) = \left(\frac{p_3}{p_2}\right)^n {}_2F_1\left(-n, r; 1; 1 + \frac{p_1 p_2}{p_3}\right) . \tag{11}$$

This Gauss hypergeometric series reduces to a polynomial of degree $n(r)$ in $(1 + (p_1p_2/p_3))$ for $n(r) = 0, 1, 2, \cdots$. From this relation (11) we can derive many results, of which we give only one below.

$$\begin{aligned} g(n, r) = \frac{1}{n}\Big[&(n - 1)\frac{p_1p_2}{p_3} g(n - 2, r) \\ &- \left\{(n - 1 - r)\left(1 + \frac{p_1p_2}{p_3}\right) - 2n + 1\right\} g(n - 1, r)\Big] . \end{aligned}$$

This relation corresponds to Lemma 5 of [5] and is useful in computing the numbers $g(n, r)$. Results corresponding Lemma 6 of [5] and many more can be similarly obtained.

3. **Further generalisation.** In general $g(n, 0) = d_n \in R$. In §2 we have treated the special case when $d_n = p_1^n$. In fact different initial conditions give rise to (i) an Arithmetic progression and (ii) Geometric progression for the numbers $g(n, 0)$. However in general we can prove the following.

THEOREM 1. *We have,*

$$g(n, r) = \sum_{\substack{u,v,w \\ u+v+w=r}} \binom{r}{u, v, w} p_1^u p_2^v p_3^w g(n - r + v, u) \tag{12}$$

where

$$\binom{r}{u, v, w} = \frac{r!}{u!v!w!} \; .$$

With initial conditions explicit formulae can be obtained.

We can further generalise by considering

$$g\colon I \times I \times I^{+} \longrightarrow R$$

where

$$g(n, m, r) = p_1 g(n, m, r-1) + p_2 g(n, m-1, r-1) + p_3 g(n-1, m-1, r-1) .$$

Then $g(n, m, r)$ satisfies the following theorem, (see [1]).

THEOREM 2. *We have*

$$g(n, m, r) - \sum_{\substack{u, v, w \\ u+v+w=r}} \binom{r}{u, v, w} p_1^u p_2^v p_3^w g(n-w, m-v-w, 0) .$$

Many generalisations are possible by increasing the dimension of the lattice, and/or by redefining the functional equation that gives the recurrence relation.

4. REMARK. Earlier in [2] and [3] the value of $g(n, r)$ found its applications in sphere packing, coding metrics and chess puzzles. Since the present function is a generalisation of this earlier result, it is hoped that these numbers too will be of use in such context.

REFERENCES

1. Charles Cadogan, *Some generalisations of the Pascal triangle,* Math. Mag., **45** (1972), 158–162.
2. S. W. Golomb, *Sphere Packing, Coding Metrics, and Chese Puzzless,* Proceedings of the Second Cahpel Hill Conference on Combinatorial Mathematics and its Applications, University of North Carolina (1970), 176–189.
3. S. W. Golomb and L. R. Welch, *Algebraic Coding and the Lie Metric,* Error Correcting Codes, (edited by H. B. Mann), John Wiley and Sons, Inc., New York, 1968, 175–194.
4. A. K. Gupta, *On Pascal's triangle of the third kind,* Biom. Z., **15** (1973), 389-392.
5. ———, *On a 'square' functional equation,* Pacific J. Math., **50** (1974), 447–454.
6. R. G. Stanton and D. D. Cowan, *Note on a 'square' functional equation,* SIAM Review, **12** (1970), 277–279.

Received August 26, 1974.

THE UNIVERSITY OF MICHIGAN