PROVABLE EQUALITY IN PRIMITIVE RECURSIVE ARITHMETIC WITH AND WITHOUT INDUCTION

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The set of identities provable in primitive recursive arithmetic without induction (PRE), with or without standard quantifier free successor axioms, is recursive. A finite number of identities can be added to PRE such that the set of identities provable become complete r.e. (with or without successor axioms). If the successor axiom $y \neq 0 \to \exists x(S(x) = y)$ is added to PRE, then the set of identities provable become complete r.e. (with or without $1 \neq 0$). If PRE is augmented by definition by cases, then the set of identities provable become complete r.e. (with or without successor axioms). Equivalents of primitive recursive arithmetic (PRA) are given, involving the rule of induction restricted to identities.

We begin with the definition of the n-ary primitive recursion indices, $0 < n$. These are given by (i) $0, 1$ are 1-ary p.r. indices. (ii) $(2, n, m)$, $1 \leq m \leq n$, is an n-ary p.r. index. (iii) $(3, x, y_1, \cdots, y_k)$ is an n-ary p.r. index if $y_1, \cdots, y_k$ are n-ary p.r. indices and x is a k-ary p.r. index, $1 \leq n, k$. (iv) $(4, x, y)$ is a 1-ary p.r. index if y is a 2-ary p.r. index. (v) $(5, x, y)$ is an n-ary p.r. index if x is an $(n - 1)$-ary p.r. index and y is an $n + 1$-ary p.r. index, $1 < n$.

The n-ary primitive recursive function symbols are the expressions F_x, where x is an n-ary p.r. index. We let S abbreviate F_1, and $\bar{0}$ be 0, $\overline{n + 1}$ be $S(\bar{n})$.

The theory PRE (primitive recursion equations) is given by $F_0(a) = 0$, $F_{(2,n,m)}(a_1, \cdots, a_n) = a_m$, $F_{(3,x,y_1,\cdots y_k)}(a_1, \cdots, a_n) = F_x(F_{y_1}(a_1, \cdots, a_n), \cdots, F_{y_k}(a_1, \cdots, a_n))$, $F_{(4,x,y)}(0) = \bar{x}$, $F_{(4,x,y)}(S(a)) = F_y(a, F_{(4,x,y)}(a))$, $F_{(5,z,w)}(a_1, \cdots, a_n, 0) = F_z(a_1, \cdots, a_n)$, $F_{(5,z,w)}(a_1, \cdots, a_n, S(a_{n+1})) = F_w(a_1, \cdots, a_n, a_{n+1}, F_{(5,z,w)}(a_1, \cdots, a_n, a_{n+1}))$, where $(2, n, m)$, $(3, x, y_1, \cdots, y_k)$, $(4, x, y)$, and $(5, z, w)$ are respectively n-ary, n-ary, 1-ary, and $n + 1$-ary p.r. indices, $1 \leq n$.

We will firstly consider relationships between various successor axioms.

LEMMA 1. $\text{PRE} \vdash S(x) = S(y) \to x = y$. $\text{PRE} \vdash S(x) = 0 \to y = 0$.

Proof. Let P be such that $\text{PRE} \vdash (P(0) = 0 \ \& \ P(S(x)) = x)$. Then $\text{PRE} \vdash (S(x) = S(y) \to P(S(x)) = P(S(y)) \to x = y)$. Let F be such that $\text{PRE} \vdash (H(y, 0) = 0 \ \& \ H(y, S(x)) = y)$. Then $\text{PRE} \vdash (S(x) = 0 \to F(y, 0) = y \to y = 0)$.

TEOREM 1.   PRE $\vdash$ $(S(x) = S(y) \rightarrow x = y)$.   *In addition, the following are provably equivalent in* PRE.   (i) $(\exists x)(x \neq 0)$.   (ii) $(\exists x)(S(x) \neq x)$. (iii) $(\exists x)(S(x) \neq 0)$.   (iv) $S(0) \neq 0$.   (v) $S(x) \neq 0$.   PRE $+ S(0) \neq 0 +$ $(y \neq 0 \rightarrow (\exists x)(S(x) = y)) \vdash S(x) \neq x$.   PRE $+ S(0) \neq 0$ *does not prove* $S(x) \neq x$.   PRE $+ S(x) \neq x$ *does not prove* $(y \neq 0 \rightarrow (\exists x)(S(x) = y))$. PRE $+ (y \neq 0 \rightarrow (\exists x)(S(x) = y))$ *does not prove* $S(0) \neq 0$.   PRE $+ (y \neq 0 \rightarrow (\exists x)(S(x) = y))$ *has a quantifier free axiomatization.*

Proof.   The first part is Lemma 1.   For the second part, note that (v) $\rightarrow$ (iv) $\rightarrow$ (iii) $\rightarrow$ (i).   By Lemma 1, (i) $\rightarrow$ (v).   Thus (i), (iii), (iv), (v) are equivalent.   Clearly (iv) $\rightarrow$ (ii).   To see that (ii) $\rightarrow$ (iv), let PRE $\vdash$ $(x + 0 = x$ & $x + S(y) = S(x + y))$.   Then PRE $\vdash$ $(S(0) = 0 \rightarrow$ $x + S(0) = x + 0 = x = S(x + 0) = S(x))$.

We now argue in PRE $+ S(0) \neq 0 + (y \neq 0 \rightarrow (\exists x)(S(x) = y))$.   Let $F(0) = S(0)$, $F(S(x)) = 0$.   Then $y \neq 0 \rightarrow F(y) = 0$.   Hence $F(y) \neq y$. Let $E(0) = 0$, $E(S(x)) = F(E(x))$.   Then $S(x) = x \rightarrow E(x) = F(E(x))$. But $F(E(x)) \neq E(x)$.   So $S(x) \neq x$.

We sketch a proof that PRE $+ S(0) \neq 0$ does not prove $S(x) \neq x$. Consider the type free λ-calculus with the three constant 0, S, R, whose axioms and rules are given by the same clauses as those for the λ-calculus of [1], except that we add the axioms $R\alpha\beta0 = \beta$, $R\alpha\beta(S\gamma) = \alpha\gamma(R\alpha\beta\gamma)$, for all terms α, β, γ.   Then for each p.r. function symbol F, we may naturally associate a closed term F^* in such a way that the set of equivalence classes $[\alpha]$ of closed terms under convertibility form a model of PRE, where 0 is interpreted as $[0]$, and $F([\alpha_1], \cdots, [\alpha_k])$ as $[F^*\alpha_1 \cdots \alpha_k]$.   This model will obey $(\exists x)(Sx = x)$ by the fixed point theorem.   It is also clear that $S(0) \neq 0$ also holds, since $S(0)$ is not convertible to 0 (by the Church Rosser theorem).

It will be established in Theorem 2 that PRE $+ S(x) \neq x$ does not prove $(y \neq 0 \rightarrow (\exists x)(S(x) = y))$.

The last independence result is obvious, using the one element model of PRE.

PRE $+ (y \neq 0 \rightarrow (\exists x)(S(x) = y))$ has the axiomatization PRE $+$ $(y = 0 \vee S(P(y)) = y)$, where P is as in the proof of Lemma 1 (see Lemma 11).

A set X of p.r. function symbols is said to be closed just in case (1) $S \in X$.   (2) if $F_{(3, x, y_1, \dots, y_k)} \in X$ then $F_x, F_{y_1}, \cdots, F_{y_k} \in X$.   (3) if $F_{(4, x, y)} \in X$ then $F_y \in X$.   (4) if $F_{(5, x, y)} \in X$ then $F_x, F_y \in X$.   Let PRE be the set of all p.r. function symbols.

For closed X, we let PRE_X consist of those axioms of PRE all of whose function symbols are from X.   An X-term is a term in the language of PRE, all of whose function symbols are from X. Below, α, β will denote terms in the language of PRE.

We say that α is the atomic reduct of β just in case one of the following holds: (a) α is $F_0(\gamma)$ and β is 0. (b) α is $F_{(2,n,m)}(\alpha_1, \cdots, \alpha_n)$ and β is α_m. (c) α is $F_{(3,x,y_1,\cdots,y_k)}(\alpha_1, \cdots, \alpha_n)$ and β is

$$F_x(F_{y_1}(\alpha_1, \cdots, \alpha_n), \cdots, F_{y_k}(\alpha_1, \cdots, \alpha_n)) .$$

(d) α is $F_{(4,x,y)}(0)$ and β is $\bar{x}$. (e) α is $F_{(5,x,y)}(\alpha_1, \cdots, \alpha_n, 0)$ and β is $F_x(\alpha_1, \cdots, \alpha_n)$. (f) α is $F_{(5,x,y)}(\alpha_1, \cdots, \alpha_n, S(\alpha_{n+1}))$ and β is

$$F_y(\alpha_1, \cdots, \alpha_n, \alpha_{n+1}, F_{(5,x,y)}(\alpha_1, \cdots, \alpha_n, \alpha_{n+1})) .$$

We say that α is an immediate reduct of β just in case α is the result of replacing one particular occurrence of a subterm γ of β by the atomic reduct of γ.

A sequence of terms $\{\alpha_n\}$ is called a reduction sequence just in case each α_{n+1} is an immediate reduct of α_n. Observe that if α_0 is an X-term, then each α_n must be an X-term.

LEMMA 2. *Every reduction sequence is finite in length. Any two maximal reduction sequences starting with α, end in the same term.*

Proof. This can either be obtained as a consequence of [4], or proved directly by the same methods. Also see [6].

Let us call α a reduct of β just in case α is a term in some reduction sequence beginning with β. If γ has no immediate reduct, then γ is called irreducible.

The following is immediate from Lemma 2.

LEMMA 3. *To each PR-term α, there is a unique irreducible term α^* which is a reduct of α. Moreover, α^* can be effectively obtained from α. If α is an X-term, X closed, then α^* is an X-tem.*

LEMMA 4. *Let X be closed. If $\alpha^* = \beta^*$, and α, β are X-terms, then $\mathrm{PRE}_X \vdash \alpha = \beta$.*

Proof. Show by induction on n, that if $(\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n)$ is a reduction sequence, then $\mathrm{PRE}_X \vdash \alpha = \alpha_n$.

Fix X to be a closed set of p.r. function symbols. We define the structure $\mathscr{A}_X$ so that (a) the domain is the set of all irreducible X-terms. (b) each n-ary $F \in X$ is interpreted by: $F(\alpha_1, \cdots, \alpha_n)$ is the term $(F(\alpha_1, \cdots, \alpha_n))^*$. (c) $\mathscr{A}_X$ interprets the constant 0 as the term 0, and interprets no other symbols. It is easy to see that $\mathscr{A}_X \models \mathrm{PRE}_X$.

LEMMA 5. *$\mathscr{A}_X$ interprets an irreducible X-term α as α, under*

the valuations which interpretes each variable x as the term x.

Proof. Use induction on the construction of the irreducible term α, and the fact that subterms of irreducible terms are irreducible.

LEMMA 6. *For X-terms α, β, if $\mathscr{A}_X \models \alpha = \beta$ then $\alpha^* = \beta^*$.*

Proof. Suppose $\mathscr{A}_X \models \alpha = \beta$. Then $\mathscr{A}_X \models \alpha^* = \beta^*$, by Lemma 4. Hence $\alpha^* = \beta^*$, by Lemma 5.

THEOREM 2. *Let X be a closed set of p.r. function symbols. For X-terms α, β, the following are equivalent.* (i) $\alpha^* = \beta^*$. (ii) $\mathrm{PRE}_X \vdash \alpha = \beta$. (iii) $\mathrm{PRE} \cup \{S(x) \neq x, (\exists x)(x \neq 0 \ \& \ (\forall y)(S(y) \neq x))\} \vdash \alpha = \beta$.

Proof. (i) $\rightarrow$ (ii) is by Lemma 3 (ii) $\rightarrow$ (iii) is obvious. To see that (iii) $\rightarrow$ (i), it suffices to quote Lemma 6, and show that $\mathscr{A}_X \models (S(x) \neq x \ \& \ (\exists x)(x \neq 0 \ \& \ (\forall y)(S(y) \neq x)))$. The latter is a consequence of the fact that $S(\alpha)$ is always irreducible if α is.

COROLLARY 1. *Let X be a closed set of p.r. function symbols. Then PRE and PRE_X have the same set of consequences of the form $\alpha = \beta$, for X-terms α, β, and this set is recursive.*

The rest of this paper is devoted to undecidability results. We first present two recursion theoretic lemmas. Recall the definition of effectively inseparable sets in [3].

LEMMA 7. *There are effectively inseparable recursively enumerable sets $A, B \subset \omega$. If A, B are effectively inseparable recursively enumerable sets, $A \subset C$, $C \cap B = \varnothing$, and C is recursively enumerable, then C is creative.*

Proof. The existence of A, B is proved in [3]. The rest is immediate.
We fix A, B.

LEMMA 8. *Every nonempty r.e. set is the range of some unary primitive recursive function. There is a primitive recursive binary function $g(n, m)$ such that $n \in B \leftrightarrow (\exists m)(g(n, m) = 1)$, and g is $0, 1$-valued.*

Proof. Use the Kleene T-predicate.

Recall that a theory T is called essentially undecidable just

in case it is consistent, and any extension of T whose set of consequences in the language of T is recursive, is inconsistent. We will call a theory T equationally essentially undecidable just in case it does not imply $x = y$, and any extension of T whose set of equational consequences in the language of T is recursive implies $x = y$.

To our knowledge, the existence of finitely axiomatized equationally essentially undecidable theories was first proved in [2], where it is shown that CL, a system of combinatory logic, viewed as an equational theory, is equationally essentially undecidable. Also, see [1].

In fact, one can see that CL has an even stronger property: let T be an extension of CL not implying $x = y$. If the set of consequences of T in the language of CL is r.e., the set is complete r.e.. We call such a theory completely equationally essentially undecidable.[1]

An n-ary p.r. function symbol F is said to represent the n-ary function f just in case $f(x_1, \cdots, x_n) = y$ implies $\mathrm{PRE} \vdash F(\bar{x}_1, \cdots, \bar{x}_n) = \bar{y}$.

LEMMA 9. *If f is an n-ary primitive recursive function, then f is represented by some p.r. function symbol F.*

Proof. This is well known.

By Lemma 8 fix f to be a unary primitive recursive function whose range is A. By Lemma 9, fix F, G to be p.r. function symbols which respectively represent f, g (of Lemma 8).

A glance at the proof of the second part of Lemma 1 reveals that there is a function symbol H such that for the closed X with $H \in X$, $\mathrm{PRE}_X \vdash S(x) = 0 \rightarrow y = 0$.

LEMMA 10. *Let X be a closed set of p.r. function symbols, F, G, $H \in X$. If $n \in A$ then $\mathrm{PRE}_X + G(F(x), y) = 0 \vdash G(\bar{n}, y) = 0$. If $n \in B$ then $\mathrm{PRE}_X + G(F(x), y) = 0 \vdash (\exists y)(G(\bar{n}, y) = S(0))$.*

Proof. If $n \in A$ then $n \in \mathrm{Rng}\,(f)$. Let $f(k) = n$. Then $\mathrm{PRE}_X \vdash F(\bar{k}) = \bar{n}$. Hence $\mathrm{PRE}_X + G(F(x), y) = 0 \vdash G(\bar{n}, y) = 0$. If $n \in B$ then $(\exists m)(g(n, m) = 1)$. Hence $\mathrm{PRE}_X \vdash G(\bar{n}, \bar{m}) = S(0)$. Hence

$$\mathrm{PRE}_X \vdash (\exists y)(G(\bar{n}, y) = S(0)) \,.$$

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[1] There are no essentially undecidable equational theories, since every consistent equational theory has a unique model with one element. If T is equationally essentially undecidable, then $T + \exists x \exists y(x \neq y)$ is essentially undecidable. If T is undecidable in either of the three senses, and T extends T', then T' is undecidable in that same sense (even if the language of T' extends the language of T). If the same theory T is viewed as having different languages, then these three notions of undeciability remain unaffected (as long as the languages in question are recursive).

THOREM 3. PRE *can be extended by a finite number of equations in its language, so that the result is a completely equationally essentially undecidable theory. Moreover, the same is true of any* PRE_X, *provided X is a closed set of* p.r. *function symbols containing a fixed finite list of* p.r. *function symbols.*

Proof. Let X be closed, $F, G, H \in X$. Let T be an extension of $\mathrm{PRE}_X + G(F(x), y) = 0$ not implying $x = y$. Let

$$C = \{n: T \vdash G(\bar{n}, y) = 0\} \ .$$

By Lemma 10, $A \subset C$. Let $n \in B$. By Lemma 10, $T \vdash (\exists y)(G(\bar{n}, y) = S(0))$. If $n \in C$ then $T \vdash S(0) = 0$, and hence $T \vdash x = y$. Hence $n \notin C$. So $B \cap C = \varnothing$. Assume that the set of equational consequences of T in the language of PRE_X is r.e. Then C is r.e. By Lemma 7, C is complete r.e. Hence the set of equational consequences of T in the language of PRE_X is complete r.e. To see that $\mathrm{PRE}_X + G(F(x), y) = 0$ does not imply $x = y$, note that is has a (unique) model with domain ω, in which 0 is interpreted as 0 and S is interpreted as successor.

COROLLARY 2. *There is a finitely axiomatized equational theory which is completely equationally essentially undecidable, and which has a model with domain ω in which all function symbols are interpreted* (primitive) *recursively.*

Now let i, j be binary primitive recursive functions such that $i(n, 0) = 1$, $A = \{n: (\exists m)(i(n, m) = 0)\}$, $B = \{n: (\exists m)(j(n, m) = 0)\}$. Let I, J represent h, i. Let K be such that $\mathrm{PRE} \vdash (K(z, y, 0) = x$ & $K(x, y, S(z)) = y)$. Let $\mathrm{PRE} \vdash (P(0) = 0$ & $P(S(x)) = x)$.

In view of the first three parts of Theorem 1, it is appropriate to let SA (successor axioms) be $\{S(0) \neq 0, y \neq 0 \rightarrow (\exists x)(S(x) = y)\}$.

LEMMA 11. $\mathrm{PRE} + \mathrm{SA} \vdash (K(x, y, 0) = x$ & $(z \neq 0 \rightarrow K(x, y, z) = y))$. $\mathrm{PRE} + \mathrm{SA} \vdash (x \neq 0 \rightarrow S(P(x)) = x)$.

Proof. Arguing in $\mathrm{PRE} + \mathrm{SA}$, if $z \neq 0$, let $z = S(a)$. Then $K(x, y, z) = K(x, y, S(a)) = y$. Second part is similar.

Let $\mathrm{PRE} \vdash (L(x, y, 0) = S(y)$ & $L(x, y, S(z)) = K(0, P(L(x, y, z)), I(x, L(x, y, z))))$. Let $P^0(x)$ be x, $P^{n+1}(x)$ be $P(P^n(x))$. Let $S^0(x)$ be x, $S^{n+1}(x)$ be $S(S^n(x))$.

LEMMA 12. $\mathrm{PRE} + \mathrm{SA}$ *proves the following.*
(i) $L(x, y, 0) = S(y)$.
(ii) $I(x, L(x, y, z)) = 0 \rightarrow L(x, y, S(z)) = 0$.
(iii) $I(x, L(x, y, z)) \neq 0 \rightarrow L(x, y, S(z)) = P(L(x, y, z))$.

LEMMA 13. *Let* $0 < n$. *Then* $\text{PRE} + \text{SA} \vdash (L(x, y, z) = S(0)$ &
$\bigwedge_{k<n} y \neq \bar{k}) \rightarrow (L(x, y, P^n(y)) = \overline{n+1}$ & $I(x, \overline{n+1}) \neq 0)$.

Proof. By induction n. For the basis case $n = 1$, let

$$L(x, y, z) = S(0) \, , \qquad y \neq 0 \, .$$

Then $y = S(P(y))$. Hence $I(x, L(x, y, P(y))) \neq 0$. Hence $(L(x, y, y) = P(L(x, y, P(y))) = S(0)$. Therefore $L(x, y, P(y)) = S(S(0))$. Hence $I(x, S(S(0))) \neq 0$.

Suppose this is true for n. Assume $L(x, y, y) = S(0)$, $\bigwedge_{k \leq n} y \neq \bar{k}$. Then $L(x, y, P^n(y)) = \overline{n+1}$. Now $P^n(y) \neq 0$, and so $P^n(y) = S(P^{n+1}(y))$. Now $(L(x, y, P^n(y)) \neq 0$, and so $I(x, L(x, y, P^{n+1}(y))) \neq 0$. Hence

$$L(x, y, P^n(y)) = P(L(x, y, P^{n+1}(y))) = \overline{n+1} \, .$$

Therefore $L(x, y, P^{n+1}(y)) = \overline{n+2}$. Hence $I(x, \overline{n+2}) \neq 0$.

LEMMA 14. *If* $n \notin A$ *then* $\text{PRE} + \text{SA} \vdash L(\bar{n}, \bar{m}, \bar{m}) = S(0)$.

Proof. If $n \notin A$ then $\text{PRE} + \text{SA} \vdash I(n, m) \neq 0$.

Now fix T to be any consistent extension of $\text{PRE} + \text{SA}$. Let $C = \{n: T \vdash (J(\bar{n}, y) = 0 \rightarrow L(\bar{n}, y, y) \neq S(0))$.

LEMMA 15. $A \subset C, C \cap B \neq \varnothing$.

Proof. Let $n \in A$. Let $i(n, m) = 0$. Then $n \notin B, 1 < m$. Arguing in T, assume $J(\bar{n}, y) = 0$, $L(\bar{n}, y, y) = S(0)$. Now, since $n \notin B$, we see that $\bigwedge_{k<m-1} y \neq \bar{k}$. Hence by Lemma 13, $I(\bar{n}, \bar{m}) \neq 0$, which is a contradiction. This establishes that $n \in C$. Hence $A \subset C$.

Let $n \in B, C$. Let $j(n, m) = 0$. Arguing in T, we have $J(\bar{n}, \bar{m}) = 0$, $J(\bar{n}, y) = 0 \rightarrow L(\bar{n}, y, y) \neq S(0)$. Hence $L(\bar{n}, \bar{m}, \bar{m}) \neq S(0)$. Since $n \in A$, we have $L(\bar{n}, \bar{m}, \bar{m}) = S(0)$, by Lemma 14. This is a contradiction. Hence $B \cap C = \varnothing$.

LEMMA 16. *C is of the form* $\{n: T \vdash M(\bar{n}, y) = 0\}$, *for some* p.r. *function symbol M*.

Proof. Note that $\text{PRE} + \text{SA} \vdash (y = S(0) \leftrightarrow (y \neq 0$ & $P(y) = 0))$. Let $\text{PRE} + \text{SA} \vdash Q(x, y) = K(K(y, 0, P(y)), 0, x)$. Then $\text{PRE} + \text{SA} \vdash (x \neq 0 \vee y = 0 \vee P(y) \neq 0) \leftrightarrow Q(x, y) = 0$. Hence $\text{PRE} + \text{SA} \vdash (Q(y, x) = 0 \leftrightarrow (x = 0 \rightarrow y \neq S(0)))$. Finally, let $\text{PRE} + \text{SA} \vdash R(x, y) = Q(J(y, x), L(x, y, y))$. Then $C = \{n: T \vdash R(\bar{n}, y) = 0\}$.

THEOREM 4. $\text{PRE} + \text{SA}$ *is completely equationally essentially undecidable.*

Proof. Clear from Lemmas 15, 16.

COROLLARY 3.   PRE $+ (y \neq 0 \rightarrow (\exists x)(S(x) = y))$ *is completely equationally essentially undecidable.   There is a finite list of* **p.r.** *function symbols such that* PRE$_x + (y \neq 0 \rightarrow (\exists x)(S(x) = y))$ *is completely equationally essentially undecidable, for all closed X containing that list.*

Proof. If T is any theory extending

$$\text{PRE} + (y \neq 0 \longrightarrow (\exists x)(S(x) = y))$$

not implying $x = y$, then T has the same equational consequences as $T + S(0) \neq 0$.   To see this, note that the models of T and $T + S(0) \neq 0$ are precisely the same, except that T may have a one element model.   For the second part, an analysis reveals that only finitely many symbols are needed for the proofs.

We now consider an extension of PRE obtained by adding the definition by cases function $\psi(a, b, c, d) = a$ if $c = d$; b if $c \neq d$. More formally, let the extended primitive recursion indices be given by the same clauses as the primitive recursion indices, together with the clause (vi) 6 is a 4-ary extended primitive recursion index.   The extended primitive recursive function symbols (e.p.r. function symbols) are written F_x, where x is an extended primitive recursion index.

A set X of e.p.r. function symbols is called closed if (i) $S \in X$. (ii) if $F_{(3,x,y_1,\cdots,y_k)} \in X$ then $F_x, F_{y_1}, \cdots, F_{y_k} \in X$.   (iii) if $F_{(4,x,y)} \in X$ then $F_y \in X$.   (iv) if $F_{(5,x,y)} \in X$ then $F_x, F_y \in X$.   Let PR$'$ be the set of all e.p.r. function symbols.

We use ψ for F_6.   We let PRE$'$ be the theory whose axioms are given by the same clauses as those for PRE, together with the clauses $\psi(x, y, z, z) = x$, $z \neq w \rightarrow \psi(x, y, z, w) = y$.

THEOREM 5.   PRE$' + S(0) \neq 0 \vdash S(x) \neq x$.   PRE$' + S(0) \neq 0$ *does not prove* $x \neq 0 \rightarrow (\exists y)(S(y) = x)$.   *For closed sets X containing* ψ, *every quantifier free X-formula is provably equivalent* PRE$'_x$, *to some equality between X-terms.*

Proof.   Assume PRE$' + S(0) \neq 0$.   Let E be such that $E(0) = 0$, $E(S(x)) = \psi(S(0), 0, E(x), 0)$.   Then $E(S(x)) \neq E(x)$.   Hence $S(x) \neq x$.

A model of PRE$' + S(0) \neq 0 + (\exists x)(x \neq 0$ & $(\forall y)(S(y) \neq x))$ can be constructed with domain $\{0, 1, 2, \cdots, \infty + 0, \infty + 1, \infty + 2, \cdots\}$ as follows.   Define 0 as 0, $S(k)$ as $k + 1$, $S(\infty + k)$ as $\infty + (k + 1)$, F_0 and $F_{(2,n,m)}$ as usual, $F_{(3,x,y_1,\cdots y_k)}$ in terms of $F_x, F_{y_1}, \cdots, F_{y_k}$ as usual, $F_{(4,x,y)}(0) = F_{(4,x,y)}(\infty + 0) = \bar{x}$, $F_{(4,x,y)}(n + 1) = F_{(4,x,y)}(\infty +$

$(n + 1)) = F_y(n, F_{(4,x,y)}(n))$, and $F_{(5,x,y)}$ analogously.

The last part is left to the reader.

We now wish to prove that PRE^+ is completely equationally essentially undecidable. The proof is similar to the proofs for $PRE + SA$, but more complicated.

LEMMA 17. *There is an e.p.r. function symbol U such that the following are provable in* $PRE' + S(0) \neq 0$.

(i) $U(x, y, 0) = S(y)$.

(ii) $((S(P(z)) = z \lor z = 0)$ & $I(x, U(x, y, z)) \neq 0$ & $S(P(U(x, y, z))) = U(x, y, z)) \rightarrow U(x, y, S(z)) = P(U(x, y, z))$.

(iii) $((S(P(z)) \neq z$ & $z \neq 0) \lor I(x, U(x, y, z)) = 0 \lor S(P(U(x, y, z))) \neq U(x, y, z)) \rightarrow U(x, y, S(z)) = 0$.

Proof. Left to the reader. Obviously ψ is vital here. The old K is not enough.

LEMMA 18. $PRE' + S(0) \neq 0 \vdash ((U(x, S(y), S(y)) = S(0)$ & $\bigwedge_{k \leq n} y \neq \bar{k}) \rightarrow (S(P^{n+1}(y)) = P^n(y)$ & $I(x, \overline{n+2}) \neq 0$ & $U(x, S(y), P^n(y)) = \overline{n+2})$, for $0 \leq n$.

Proof. By induction on n. For the basis case $n = 0$, let $U(x, S(y), S(y)) = S(0)$, $y \neq 0$. Then $U(x, S(y), S(y)) \neq 0$, and so $S(P(y)) = y \lor y = 0$, $I((x, U(x, S(y), y)) \neq 0$, $S(P(U(x, S(y), y))) = U(x, S(y), y)$, $U(x, S(y), S(y)) = S(0) = P(U(x, S(y), y))$. Hence $U(x, S(y), y) = S(S(0))$. Therefore $I(x, S(S(0))) \neq 0$, $S(P(y)) = y$.

Suppose this is true for all $m \leq n$. Let $U(x, S(y), S(y)) = S(0)$, $\bigwedge_{k \leq n+1} y \neq k$. Then $U(x, S(y), P^n(y)) = \overline{n+2}$, $S(P^{n+1}(y)) = P^n(y)$. Hence $U(x, S(y), P^n(y)) \neq 0$, and so $S(P^{n+2}(y)) = P^{n+1}(y) \lor P^{n+1}(y) = 0$, $I(x, U(x, S(y), P^{n+1}(y)) \neq 0$, $S(P(U(x, S(y), P^{n+1}(y)))) = U(x, S(y), P^{n+1}(y))$,

$$U(x, S(y), P^n(y)) = P(U(x, S(y), P^{n+1}(y))) = \overline{n+2} .$$

Hence $U(x, S(y), P^{n+1}(y)) = \overline{n+3}$. Therefore $I(x, \overline{n+3}) \neq 0$. Now by induction hypotheses, it is easily seen that $S(P^{m+1}(y)) = P^m(y)$, for all $m \leq n$. Hence $P^{n+1}(y) = 0 \rightarrow y = \overline{n+1}$. Since $y \neq \overline{n+1}$, we have $P^{n+1}(y) \neq 0$, and so $S(P^{n+2}(y)) = p^{n+1}(y)$.

LEMMA 19.   *If $n \notin A$ then* $PRE' + S(0) \neq 0 \vdash U(\bar{n}, \bar{m}, \bar{m}) = S(0)$.

Proof. Like Lemma 13.

Now fix Y to be any consistent extension of $PRE^+ + S(0) \neq 0$. Let $C = \{n: Y \vdash (J(\bar{n}, y) = 0 \rightarrow U(\bar{n}, S(y), S(y)) \neq S(0))\}$.

LEMMA 20. $A \subset Y$, $Y \cap B = \varnothing$.

Proof. Like Lemma 15.

THEOREM 6. PRE$'$ $+$ $S(0) \neq 0$ *is completely equationally essentially undecidable.*

Proof. Like Theorem 4.

COROLLARY 4. PRE$'$ *is completely equationally essentially undecidable. There is a finite list of* e.p.r. *function symbols such that* PRE$'_X$ *is completely equationally essentially undecidable, for all closed sets X of* e.p.r. *function symbols containing that list.*

Proof. Like Corollary 3.

We now consider some forms of induction. We follow the terminology of [5] by referring to quantifier free formulae as open formulae.

Firstly, we have the axiom of induction for open formulae:

$$(A(0) \ \& \ (\forall x)(A(x) \longrightarrow A(S(x)))) \longrightarrow Ax$$

for open PR(PR$'$)-formulae A, which we call AIO(AIO$'$).

Secondly, we will consider the axiom of induction for equations:

$$(\alpha(0) = \beta(0) \ \& \ (\forall x)(\alpha(x) = \beta(x) \to \alpha(S(x)) = \beta(S(x))) \to \alpha(x) = \beta(x)$$

for PR(PR$'$)-terms α, β, which we will call AIE(AIE$'$).

Thirdly, we have the rule of induction for open formulae:

$$\text{from } A(0) , \quad A(x) \longrightarrow A(S(x)) \text{ infer } A(x)$$

for open PR(PR$'$)-formulae A, which we will call RIO(RIO$'$).

Lastly, we will consider the rule of induction for equations:

$$\text{from } \alpha(0) = \beta(0), \ \alpha(x) = \beta(x) \longrightarrow \alpha(S(x)) = \beta(S(x)) \text{ infer } \alpha(x) = \beta(x)$$

for PR(PR$'$)-terms α, β, which we will call RIE(RIE$'$).

By primitive recursive arithmetic (PRA), Skolem arithmetic, or recursive arithmetic, one usually means the system based on the successor axiom $S(x) \neq 0$, the equations PRE, and the rule RIO.

In PRA, the axioms PRE and $S(x) \neq 0$ are to be closed under the rule of induction. Thus we write PRA = RIO(PRE $+$ $S(x) \neq 0$). This is to be distinguished from, say, RIO(PRE) $+$ $S(x) \neq 0$, in which PRE alone is first closed under the rule of induction, and then $S(x) \neq 0$ is added to the closure.

LEMMA 21. *There are* p.r. *function symbols,* P, E, $[x/2]$, $+$, $'$, $^{-1}$, $-$, *and* $\oplus$ *such that the following are provable in* PRE + SA.

(i) $x \neq 0 \rightarrow P(S(x)) = x$, $P(0) = 0$.

(ii) $E(0) = 0$, $E(x) = 0 \rightarrow E(S(x)) = S(0)$, $E(x) = S(0) \rightarrow E(S(x)) = 0$.

(iii) $[0/2] = 0$, $E(x) = 0 \rightarrow [S(x)/2] = x$, $E(x) = S(0) \rightarrow [S(x)/2] = S(x)$.

(iv) $x + 0 = x$, $x + S(y) = S(x + y)$.

(v) $E(x) = 0 \rightarrow x' = S(S(x))$, $E(x) = S(0) \rightarrow x' = P(P(x))$.

(vi) $E(x) = S(0) \rightarrow x^{-1} = S(S(x))$, $(E(x) = 0 \ \& \ x \neq 0) \rightarrow x^{-1} = P(P(x))$, $0^{-1} = S(0)$.

(vii) $x - 0 = x + x$, $x - S(y) = (x - y)^{-1}$.

(viii) $x \oplus 0 = x$, $x \oplus S(y) = (x \oplus y)'$.

LEMMA 22. *The following are provable in* RIE(PRE+SA). (i) $x + y = y + x$. (ii) $(x + y) + z = x + (y + z)$. (iii) $E(x + x) = 0$. (iv) $[x + x/2] = x$. (v) $(x + x = y + y) \rightarrow x = y$. (vi) $0 \oplus x = x + x$. (vii) $(x')^{-1} = (x^{-1})' = x$. (viii) $S(y) - x = (y - x)'$. (ix) $x - x = 0$. (x) $x^{-1} \oplus y = (x \oplus y)^{-1}$. (xi) $(x - y) \oplus y = x + x$. (xii) $x = y \leftrightarrow x - y = 0$.

Proof. (i) $0 + 0 = 0$. $0 + y = y \rightarrow 0 + S(y) = S(0 + y) = S(y)$. Hence $0 + y = y + 0$. $S(x) + 0 = S(x + 0)$. $S(x) + y = S(x + y) \rightarrow S(x) + S(y) = S(S(x) + y) = S(S(x + y)) = S(x + S(y))$. Hence $S(x) + y = x + S(y)$. $x + y = y + x \rightarrow S(x) + y = x + S(y) = S(x + y) = S(y + x) = y + S(x)$. Hence $x + y = y + x$.

(ii) $(x + y) + 0 = x + y = x + (y + 0)$. $(x + y) + z = x + (y + z) \rightarrow (x+y)+S(z) = S((x+y) + z) = S(x + (y+z)) = x+S(y+z) = x+(y+S(z))$. Hence $(x + y) + z = x + (y + z)$.

(iii) $E(0 + 0) = 0$. $S(x) + S(x) = S(S(x + x))$. $E(x + x) = 0 \rightarrow E(S(x + x)) = S(0) \rightarrow E(S(S(x + x))) = 0 = E(S(x) + S(x))$.   Hence $E(x + x) = 0$.

(iv) $[0 + 0/2] = [0/2] = 0$. $E(x + x) = 0$, $E(S(x + x)) = S(0)$. $E(x + x/2) = x \rightarrow E(S(x + x)/2) = x \rightarrow E(S(S(x + x))/2) = S(x) = E(S(x) + S(x)/2)$. Hence $[x + x/2] = x$.

(v) $x + x = y + y \rightarrow [x + x/2] = [y + y/2] \rightarrow x = y$.

(vi) $0 \oplus 0 = 0 + 0$. $0 \oplus x = x + x \rightarrow 0 \oplus S(x) = (0 \oplus x)' = (x+x)' = S(S(x + x)) = S(x) + S(x)$. Hence $0 \oplus x = x + x$.

(vii) By inspection.

(viii) $S(y) - 0 = S(y) + S(y) = (y + y)' = (y - 0)'$. $S(y) - x = (y - x)' \rightarrow S(y) - S(x) = (S(y) - x)^{-1} = ((y - x)')^{-1} = y - x = ((y - x)^{-1})' = (y - S(x))'$. Hence $S(y) - x = (y - x)'$.

(ix) $0 - 0 = 0$. $x - x = 0 \rightarrow S(x) - S(x) = (S(x) - x)^{-1} = ((x - x)^{-1})' = x - x = 0$. Hence $x - x = 0$.

(x) $x^{-1} \oplus 0 = x^{-1} = (x \oplus 0)^{-1}$. $x^{-1} \oplus y = (x \oplus y)^{-1} \rightarrow x^{-1} \oplus S(y) = (x^{-1} \oplus y)' = ((x \oplus y)^{-1})' = x \oplus y = ((x \oplus y)')^{-1} = (x \oplus S(y))^{-1}$. Hence $x^{-1} \oplus y = (x \oplus y)^{-1}$.

(xi) $(x - 0) \oplus 0 = (x + x) \oplus 0 = x + x.$ $(x - y) \oplus y = x + x \rightarrow$
$(x - S(y)) \oplus S(y) = (x-y)^{-1} \oplus S(y) = ((x-y) \oplus S(y))^{-1} = (((x-y) \oplus y)')^{-1} =$
$(x - y) \oplus y = x + x.$ Hence $(x - y) \oplus y = x + x.$

(xii) $x = y \rightarrow x - y = 0$ by (ix). $x - y = 0 \rightarrow (x - y) \oplus y = 0 \oplus y =$
$y + y = x + x \rightarrow x = y$ by (v).

LEMMA 23. *Every open* PR-*formula is provably equivalent, in*
RIE(PRE + SA), *to an equation between* PR-*terms.*

Proof. From (xii) of Lemma 22, using the K of Lemma 11.

THEOREM 7. PRA *is provably equivalent to each of the following.*
(i) (the PR-consequences of) PRE' + $S(x) \neq 0 + AIO'$ (ii) RIE'(PRE') +
$S(0) \neq 0.$ (iii) PRE + $S(0) \neq 0$ + AIE.   (iv) RIE(PRE + $(x = 0 \vee$
$S(P(x))=x))+S(0)\neq0.$ (v) RIE(PRE+$(x \neq 0 \rightarrow (\exists y)(S(y)=x)))+S(0)\neq0.$
(vi) RIO(PRE) + $S(0) \neq 0.$

Proof. (i) $\rightarrow$ (ii), (i) $\rightarrow$ (iii) are obvious. Since RIE'(PRE') $\vdash$ $(x \neq$
$0 \rightarrow (\exists y)(S(y) = x))$, we have (ii) $\rightarrow$ (iv). To see that (iii) $\rightarrow$ (iv), it
suffices to show that PRE + $S(0) \neq 0$ + AIE $\vdash$ $(x \neq 0 \vee S(P(x)) = x).$
Arguing in PRE + $S(0) \neq 0$ + AIE, assume $x \neq 0$, $S(P(x)) \neq x.$ Now
$x \neq 0$, $x \neq y \rightarrow x \neq S(y).$ Hence $x \neq y$, which is a contradiction.

Obviously, (iv) $\rightarrow$ (v). Towards showing (v) $\rightarrow$ (vi), we first show
that (v) $\rightarrow$ RIE(PRE + SA). This follows immediately from the fact
that the set of equational consequences of T is always the same as
the set of equational consequences of $T + S(0) \neq 0$, whenever $T \vdash S(0) =$
$0 \rightarrow x = 0.$

Next we show that RIE(PRE + SA) $\vdash$ RIO(PRE). It suffices to
show that the set of consequences of RIE(PRE + SA) is closed under
RIO.   Assume RIE(PRE + SA) $\vdash$ $A(0)$ & $(A(x) \rightarrow A(S(x)))$, where A is
open. By Lemma 23, let α, β, be such that RIE(PRE + SA) $\vdash$ $A(x) \leftrightarrow$
$\alpha(x) = \beta(x).$ Then RIE(PRE + SA) $\vdash$ $\alpha(0) = \beta(0)$ & $(\alpha(x) = \beta(x) \rightarrow$
$\alpha(S(x)) = \beta(S(x))).$ Hence RIE(PRE + SA) $\vdash$ $\alpha(x) = \beta(x)$, and so
RIE(PRE + SA) $\vdash$ $A.$ Therefore (v) $\rightarrow$ (vi).

To see that (vi) $\rightarrow$ PRA, it suffices to prove that the set of
consequences of RIO(PRE) + $S(0) \neq 0$ is closed under RIO. This is
an immediate consequence of the fact that whenever an open sentence
is added to a theory closed under RIO, the result is closed under
RIO.

We have now shown that (i) $\rightarrow$ (ii) $\rightarrow$ (iv) $\rightarrow$ (v) $\rightarrow$ (vi) $\rightarrow$ PRA, and
(i) $\rightarrow$ (iii) $\rightarrow$ (iv) $\rightarrow$ (v) $\rightarrow$ (vi) $\rightarrow$ PRA. To complete the proof, we have
only to show that PRA $\rightarrow$ (i).

In [5], p. 349, it is shown that PRA $\vdash$ AIO. (We call the very
interesting related Theorem 2.2 of [5] to the attention of the reader.)

It therefore suffices to show that $PRE' + S(x) \neq 0 + AIO'$ and $PRE + S(x) \neq 0 + AIO$ have the same provable PR-formulae. It is well known that there is a p.r. function symbol H which plays the role of ψ in PRA; i.e., $PRA \vdash H(x, y, z, z) = x \,\&\, (z \neq w \rightarrow H(x, y, z, w) = y)$. Using H, we can translate each PR'-formula A into a PR-formula A^*, (so that $A^* = A$ for PR-formulae A), and verify by induction that $PRE' + S(x) \neq 0 + AIO' \vdash A$ implies $PRE + S(x) \neq 0 + AIO \vdash A^*$.

As would be expected, PRA is not equivalent to $PRE' + SA$. In fact, $PRE' + SA$ does not prove $G(x) = S(x)$, where G is introduced by $G(0) = S(0)$, $G(S(x)) = S(G(x))$. This will be an immediate consequence of Theorem 8.

A model $\mathscr{A}$ of $PRE'_X + SA$ is called *weakly saturated* just in case its relational type is X, and for each $k + 2$-ary function symbol $H \in X$, $0 \leq k$, and each $x_1, \cdots, x_k, y \in |\mathscr{A}|$, there is a sequence $\{y_0, y_1, \cdots\}$ of elements of $|\mathscr{A}|$ such that for $n \geq 0$, we have $\mathscr{A} \models H(x_1 \cdots, x_k, P^{n+1}(y), y_{n+1}) = y_n$.

LEMMA 24. *Every model of* $PRE'_X + SA$ *has an elementary extension which is a weakly saturated model of* $PRE'_X + SA$.

Proof. By a routine use of the compactness and elementary chain theorems.

LEMMA 25. *Let X be closed (with or without ψ). Then every model of* $PRE'_X + SA$ *has an elementary extension which is a model of* $PRE' + SA$.

Proof. By an elementary chain argument. Let $\mathscr{A}_0$ be the original model of $PRE'_X + SA$. Take $\mathscr{A}_{2n+1}$ to be a weakly saturated elementary extension of $\mathscr{A}_{2n}$ in the language of $\mathscr{A}_{2n}$. Take $\mathscr{A}_{2n+2}$ to be an expansion of $\mathscr{A}_{2n+1}$ which satisfies PRE'_Y, where Y includes all the function symbols interpreted by $\mathscr{A}_{2n+1}$, together with all the function symbols immediately derivative of these.

THEOREM 8. *Let X be a closed set of extended* p.r. *function symbols (with or without ψ). Then $PRE' + SA$ is a conservative extension of* $PRE'_X + SA$ *for all X-formulae.*

Proof. Immediate from Lemma 25.

Another easy consequence of Theorem 8 is that the commutative law of addition is not provable in $PRE' + SA$.

We conclude with a couple of open questions.

Is the set of all open consequences of PRE decidable? We do not even know whether the set of all consequences of PRE of the form $\alpha = \beta \rightarrow S(0) = 0$ is decidable.

Is RIE(PRE) $+ S(0) \neq 0$ provably equivalent to PRA? We do not even know whether its open or equational conseqences are decidable, or even whether it implies $S(x) \neq x$, or $(x \neq 0 \rightarrow (\exists y)(S(y) = x))$.

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