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IMAGES AND PRE-IMAGES OF LOCALIZATION MAPS

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This paper treats the induced function $\mathscr{L}_*: [X, Y] \to [X, Y_P]$ when X and Y are spaces having "just enough" structure and $\mathscr{L}: Y \to Y_P$ is a localization map. Special attention is paid to the images of finite subsets of $[X, Y]$ and the preimages of finite subsets of $[X, Y_P]$. While our results are not restricted to finite $[X, Y]$ we do, as an immediate corollary of our main theorem, establish necessary and sufficient conditions for the finiteness of $[X, Y_P]$.

1. Introduction. If Y is a homotopy-associative H-space[1] it is relatively easy to study properties of $\mathscr{L}_*: [X, Y] \to [X, Y_P]$ where P is a set of prime integers and $\mathscr{L}: Y \to Y_P$ is a localization map [1], [7]. The most obvious technique to use is to note that $[X, Y]$ and $[X, Y_P]$ are nilpotent groups, $\mathscr{L}_*$ is a homomorphism and $[X, Y_P]$ is isomorphic to $[X, Y]_P$, the localization of the group $[X, Y]$. For example if Y is a homotopy associative H-space of type $(n_1, \cdots, n_r)$ and X is finite CW, it is immediate that $[X, Y]$ is finite if, and only if $\bigoplus_{i=1}^r H^{n_i}(X; Q) = 0$. Even without finiteness restrictions there are numerous results which can be based on the structure of finitely generated nilpotent groups [2].

At the other exteme, there is in general little that one can say about the system $\mathscr{L}_*: [X, Y] \to [X, Y_P]$ if there are no restrictions on Y. A notable exception to this statement are the results of Hilton, Mislin and Roitberg [3].

In the present paper, while some of our results are general and overlap [3], we are primarily concerned with spaces Y which have "just enough structure." This class of spaces is essentially the p-universal spaces of [6], and in particular contains all rational H-spaces and co-H-spaces (that is, spaces whose localization at the empty set is an H-space or co-H-space). While our results are not restricted to finite $[X, Y]$ we do, as an immediate corollary of our main theorem, prove:

Let Y be a finitely generated rational H-space or co-H-space, let X be a finite CW-space and let $\mathscr{L}: Y_P \to Y_S$ be a localization map ($P \supset S$ are subsets, not necessarily proper, of the set of prime integers). Then $[X, Y_P]$ is finite if, and only if $[X, Y_S]$ is finite. Furthermore, $\mathscr{L}_*: [X; Y_P] \to [X; Y_S]$ is epic.

[1] Unless otherwise noted all spaces are assumed to be of finite type and simple, or the P-localizations of such spaces.

The paper is divided as follows. Section 2 contains the notation. definitions, and basic results we need. The third section is devoted to proving an algebraic result concerning the relationship between homotopy-principal fibrations and localizations. In §4 we prove the basic theorem of this paper. The final section is devoted to corollaries and applications of this theorem.

2. **Notation.** If P is a set of prime numbers, let P' denote the complementary set of primes, and $\langle P\rangle$ the multiplicative set of integers generated by P(i.e., all products of powers of members of P). Let Z be the group (or ring) of integers, Q the rational numbers, and Z_P the rational numbers with denominators in $\langle P'\rangle$. If n is an integer, let Z/n denote the group of integers modulo n. An integer is said to be *prime* to P if it is prime to each member of P, and a finite group is *prime* to P if its order is prime to P.

A *map* is a continuous, pointed function between pointed topological spaces. We do not make a notational distinction between maps and their homotopy classes. As usual, $[X, Y]$ denotes the set of homotopy classes of maps $X \rightarrow Y$.

An abelian group G is called *P-local* if G is isomorphic to $G \otimes Z_P$. The reader is referred to [2] for the extension of this notion to nilpotent groups. A pointed space is called *P-local* if its homotopy groups (hence its homology groups) are *P-local*. A map $\mathscr{L}\colon X \rightarrow Y_P$ is called a *localization map* if

(1) X_P is a *P-local* pointed space, and

(2) For each map f of X into a P-local space Y, there is a unique map $g\colon X_P \rightarrow Y$ with $f = g \circ \mathscr{L}$.

The reader is referred to [7], [1] and [5] for more complete discussions of this topic. In particular, every simple, pointed CW-space X admits a localization map $\mathscr{L}\colon X \rightarrow X_P$ with X_P a pointed, simple CW-space. The space X_P is unique up to homotopy equivalences and is called the *localization* of X. It is immediate that, for spaces X and Y admitting localizations, a map $h\colon X \rightarrow Y$ induces a map $h_P\colon X_P \rightarrow Y_P$ that commutes with the given localization maps. The map h is a *P-equivalence* provided h_P is a weak homotopy equivalence.

A simple, pointed CW-space Y is called *P-universal* if, for each finite abelian group G prime to P and for each $N > 0$, there is a P-equivalence h on Y such that $\pi_n(h) \otimes 1$ is trivial on $\pi_n(Y) \otimes C$ for all $n \leqq N$. It is known that H-spaces and co-H-spaces are P-universal for all P [6]. Furthermore, if Y is P-universal and $\{Y_n\}$ is a Postnikov system for Y, then each Y_n is P-universal and the induced maps h_n are compatible with the system [4].

Two path-connected, pointed CW-spaces X and Y form a *finite obstruction pair* if the groups

$$H^{n+i}(X; \pi_n(Y)) \qquad i = 0, 1$$

are finitely generated for all n and are trivial for large n. The least integer N such that $H^{n+i}(X; \pi_n(Y)) = 0$ for $n > N$ and $i = 0, 1$ is called the *dimension* of the pair.

THEOREM 2.1. *If X, Y is a finite obstruction pair and if Y is a homotopy-associative H-space, then $[X, Y]$ is a finitely generated nilpotent group.*

Proof. The proof is by induction on the index n of the Postnikov complex Y_n of Y. The result is trivial for the one-point space Y_0. Suppose $[X; Y_{n-1}]$ is finitely generated and nilpotent. Let $p: Y_n \to Y_{n-1}$ be the fibering induced by the Postnikov invariant $k: Y_{n-1} \to K(\pi_n, n+1)$ and let $i: K(\pi_n, n) \to Y_n$ be the inclusion map of the fibre into the total space. Since i and p are multiplicative maps of H-spaces, the sequence,

$$[X; K(\pi_n, n)] \xrightarrow{i_*} [X; Y_n] \xrightarrow{p_*} [X; Y_{n-1}]$$

is an exact sequence of groups and homomorphisms. Note that

$$[X; K(\pi_n, n)] = H^n(X; \pi_n(Y))$$

is a finitely generated abelian group. Thus $[X; Y_n]$ is a central extension of the finitely generated abelian group $im(i_*)$ by a finitely generated nilpotent group $im(p_*)$, and hence is finitely generated and nilpotent. Since $[X; K(\pi_n, n)]$ and $[X; K(\pi_n, n+1)]$ are eventually trivial, $[X; Y] = [X; Y_n]$ for n sufficiently large, whence the result follows. It should be noted that this result is a generalization of a well known result of G. W. Whitehead [8].

3. **Principal fibrations and localizations.** Let (E, F) be a pair of path-connected pointed spaces with $F \subset E$. A map $\phi: F \times E \to E$ is called a *principal action* provided:

(1) There is a homotopy-associative H-structure $\mu: F \times F \to F$ whose composite with the inclusion $F \subset E$ is homotopic to $\phi \mid F \times F$, and

(2) The diagram

$$\begin{array}{ccc} F \times F \times E & \xrightarrow{\mu \times 1} & F \times E \\ \downarrow{\scriptstyle 1 \times \phi} & & \downarrow{\scriptstyle \phi} \\ F \times E & \xrightarrow[\phi]{} & E \end{array}$$

is homotopy-commutative.

This situation arises when F is the space of loops on some space C and E is the total space of a fibering $p: E \to B$ induced by a map $B \to C$. Our applications are to such fiberings.

Suppose P is a collection of primes. Let $\mathscr{L}: E \to E_P$ and $\mathscr{L}': F \to F_P$ be localization maps. Then there is no loss of generality in regarding F_P as a subspace of E_P and $\mathscr{L}'$ as a restriction of $\mathscr{L}$. Recall that $(F \times E)_P$ has the homotopy type of $F_P \times E_P$. [1, p. 34], [7], [5].

LEMMA 3.1. *The localization* $\phi_P: F_P \times E_P \to E_P$ *is a principal action.*

The proof is a sequence of routine applications of the universality of localization maps.

Let X be a CW-space. If $f \in [X, E]$ and $\tau \in [X, F]$, let $\tau_* f \in [X, E]$ be the homotopy class, $\tau_* f = \phi_*(\tau \times f)$. Define

$$K_f = \{\tau \in [X, F] \mid \tau_* f = f\} \quad \text{and} \quad [X, E]_f = \{\tau_* f \mid \tau \in [X, F]\} .$$

It is immediate that K_f is a subgroup of the group $[X, F]$ and that the map $\Phi: [X, F] \to [X, E]_f$ defined by $\Phi(\tau) = \tau_* f$ is surjective.

LEMMA 3.2. *If* K_f *is normal in* $[X, F]$, *then* $[X, E]_f$ *has a group structure; it is the quotient of* $[X, F]$ *by* K_f.

Proof. Define the product on $[X, E]_f$ by $(\tau_{1*} f)(\tau_{2*} f) = (\tau_1\tau_2)_* f$. This is well-defined since K_f is normal and the action is associative. One easily verifies that Φ is a homomorphism whose kernel is K_f.

A *morphism* of principal actions is defined to be a map $g: (E, F) \to (E', F')$ such that $\phi' \circ ((g \mid F) \times g)$ is homotopic to $g \circ \phi$. Note that this is equivalent to requiring the diagram,

$$\begin{array}{ccc} [X, F] \times [X, E] & \xrightarrow{g|_* \times g_*} & [X, F'] \times [X, E'] \\ \phi \downarrow & & \downarrow \phi' \\ [X, E] & \xrightarrow[g_*]{} & [X, E'] \end{array}$$

to commute for all spaces X. If K_f is normal in $[X, F]$ and $K_{g_* f}$ is normal in $[X, F']$, then it follows easily from Lemma 3.2 that g induces a homomorphism $g_*: [X, E]_f \to [X, E']_{g_* f}$. In particular, a localization map $\mathscr{L}: (E, F) \to (E_P F_P)$ induces such a homomorphism.

THEOREM 3.3. *Suppose* $\phi: F \times E \to E$ *is a principal action and that* X *is a* CW*-space such that* X, F *and* X, E *are finite obstruc-*

tion pairs. If K_f is normal, then a localization $\mathscr{L}: (E, F) \to (E_P, F_P)$ induces an isomorphism of the local group $([X, E]_f)_P$ onto $[X, E_P]_{\mathscr{L}_* f}$ and

$$\ker(\mathscr{L}_*: [X, E]_f \longrightarrow [X, E_P]_{\mathscr{L}_* f})$$

is finite of order prime to P.

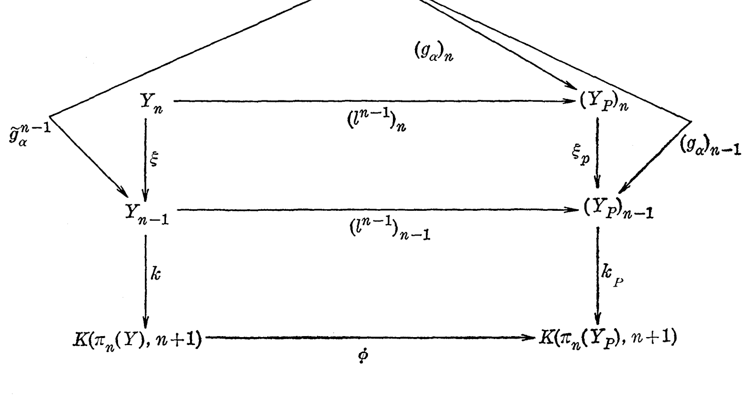
Proof. Since F is a homotopy-associative H-space and X, F is a finite obstruction pair, $[X, F]$ is a finitely generated nilpotent group. Thus $[X, F]_P$ is naturally isomorphic to $[X, F_P]$. It then follows that K_f is finitely generated and nilpotent, and that $(K_f)_P$ is isomorphic to $K_{\mathscr{L}_* f}$, whence $[K, E]_f$ is finitely generated nilpotent, and $([X, E]_f)_P$ is isomorphic to $[X, E_P]_{\mathscr{L}_* f}$, whence $\ker \mathscr{L}_*$ is finitely generated, nilpotent and every element has finite order. It follows at once that $\ker \mathscr{L}_*$ is finite. A result of Hilton [2, p. 266] shows that each element has order prime to P.

4. The main theorem.

THEOREM 4.1. *Let P be a set of primes. Suppose X, Y is a finite obstruction pair with Y P-universal. If $\{g_\alpha \mid \alpha \in A\} \subset [X, Y_P]$ is a finite set of homotopy classes, then there exist a localization map $L: Y \to Y_P$ and classes $\tilde{g}_\alpha \in [X, Y]$ such that $L_*(\tilde{g}_\alpha) = g_\alpha$. Furthermore, for any localization map $\mathscr{L}: Y \to Y_P$, the sets $\mathscr{L}_*^{-1}(g_\alpha)$ are finite.*

Proof. The proof is by induction on the dimension of the finite obstruction pair X, Y. Note that if the dimension is n, there is no loss in replacing Y and Y_P by their Postnikov complexes Y_n and $(Y_P)_n$, respectively. Since $Y_0 = (Y_P)_0$ is a single point, the initial step is trivial.

Suppose the result has been established for pairs X, Y of dimen-



sion $\leqq n-1$. In particular, given a pair X, Y of dimension n, assume we have maps $\tilde{g}_\alpha^{n-1}: X \to Y_{n-1}$ and a localization map $\mathscr{L}^{n-1}: Y \to Y_P$ such that $(\mathscr{L}^{n-1})_{n-1*}\tilde{g}_\alpha^{n-1} = (g_\alpha)_{n-1}$. These fit into the homotopy-commutative diagram above where k, k_P are maps corresponding to Postnikov invariants and $\phi = K(\pi_n(\mathscr{L}^{n-1}), n+1)$ is a map between Eilenberg-MacLane spaces corresponding to the homomorphism $\pi_n^{(\mathscr{L}^{n-1})}: \pi_n(Y) \to \pi_n(Y_P)$. The existence of maps $(g_\alpha)_{n-1}$, $(g_\alpha)_n$, $(\mathscr{L}^{n-1})_{n-1}$ and $(\mathscr{L}^{n-1})_n$ induced by g_α and $\mathscr{L}^{n-1}$ and making the diagram homotopy-commutative is insured by the construction of localization [7].

Since $(g_\alpha)_{n-1}$ has a lifting g_α, it holds that $k_P(g_\alpha)_{n-1} = \phi k \tilde{g}_\alpha^{n-1}$ is trivial for each α. Thus there is an integer m_α prime to P such that

$$m_\alpha(k\tilde{g}_\alpha^{n-1}) \in [X; K(\pi_n(Y), n+1)] = H^{n+1}(X; \pi_n(Y))$$

is trivial. Let m be the product $m = \Pi\{m_\alpha \mid \alpha \in A\}$. Since Y is P-universal, there is a P-equivalence h on Y such that $\pi_n(h) \otimes 1$ is trivial on $\pi_n(Y) \otimes (Z/m)$. Note that $kh_{n-1}\tilde{g}_\alpha^{n-1}$ is trivial for each α, whence $h_{n-1}\tilde{g}_\alpha^{n-1}$ has a lifting $f_\alpha: X \to Y_n$. Since h is a P-equivalence, the induced map h_P on Y_P is a homotopy equivalence. Let h_P^{-1} be a homotopy inverse and set $\mathscr{L}' = h_P^{-1}\mathscr{L}^{n-1}$. Then,

$$\xi_P(\mathscr{L}')_n f_\alpha = (h_P^{-1})_{n-1}(\mathscr{L}^{n-1})_{n-1}h_{n-1}\tilde{g}_\alpha^{n-1} = \xi_P(g_\alpha)_n \ .$$

Since ξ_P is a principal fibration with fibre $K(\pi_n(Y_P), n)$, there is a map $\tau_\alpha: X \to K(\pi_n(Y_P), n)$ such that

$$\tau_\alpha * ((\mathscr{L}')_n f_\alpha) = (g_\alpha)_n \ ,$$

where the asterisk denotes the action of the fibre on the total space $(Y_P)_n$. But there exists an integer q_α prime to P and a map $\tau'_\alpha: X \to K(\pi_n(Y), n)$ such that the homotopy class τ'_α maps onto $q_\alpha\tau_\alpha$ under the localization map,

$$\theta = K(\pi_n(\mathscr{L}'), n): K(\pi_n(Y), n) \longrightarrow K(\pi_n(Y_P), n) \ .$$

Let $q = \Pi\{q_\alpha \mid \alpha \in A\}$ and let H be a P-equivalence on Y such that $\pi_n(H) \otimes 1$ is trivial on $\pi_n(Y) \otimes (Z/q)$. Thus each element in the image of $\pi_n(H)$ is divisible by q, and the induced map H_* on $[X; K(\pi_n(Y), n)] = H^n(X; \pi_n(Y))$ has the same property. For each α, let $\tau''_\alpha \in [X; K(\pi_n(Y), n)]$ be an element such that $q\tau''_\alpha = H_*(\tau'_\alpha)$. Note that q/q_α is an integer, and let $\tilde{\tau}_\alpha = (q/q_\alpha)\tau''_\alpha$. As before, H induces a homotopy equivalence H_P on Y_P and we select a homotopy inverse H_P^{-1}. Let $L = H_P^{-1}\mathscr{L}'$ and set $\tilde{g}_\alpha^n = \tilde{\tau}_\alpha * (H_n f_\alpha)$. If we replace Y_n by Y we obtain

$$L\tilde{g}_\alpha = H_P^{-1}\mathscr{L}'(\tilde{\tau}_\alpha * (Hf_\alpha)) = H_P^{-1}((\theta\tilde{\tau}_\alpha)*(H_P\mathscr{L}'f_\alpha))$$

Now,

$$(H_P^{-1})_*\theta\tilde{\tau}_\alpha = (H_P^{-1})_*(q/q_\alpha)\theta_*(\tau''_\alpha) = (1/q_\alpha)(H_P^{-1})_*\theta_*H_*\tau'_\alpha = \tau_\alpha ,$$

whence,

$$L\tilde{g}_\alpha = \tau_{\alpha*}(H_P^{-1}H_P\mathscr{L}'f_\alpha) = g_\alpha .$$

This establishes the existence of a localization L such that $L_*^{-1}(g_\alpha)$ is nonempty.

Let $\mathscr{L}: Y_n \to (Y_P)_n$ be any localization and let $\alpha \in A$. By the inductive hypothesis, we know that $\xi_*(\mathscr{L}_*^{-1}(g_\alpha)n) = (\mathscr{L}_{n-1})_*^{-1}(g_\alpha)_{n-1}$ is a finite set. But if $f, f' \in \mathscr{L}_*^{-1}(g_\alpha)_n$ and $\xi_*f = \xi_*f'$, then there exists a class $\tau \in [X; K(\pi_n(Y), n)]$ such that $\tau_*f' = f$. Thus by (3.3) f' lies in the finite set $\ker(\mathscr{L}_*: [X, Y_n]_f \to [X, (Y_P)_n]_{(g_\alpha)n})$ and $\mathscr{L}_*^{-1}(g_\alpha)_n$ is finite.

Remarks. It should be noted that the commutativity of maps between Postnikov decompositions is critical at two points in the above proof. This may be seen by the fact that it is necessary to replace the initial localization map by a different one in two steps and each time require that the diagrams commute so that we may continue the induction process.

We also remark that ([5], 5.3, p. 607) contains a simply-connected version of the first part of Theorem 4.1. We would also like to note that the proof of 4.1 goes over, mutatis mutandis, to the more general class of nilpotent spaces.

Futhermore note that given $\{g_\alpha \mid \alpha \in A\}$ a finite subset of $[X, Y_P]$ the P-universality of Y was only used to assure the existence of a localization map $L: Y \to Y_P$ and of classes $\{\tilde{g}_\alpha\} \in [X, Y]$ with $L_*(\tilde{g}_\alpha) = g_\alpha$. Thus we get:

Corollary 4.2. *Let $\mathscr{L}: Y \to Y_P$ be any localization map with X, Y a finite obstruction pair. If $\{g_\alpha \mid \alpha \in A\} \subseteqq [X, Y_P]$ is finite then $\mathscr{L}_*^{-1}(\{g_\alpha \mid \alpha \in A\})$ is finite.*

This is essentially the same statement as [3, Corollary 2.2].

5. Applications.

If Y is P-universal, then Y_Q is P-universal for all sets of primes $Q \supset P$. Further, X finite and Y having finitely generated homotopy in each dimension implies that X, Y is a finite obstruction pair. Therefore as an immediate corollary of 4.1 we have:

Theorem 5.1. *Let X be a finite, pointed CW-space. Let Y be P-universal and have finitely generated homotopy in each dimension. If $\{g_\alpha \mid \alpha \in A\}$ is a finite subset of $[X, Y_P]$, then there exists a localization map $L: Y \to Y_P$ and homotopy classes $\tilde{g}_\alpha \in [X, Y]$ such that*

$L_*(\tilde{g}_\alpha) = g_\alpha$. *Further, for any localization map* $\mathscr{L}: Y \to Y_P$, *the sets* $\mathscr{L}_*^{-1}(g_\alpha)$ *are finite.*

If X, Y is a finite obstruction pair and Q is a set of primes then X_Q, Y_Q is not necessarily a finite obstruction pair. However the proof of 4.1 extends immediately to give

THEOREM 5.2 *Let X be a finite, pointed CW-space. Let Y be P-universal for some set of primes P and have finitely generated homotopy in each dimension. Let Q be a set of primes containing P. If* $\{g_\alpha \mid \alpha \in A\}$ *is a finite subset of* $[X, Y_P]$, *then there exists a localization map* $L: Y_Q \to Y_P$ *and homotopy classes* $\tilde{g}_\alpha \in [X, Y_Q]$ *such that* $L_*(\tilde{g}_\alpha) = g_\alpha$. *Further, for any localization map* $\mathscr{L}: Y_Q \to Y_P$ *the sets* $\mathscr{L}_*^{-1}(g_\alpha)$ *are finite.*

THEOREM 5.3. *Let P be a set of primes and X, Y a finite obstruction pair with Y P-universal. Let* $\mathscr{L}: Y \to Y_P$ *be any localization. Then* $[X, Y]$ *is finite if, and only if,* $[X, X_P]$ *is finite. Furthermore,* $\mathscr{L}_*: [X, Y] \to [X, Y_P]$ *is epic.*

Proof. First assume $[X, Y]$ is finite of cardinality r. If $[X, Y_P]$ is not finite, let $g_1 \cdots, g_{r+1}$ be $r+1$ distinct homotopy classes in $[X, Y_P]$. By 4.1 there exist $\tilde{g}_i \in [X, Y]$ and a localization map $L: Y \to Y_P$ with $L_*(\tilde{g}_i) = g_i$. But this implies that the cardinality of $[X, Y]$ is greater than r, a contradiction. Thus $[X, Y_P]$ is finite. If $[X, Y_P]$ is finite, then again by 4.1, there exists a localization map $L: Y \to Y_P$ which is epic. Furthermore, $L_*^{-1}(g)$ is finite for each $g \in [X, Y_P]$. Thus $[X, Y]$ is finite.

Finally, let $\mathscr{L}: Y \to Y_P$ be a localization map. Since any two localization maps differ by a homotopy equivalence, we have $\mathscr{L} = L \circ h$, where L_* is epic and h_* is an isomorphism of sets. Therefore, $\mathscr{L}_*: [X, Y] \to [X, Y_P]$ is epic.

COROLLARY 5.4. *Let X, Y be a finite obstruction pair. Then for any set of primes P, the set* $[X, Y]$ *is finite if* $[X, Y_P]$ *is finite.*

Proof. Immediate by 4.2.

The example, $X = Y = S^n$ and P any set of primes (except the set of all primes), shows that a localization need not induce a surjection when $[X, Y]$ is infinite.

As noted before, any H-space or co-H-space is P-universal for all P [6]. Therefore we get:

COROLLARY 5.5. *Let X be a finite, pointed CW-space and let Y be an H-space or co-H-space whose homotopy is finitely generated in each dimension. Then for any set P of primes and any localization map $\mathscr{L}: Y \to Y_P$, $[X, Y]$ is finite if, and only if, $[X, Y_P]$ is finite. Furthermore, $\mathscr{L}_*: [X, Y] \to [X, Y_P]$ is epic.*

In particular, if Y is a finite CWH-space of type $(n_1, \cdots, n_r)$, then the number of homotopy classes of multiplications on Y is the cardinality of $[Y \wedge Y, Y]$. Since $Y_0 \cong \prod_{i=1}^r K(Q, n_i)$ we have $[Y \wedge Y, Y] = \bigoplus_{i=1}^r H^{n_i}(Y \wedge Y; Q)$. Finally, since 0 is the only finite Q-module, we get:

COROLLARY 5.6. *Let Y be a finite CWH-space of type $(n_i, \cdots, n_r)$. Y has finitely many homotopy classes of multiplications if, and only if, $H^{n_i}(Y \wedge Y; Q) = 0$ for $i = 1, \cdots, r$.*

We close this paper with the following conjectures.

Conjecture 5.7. $[X, Y]$ is finite if and only if $[X, Y_P]$ is finite.

Note that in the light of 4.2 this is equivalent to the statement that $[X, Y_0]$ is finite if $[X, Y]$ is finite and is therefore a question about the rationalization of the spaces Y.

Conjecture 5.8. If Y is a rational space (i.e., $H_*(Y; Z) = H_*(Y; Q)$) then $[X, Y]$ is either infinite or a single element.

If Y is an H-space then $Y \simeq \prod_{i \geq 1} K(Q^{n_i}, i)$ and hence has a multiplication such that $[X, Y]$ is isomorphic to a Q module. Thus 5.8 holds for H-spaces

Theorem 3.3 when applied to the proof of 4.1 shows that at every stage of the Postnikov system the indeterminacy in the choice of the lifting of a map $X \to Y_{n-1}$ to $X \to Y_n$ is finite of order prime to P this suggests:

Conjecture 5.9. Let Y be P-universal and $[X, Y]$ finite. Then there exists an integer m such that $\#[X, Y] = m(\#[X, Y_P])$ and m is prime to P.

Combining 5.8 and 5.9 we could then prove,

Conjecture 5.10. Let Y be P-universal and $[X, Y]$ finite. Then for each prime p, $\#[X, Y_P]$ is a power of p.

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