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A THREE-TERM RELATION FOR SOME SUMS RELATED TO DEDEKIND SUMS

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The object of the paper is to evaluate

$$\sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a}-y\right]\left[\frac{c(r+x)}{a}-z\right]$$
$$+\sum_{s=0}^{b-1}\left[\frac{c(s+y)}{b}-z\right]\left[\frac{a(s+y)}{b}-x\right]$$
$$+\sum_{t=0}^{c-1}\left[\frac{a(t+z)}{c}-x\right]\left[\frac{b(t+z)}{c}-y\right].$$

where $[u]$ **is the greatest integer function,**

$$(b, c) = (c, a) = (a, b) = 1 .$$

and

$$0 \leqq x < 1 , \quad 0 \leqq y < 1 , \quad 0 \leqq z < 1 .$$

1. Introduction. Put

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}) . \end{cases} \tag{1.1}$$

The Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{h}\right)\right) . \tag{1.2}$$

It is well known that $s(h, k)$ satisfies the reciprocity relation [5]

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) , \tag{1.3}$$

for $(h, k) = 1$.

Rademacher [5] has proved the three-term relation

$$s(bc', a) + s(ca', b) = s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) , \tag{1.4}$$

where

$$(b, c) = (c, a) = (a, b) = 1 \tag{1.5}$$

and

$$aa' \equiv 1 \,(\mathrm{mod}\, bc) , \quad bb' \equiv 1 \,(\mathrm{mod}\, ca) , \quad cc' \equiv 1 \,(\mathrm{mod}\, ab) . \tag{1.6}$$

Rademacher [4] has introduced the sum

$$s(h, k \mid x, y) = \sum_{r \pmod k} \left(\left(h\frac{r+y}{k} + x\right)\right)\left(\left(\frac{r+y}{k}\right)\right) \tag{1.7}$$

and proved the reciprocity formula

$$\begin{aligned} &s(h, k \mid x, y) + s(k, h \mid y, x) \\ &\quad = ((x))((y)) + \frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y) + \frac{1}{hk}\bar{B}_2(hy + kx) + \frac{k}{h}\bar{B}_2(x)\right\}, \end{aligned} \tag{1.8}$$

where $(h, k) = 1$, x and y are not both integers and $\bar{B}_2(x) = B_2(x - [x])$, where $B_2(x) = x^2 - x + 1/6$, the Bernoulli polynomial of degree 2.

In order to get a three-term relation for the generalized sum $s(h, k \mid x, y)$, the writer [2] defined

$$s(a, b, c; x, y, z) = \sum_{r \pmod c} \Phi\left(a\frac{t+z}{c} - x\right)\Phi\left(y - b\frac{t+z}{c}\right), \tag{1.9}$$

where

$$\Phi(x) = x - [x] - \frac{1}{2},$$

and proved that

$$\begin{aligned} &s(a, b, c; x, y, z) + s(b, c, a; y, z, x) + s(c, a, b; z, x, y) \\ &\quad = \delta - \frac{a}{2bc}\bar{B}_2(cy - bz) - \frac{b}{2ca}\bar{B}_2(az - cx) - \frac{c}{2ab}\bar{B}_2(bx - ay), \end{aligned} \tag{1.10}$$

where $(b, c) = (c, a) = (a, b) = 1$ and $\delta = 1$ if integers r, s, t exist such that

$$\frac{r+x}{a} = \frac{s+y}{b} = \frac{t+z}{c}; \quad 0 \leqq r < a, \quad 0 \leqq s < b, \quad 0 \leqq t < c;$$

$\delta = 0$ otherwise. For $c = 1$, $z = 0$, it is easily verified that (1.10) reduces to (1.8).

The reciprocity Theorem (1.3) is equivalent to [5, p. 9]

$$h\sum_{r=1}^{k-1} r\left[\frac{hr}{k}\right] + k\sum_{s=1}^{h-1} s\left[\frac{ks}{h}\right] = \frac{1}{12}(h-1)(k-1)(8hk - h - k - 1), \tag{1.11}$$

where $(h, k) = 1$. It is shown in [1] that (1.4) implies

$$A(b, c; a) + A(c, a; b) + A(a, b; c) = (a-1)(b-1)(c-1), \tag{1.12}$$

where $(b, c) = (c, a) = (a, b) = 1$ and

$$A(b, c; a) = \sum_{r=1}^{a-1}\left[\frac{br}{a}\right]\left[\frac{cr}{a}\right]. \tag{1.13}$$

It is not difficult to give a direct proof of (1.12) and indeed considerably more. It is also proved in [1] that

$$(1.14)\quad \begin{aligned} &6k\sum_{r=0}^{k-1}\left[\frac{hr+z}{k}\right]^2 + 6h\sum_{s=0}^{h-1}\left[\frac{ks+z}{h}\right]^2 \\ &\quad = (h-1)(2h-1)(k-1)(2k-1) + 6[z]([z]+2hk-h-k+1)\,, \end{aligned}$$

where $(h, k) = 1$ and $0 \leqq z < h + k$.

Generalizing (1.13), we define

$$(1.15)\qquad A(b, c; a \mid y, z; x) = \sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a} - y\right]\left[\frac{c(r+x)}{a} - z\right]^2 .$$

In Theorem 2 below we evaluate

$$R \equiv A(a, b; c \mid x, y; z) + A(b, c; a \mid y, z; x) + A(c, a; b \mid z, x; y)\,,$$

where $(b, c) = (c, a) = (a, b) = 1$ and

$$0 \leqq x < 1\,,\quad 0 \leqq y < 1\,,\quad 0 \leqq z < 1\,.$$

It is however convenient to first consider the sum

$$(1.16)\qquad B(b, c; a \mid y, z; x) = \sum_{r=0}^{a-1}\left[y - \frac{b(r+x)}{a}\right]\left[z - \frac{c(r+x)}{a}\right],$$

In Theorem 1 below we evaluate

$$S \equiv B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + A(c, a; b \mid z, x; y)\,,$$

where the parameters satisfy the same conditions as above.

The results are particularly simple if no two of the fractions

$$\frac{r+x}{a}\,,\quad \frac{s+y}{b}\,,\quad \frac{t+z}{c}\qquad (0 \leqq r < a,\ 0 \leqq s < b,\ 0 \leqq s < c)$$

are equal. In this case we show that

$$(1.17)\qquad S = abc\,,\quad R = (a-1)(b-1)(c-1) + 1\,.$$

Also, if $x = y = z$, we have

$$(1.18)\qquad S = abc - 1\,,\quad R = (a-1)(b-1)(c-1)\,.$$

The last form evidently includes (1.12).

It should be noted that (1.14) is not contained in the results of the present paper.

2. Preliminaries.

LEMMA 1. *For* $a \geqq 1$,

$$[ax] = \sum_{r=0}^{a-1}\left[x + \frac{r}{a}\right]. \tag{2.1}$$

LEMMA 2. *For $x \neq$ integer,*

$$[-x] = -[x] - 1\,. \tag{2.2}$$

LEMMA 3. *Put $\Phi(x) = x - [x] - 1/2$. Then $\Phi(x+1) = \Phi(x)$ and*

$$\Phi(ax) = \sum_{r \,(\mathrm{mod}\, a)} \Phi\left(x + \frac{r}{a}\right). \tag{2.3}$$

LEMMA 4. *For $a \geqq 1$, $(b, a) = 1$,*

$$\sum_{r=0}^{a-1}\left[x + \frac{br}{a}\right] = [ax] + \frac{1}{2}(a-1)(b-1)\,.$$

Lemmas 1, 2, 3 are well-known. To prove Lemma 4, take

$$\begin{aligned}\sum_{r=0}^{a-1}\left[x + \frac{br}{a}\right] &= \sum_{r=0}^{a-1}\left\{x + \frac{br}{a} - \Phi\left(x + \frac{br}{a}\right) - \frac{1}{2}\right\} \\ &= ax + \frac{1}{2}b(a-1) - \Phi(ax) - \frac{1}{2}a \\ &= [ax] + \frac{1}{2}ab - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2} \\ &= [ax] + \frac{1}{2}(a-1)(b-1)\,.\end{aligned}$$

3. Three term relation for $B(a, b; c \mid x, y; z)$. We assume in the remainder of the paper that

$$(b, c) = (c, a) = (a, b) = 1 \tag{3.1}$$

and

$$0 \leqq x < 1\,, \quad 0 \leqq y < 1\,, \quad 0 \leqq z < 1\,. \tag{3.2}$$

Also we write

$$\sum_{r,s,t} \equiv \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{t=0}^{c-1}.$$

It is convenient to define

$$B(a, b; c \mid x, y; z) = \sum_{t=0}^{c-1}\left[x - \frac{a(t+z)}{c}\right]\left[y - \frac{b(t+z)}{c}\right]. \tag{3.3}$$

By Lemma 1, this gives

$$B(a, b; c \mid x, y, z) = \sum_{r,s,t} \left[\frac{r+x}{a} - \frac{t+z}{c}\right]\left[\frac{s+y}{b} - \frac{t+z}{c}\right]. \tag{3.4}$$

Put

$$\xi = \frac{r+x}{a}, \quad \eta = \frac{s+y}{b}, \quad \zeta = \frac{t+z}{c}.$$

Thus (3.4) becomes

$$B(a, b; c \mid x, y; z) = \sum_{r,s,t} [\xi - \zeta][\eta - \zeta]. \tag{3.5}$$

To begin with, we assume that no two of the fractions

$$\frac{r+x}{a}, \quad \frac{s+y}{b}, \quad \frac{t+z}{c} \qquad (0 \leqq r < a,\ 0 \leqq s < b,\ 0 \leqq t < c) \tag{3.6}$$

differ by an integer. It is easily seen that this is equivalent to requiring that no two of the fractions are equal. We shall refer to this as Case 1. Case 2 is that in which exactly two of the fractions are equal, Case 3 that in which all three are equal. If a pair r, s exists such that

$$\frac{r+x}{a} = \frac{s+y}{b},$$

it is easily shown to be unique.

Thus in Case 1 we have, by Lemma 2,

$$[\xi - \zeta] = -[\zeta - \xi] - 1, \tag{3.7}$$

so that (3.5) becomes

$$B(a, b; c \mid x, y; z) = -\sum_{r,s,t} [\eta - \zeta][\zeta - \xi] - \sum_{r,s,t} [\eta - \zeta].$$

Hence

$$\begin{aligned} &B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + B(c, a; b \mid z, x; y) \\ &= -\sum_{r,s,t} \{[\eta - \zeta][\zeta - \xi] + [\zeta - \xi][\xi - \eta] + [\xi - \eta][\eta - \zeta] \\ &\quad + [\eta - \zeta] + [\zeta - \xi] + [\xi - \eta]\} \\ &= -\sum_{r,s,t} \left\{\left([\eta - \zeta] + \frac{1}{2}\right)\left([\zeta - \xi] + \frac{1}{2}\right)\right. \\ &\quad + \left([\zeta - \xi] + \frac{1}{2}\right)\left([\xi - \eta] + \frac{1}{2}\right) \\ &\quad \left. + \left([\xi - \eta] + \frac{1}{2}\right)\left([\eta - \zeta] + \frac{1}{2}\right)\right\} + \frac{3}{4} abc. \end{aligned} \tag{3.8}$$

Hence if we put

$$T = \sum_{r,s,t} \left\{ [\eta - \zeta] + [\zeta - \xi] + [\xi - \eta] + \frac{3}{2} \right\}^2 , \tag{3.9}$$

it is clear that

$$\begin{aligned} T = \sum_{r,s,t} \left\{ \left([\eta - \zeta] + \frac{1}{2} \right)^2 + \left([\zeta - \xi] + \frac{1}{2} \right)^2 \right. \\ \left. + \left([\xi - \eta] + \frac{1}{2} \right)^2 \right\} + \frac{3}{2} abc - 2S , \end{aligned} \tag{3.10}$$

where

$$S = B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + B(c, a; b \mid z, x; y) . \tag{3.11}$$

Since

$$[\eta - \zeta] + [\zeta - \xi] + [\xi - \eta] = -1 \text{ or } -2 \text{ (Case 1 or 2)} ,$$

it follows that

$$T = \frac{1}{4} abc \qquad \text{(Case 1 or 2)} . \tag{3.12}$$

Also since each of $[\eta - \zeta]$, $[\zeta - \xi]$, $[\xi - \eta] = 0$ or -1, we get

$$\sum_{r,s,t} \left\{ \left([\eta - \zeta] + \frac{1}{2} \right)^2 + \left([\zeta - \xi] + \frac{1}{2} \right)^2 + \left([\xi - \eta] + \frac{1}{2} \right)^2 \right\} = \frac{3}{4} abc .$$

Thus (3.11) reduces to simply

$$S = abc \qquad \text{(Case 1)} . \tag{3.13}$$

Turning next to Case 2, let r_0, s_0 satisfy

$$\xi_0 = \frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \eta_0 .$$

For this pair we have

$$[\eta_0 - \xi_0] = -[\xi_0 - \eta_0] = 0$$

rather than (3.7). Thus

$$\begin{aligned} B(b, c; a \mid y, z; x) = - \sum_{r,s,t} ([\zeta - \xi][\xi - \eta] \\ + [\zeta - \xi]) + \sum_t [\zeta - \xi_0] . \end{aligned} \tag{3.14}$$

Since

$$\sum_t [\zeta - \xi_0] = \sum_{t=0}^{c-1} \left[\frac{t + z}{c} - \xi_0 \right] = [z - c\xi_0] ,$$

we get, in place of (3.13),

(3.15) $$S = abc + [z - c\xi_0] \qquad (\text{Case } 2) .$$

For Case 3, let r_0, s_0, t_0 satisfy $\xi_0 = \eta_0 = \zeta_0$, that is

$$\frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \frac{t_0 + z}{c} .$$

Equation (3.14) remains unchanged. Since

$$T = \frac{1}{4} abc + 2 \qquad (\text{Case } 3) ,$$

we therefore get

(3.16) $$S = abc - 1 + [x - a\zeta_0] + [y - b\xi_0] + [z - c\eta_0] \quad (\text{Case } 3) .$$

Since

$$[x - a\zeta_0] = [x - a\xi_0] = [x - (r_0 + x)] = -r_0 , \quad \text{etc.} ,$$

we may replace (3.16) by

(3.16)′ $$S = abc - 1 - r_0 - s_0 - t_0 \qquad (\text{Case } 3) .$$

We may now state

THEOREM 1. *Let* $(b, c) = (c, a) = (a, b) = 1$ *and*

$$0 \leqq x < 1 , \quad 0 \leqq y < 1 , \quad 0 \leqq z < 1 .$$

Then

$$S = B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + B(c, a; b \mid z, x; y)$$

is evaluated by (3.13), (3.15) *and* (3.16).

In particular, for $x = y = z$, we have

COROLLARY 1.

(3.17) $$\begin{aligned} &\sum_{r=0}^{a-1}\left[x - \frac{b(r + x)}{a}\right]\left[x - \frac{c(r + x)}{a}\right] \\ &+ \sum_{s=0}^{b-1}\left[x - \frac{c(s + x)}{b}\right]\left[x - \frac{b(s + x)}{b}\right] \\ &+ \sum_{t=0}^{c-1}\left[x - \frac{a(t + x)}{c}\right]\left[x - \frac{b(t + x)}{c}\right] = abc - 1 . \end{aligned}$$

4. **Three-term relations for** $A(a, b; c \mid x, y; z)$. We again first consider Case 1. Thus in

$$A(a, b; c \mid x, y; z) = \sum_{t=0}^{c-1}\left[\frac{a(t+z)}{c} - x\right]\left[\frac{b(t+z)}{c} - y\right],$$

none of the quantities

$$\frac{a(t+z)}{c} - x\,, \quad \frac{b(t+z)}{c} - y \qquad (0 \leqq t < c)$$

is an integer. It follows that

$$\begin{aligned} A(a, b; c \mid x, y; z) &= \sum_{t=0}^{c-1}\left\{1 + \left[x - \frac{a(t+z)}{c}\right]\right\}\left\{1 + \left[y - \frac{b(t+z)}{c}\right]\right\} \\ &= c + \sum_{t=0}^{c-1}\left[x - \frac{a(t+z)}{c}\right] + \sum_{t=0}^{c-1}\left[y - \frac{b(t+z)}{c}\right] \\ &\quad + B(a, b; c \mid x, y; z)\,. \end{aligned}$$

Thus by Lemma 4, we get

$$\begin{aligned} &A(a, b; c \mid x, y; z) \\ &= [cx - az] + [cy - bz] - \frac{1}{2}ac - \frac{1}{2}bc + \frac{1}{2}a + \frac{1}{2}b + 1 \\ &\quad + B(a, b; c \mid x, y; z)\,. \end{aligned} \tag{4.1}$$

Let

$$R = A(a, b; c \mid x, y; z) + A(b, c; a \mid y, z; x) + A(c, a; b \mid z, x; y)\,. \tag{4.2}$$

Then by (4.1) and (3.13) we get

$$\begin{aligned} R &= [cx - az] + [cy - bz] + [ay - bx] + [az - cx] \\ &\quad + [bz - cy] + [bx - ay] - (bc + ca + ab) \\ &\quad + (a + b + c) + 3 + S\,. \end{aligned} \tag{4.3}$$

Applying Lemma 2, this reduces to

$$R = (a-1)(b-1)(c-1) + 1 \qquad \text{(Case 1)}\,. \tag{4.4}$$

As for Case 2, let r_0, s_0 denote the exceptional pair, that is,

$$\xi_0 = \frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \eta_0\,.$$

$$\begin{aligned} A(b, c; a \mid y, z; x) &= \sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a} - y\right]\left[\frac{c(r+x)}{a} - z\right] \\ &= \sum_{r=0}^{a-1}\left\{1 + \left[y - \frac{b(r+x)}{a}\right]\right\}\left\{1 + \left[z - \frac{c(r+x)}{a}\right]\right\} \\ &\quad - \left\{1 + \left[z - \frac{c(r_0+x)}{a}\right]\right\}. \end{aligned} \tag{4.5}$$

Then, as above, it follows that

$$\begin{aligned}A(b,\, c;\, a \mid y,\, z;\, x) &= B(b,\, c;\, a \mid y,\, z;\, x) + [ay - bx] + [az - cx] \\ &\quad + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ab - \frac{1}{2}ac - [z - c\xi_0] \,.\end{aligned} \tag{4.6}$$

Similarly

$$\begin{aligned}A(c,\, a;\, b \mid z,\, x;\, y) &= B(c,\, a;\, b \mid z,\, x;\, y) + [bx - ay] + [bz - cy] \\ &\quad + \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ab - \frac{1}{2}bc - [z - c\eta_0] \,.\end{aligned} \tag{4.7}$$

However (4.1) remains unchanged.
Thus

$$R = S - (bc + ca + ab) + (a + b + c) - 1 - 2[z - c\eta_0] \,.$$

Therefore, by (3.15), we have

$$R = (a - 1)(b - 1)(c - 1) - [z - c\eta_0] \quad \text{(Case 2)} \,. \tag{4.8}$$

Finally, for Case 3, let r_0, s_0, t_0, be the exceptional triple:

$$\frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \frac{t_0 + z}{c} \quad (\xi_0 = \eta_0 = \zeta_0) \,. \tag{4.9}$$

We now have

$$\begin{aligned}A(b,\, c;\, a \mid y,\, z;\, x) &= \sum_{r=0}^{a-1}\left\{1 + \left[y - \frac{b(r + x)}{a}\right]\right\}\left\{1 + \left[z - \frac{c(r + x)}{a}\right]\right\} \\ &\quad - 1 - [y - b\xi_0] - [z - b\xi_0] \,.\end{aligned}$$

Thus in place of (4.6) we get

$$\begin{aligned}&A(b,\, c;\, a \mid y,\, z;\, x) \\ &\quad = B(b,\, c;\, a \mid y,\, z;\, x) + [ay - bx] + [az - cx] \\ &\quad + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ab - \frac{1}{2}ac - [y - b\xi_0] - [z - c\xi_0] \,.\end{aligned} \tag{4.10}$$

There are similar formulas for $A(c,\, a;\, b \mid z,\, x;\, y)$ and $A(a,\, b;\, c \mid x,\, y;\, z)$. Since

$$\begin{aligned}[cy - bz] + [bz - cx] &= [az - cx] + [cx - az] \\ &= [bx - ay] + [ay - bx] = 0 \,,\end{aligned}$$

it follows that

$$\begin{aligned} R &= S + a + b + c - bc - ca - ab - [y - b\xi_0] - [z - c\xi_0] \\ &\quad - [z - c\eta_0] - [x - a\eta_0] - [x - a\zeta_0] - [y - b\zeta_0] \\ &= S + a + b + c - bc - ca - ab \\ &\quad - 2[x - a\xi_0] - 2[y - b\eta_0] - 2[z - c\zeta_0] \, . \end{aligned}$$

Making use of (3.16) we get

$$R = (a-1)(b-1)(c-1) - [x - a\xi_0] - [y - b\eta_0] - [z - c\zeta_0] \quad \text{(Case 3)} \tag{4.11}$$

or, if we prefer,

$$R = (a-1)(b-1)(c-1) + r_0 + s_0 + t_0 \quad \text{(Case 3)} \, . \tag{4.11$'$}$$

This completes the proof of

THEOREM 2. *Let* $(b, c) = (c, a) = (a, b) = 1$ *and*

$$0 \leqq x < 1 \, , \quad 0 \leqq y < 1 \, , \quad 0 \leqq x < 1 \, .$$

Then

$$R = A(a, b; c \mid x, y; z) + A(b, c; a \mid y, z; x) + A(c, a; b \mid z, x; y)$$

is evaluated by (4.4), (4.8) *and* (4.11).

In particular, for $x = y = z$, we have

COROLLARY 2.

$$\begin{aligned} &\sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a} - x\right]\left[\frac{c(r+x)}{a} - x\right] \\ &+ \sum_{s=0}^{b-1}\left[\frac{c(s+x)}{b} - x\right]\left[\frac{a(s+x)}{b} - x\right] \\ &+ \sum_{t=1}^{c-1}\left[\frac{a(t+x)}{c} - x\right]\left[\frac{b(t+x)}{c} - x\right] = (a-1)(b-1)(c-1) \, . \end{aligned} \tag{4.12}$$

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