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ON SOLVABILITY OF GENERALIZED ORTHOMODULAR LATTICES

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The purpose of this paper is to establish a universal criterion for a generalized orthomodular lattice to belong to a primitive class of lattices.

The starting point for the investigation is the description of the reflection and the coreflection. These lattices can be determined by two lattice congruences defined by means of alleles on any lattice.

The main results permit to characterize a generalized orthomodular lattice solvable in a nontrivial primitive class $\mathscr{C}$ of lattices as a lattice belonging to $\mathscr{C}$. The characterization is further used to show that such a lattice belongs to $\mathscr{C}$ if and only if its commutator sublattice is a lattice of $\mathscr{C}$.

1. Preliminaries. Basic facts on orthomodular lattices used in this paper may be found in [4], and also most of the notation and terminology will be taken from that book. We assume familiarity with the results of Marsden [8] on solvability of generalized orthomodular lattices. The notation concerning the projectivity of quotients in a lattice is essentially the same as in [7].

Let Ω be a nonempty set of quotients of a lattice $\mathscr{L}$. A quotient b/a is called an *Ω-allele* of $\mathscr{L}$ if there exists $n \in \mathsf{N}$ and a sequence $b_0/a_0 = b/a,\ b_1/a_1,\ \cdots,\ b_n/a_n$ of quotients of Ω such that $b_i/a_i \sim b'_{i+1}/a'_{i+1}$ for every $i = 0, 1, \cdots, n-1$ where $[a'_{i+1}, b'_{i+1}] \subset [a_{i+1}, b_{i+1}]$ and $b \leqq a_n$ or $a \geqq b_n$. The set $\mathsf{A}_\Omega(\mathscr{L})$ of all Ω-alleles of $\mathscr{L}$ will be called the *Ω-allelomorph* of the lattice $\mathscr{L}$.

For the special case $\Omega = \Omega_0 = \{b/a \mid a \prec b\ (b \text{ covers } a)\}$, the Ω_0-alleles of a submodular lattice have been studied in the paper [1]. Other results concerned with Ω-alleles can be found in [3].

In the present paper we shall investigate the Ω_1-alleles where Ω_1 is the set of all quotients of $\mathscr{L}$ and we shall omit reference to Ω_1; e.g., we shall refer to an Ω_1-allele as an allele; similarly, $\mathsf{A}(\mathscr{L})$ (or simply A) will be the set of all alleles. Hence, $b/a \in \mathsf{A}(\mathscr{L})$ if and only if there exists d/c such that b/a is weakly projective into d/c, i.e. $b/a \approx_w d/c$, and $c \geqq b$ or $d \leqq a$.

2. Reflection and coreflection. For each lattice $\mathscr{L}$ there exists a smallest congruence $\varDelta$ such that $\mathscr{L}/\varDelta$ is distributive. Let $\varDelta^*$ be the pseudocomplement of $\varDelta$. We define the *reflection* of $\mathscr{L}$, denoted by Ref $\mathscr{L}$, to be the lattice $\mathscr{L}/\varDelta^*$; the lattice $\mathscr{L}/\varDelta$ is denoted by

Coref $\mathscr{L}$ and it is called the *coreflection* of $\mathscr{L}$. A well-known result asserts that a nontrivial primitive class of lattices contains the class of distributive lattices. Since $\mathscr{L}$ is a subdirect product of Ref $\mathscr{L}$ and Coref $\mathscr{L}$, it follows that a lattice belongs to a nontrivial class $\mathscr{C}$ of lattices if and only if its reflection belongs to $\mathscr{C}$.

The following lemma (which appears in [9, p. 95] as Lemma 2) is useful:

LEMMA 2.1. *Let $\mathscr{L}$ be a lattice and θ a congruence of $\mathscr{L}$. If $[b]/[a] \approx_w [d]/[c]$ in $\mathscr{L}/\theta$, then there exists ${}^\backprime a \in [a]$, ${}^\backprime b \in [b]$, ${}^\backprime c \in [c]$, ${}^\backprime d \in [d]$ such that ${}^\backprime b/{}^\backprime a \approx_w {}^\backprime d/{}^\backprime c$ in $\mathscr{L}$.*

Proof. Suppose $[b]/[a] = [b_0]/[a_0] \sim_w [b_1]/[a_1] \sim_w \cdots \sim_w [b_n]/[a_n] = [d]/[c]$ with $a_0 = a \leqq b = b_0$ and $a_n = c \leqq d = b_n$. Let ${}^\backprime d = {}^\backprime b_n = d$ and ${}^\backprime c = {}^\backprime a_n = c$, and suppose that ${}^\backprime a_j$, ${}^\backprime b_j$ for $i < j < n$ have already been defined so that ${}^\backprime a_j \in [a_j]$, ${}^\backprime b_j \in [b_j]$, ${}^\backprime a_j \leqq {}^\backprime b_j$, and ${}^\backprime b_j/{}^\backprime a_j \sim_w {}^\backprime b_{j+1}/{}^\backprime a_{j+1}$. Then if $[b_i]/[a_i] \nearrow_w [b_{i+1}]/[a_{i+1}]$, let ${}^\backprime a_i = {}^\backprime a_{i+1} \wedge b_i$, ${}^\backprime b_i = {}^\backprime b_{i+1} \wedge b_i$, and if $[b_i]/[a_i] \searrow_w [b_{i+1}]/[a_{i+1}]$, let ${}^\backprime a_i = {}^\backprime a_{i+1} \vee a_i$, ${}^\backprime b_i = {}^\backprime b_{i+1} \vee a_i$. So we get ${}^\backprime a = {}^\backprime a_0 \in [a]$, ${}^\backprime b = {}^\backprime b_0 \in [b]$, and ${}^\backprime b/{}^\backprime a \approx_w {}^\backprime d/{}^\backprime c$. Note that if $[a] \neq [b]$, we must have ${}^\backprime a \neq {}^\backprime b$.

COROLLARY 2.2. *Let $\mathscr{L}$ be a lattice and θ a congruence of $\mathscr{L}$. If $[b]/[a]$ is an allele of $\mathscr{L}/\theta$, then there exists ${}^\backprime a \in [a]$, ${}^\backprime b \in [b]$ such that ${}^\backprime b/{}^\backprime a$ is an allele of $\mathscr{L}$.*

Proof. Let $[b]/[a] \approx_w [d]/[c]$ with $a \leqq b \leqq c \leqq d$. Choose ${}^\backprime a$, ${}^\backprime b$, ${}^\backprime c$, ${}^\backprime d$ as in Lemma 2.1. It suffices to prove that ${}^\backprime b \leqq {}^\backprime c$. But we may assume that $[b]/[a] = [b_0]/[a_0] \nearrow_w [b_1]/[a_1]$, so ${}^\backprime b = {}^\backprime b_0 = {}^\backprime b_1 \wedge b_0 = {}^\backprime b_1 \wedge b \leqq b \leqq c = {}^\backprime c$.

LEMMA 2.3. (i) *Let $\mathscr{L}$ be any lattice and let $a < b$, $r < s$ be elements of $\mathscr{L}$. Then if $b/a \approx_w s/r$ and $r = r_0 < r_1 < \cdots < r_n = s$, there exist a_j, $j = 0, 1, \cdots, n$ such that $a = a_0 \leqq a_1 \leqq \cdots \leqq a_n = b$ and $a_{j+1}/a_j \approx_w r_{j+1}/r_j$ for each j.*

(ii) *Let $\mathscr{L}$ be a lattice and let γ be the binary relation defined on L by*

$$a \equiv b(\gamma) \Leftrightarrow \exists n \in \mathsf{N}\ \exists a_1, a_2, \cdots, a_n$$
$$a \wedge b = a_0 \leqq a_1 \leqq \cdots \leqq a_n = a \vee b\ ,$$

and

$$a_{j+1}/a_j \in \mathsf{A}(\mathscr{L}) \quad \textit{for every} \quad j = 0, 1, \cdots, n-1\ .$$

If $b/a \approx_w q/p$ and $p \equiv q(\gamma)$, then $a \equiv b(\gamma)$.

Proof. (i) It is enough to consider the case $b/a \sim_w s/r$. Assume $b/a \searrow_w s/r$; then $a_j = a \vee r_j$, $j = 0, 1, \cdots, n$ are the required elements.

(ii) We will first treat the following two cases:

Case I. $b/a \searrow_w q/p$, $q/p \approx_w s/r$ and $r \geqq q$. Let $r_0 = r$, $r_1 = r \vee (b \wedge s)$ and $r_2 = s$. Since $r_1/r \searrow b \wedge s/b \wedge r$ and $b/q \searrow a/a \wedge q$, we get that $r_1/r_0 \approx_w a/a \wedge q$. We also have $r_2/r_1 \approx_w b \vee s/b$. Since $b/a \approx_w s/r$, (i) implies that there exist $a_0 = a \leqq a_1 \leqq a_2 = b$ such that $a_1/a_0 \approx_w a/a \wedge q$ and $a_2/a_1 \approx_w b \vee s/b$. Thus a_1/a and b/a_1 belong to the allelomorph and so $a \equiv b(\gamma)$.

Case II. $b/a \nearrow_w q/p$, $q/p \approx_w s/r$ and $r \geqq q$. However, here we have $b/a \approx_w s/r$ and $r \geqq q \geqq p \vee b \geqq b$. This yields $b/a \in \mathsf{A}$ and, consequently, $a \equiv b(\gamma)$.

By the same argument as in the proof of (i) it is clear that $b/a \sim_w q/p$ and $p \equiv q(\gamma)$ implies $a \equiv b(\gamma)$. The general case now follows by induction.

Our next theorem provides much more information on γ.

THEOREM 2.4. *Let $\mathscr{L}$ be a lattice. Then γ is a congruence relation of $\mathscr{L}$.*

Proof. If $x \leqq y$, $x \equiv y(\gamma)$ and $t \in L$, then $y \wedge t/x \wedge t \approx_w y/x$. By Lemma 2.3 (ii) it follows that $x \wedge t \equiv y \wedge t(\gamma)$. The conclusion is now immediate from [6, Lemma 8, p. 24].

The following theorem can be proved by similar arguments; we omit the proof.

THEOREM 2.5. *Let $\mathscr{L}$ be a lattice and let β be the binary relation defined on L by*

$$a \equiv b(\beta) \Leftrightarrow \{(d/c \approx_w a \vee b/a \wedge b \ \& \ d/c \in \mathsf{A}(\mathscr{L})) \Rightarrow c = d\} .$$

Then β is a congruence of $\mathscr{L}$. Moreover, β is the pseudocomplement of γ.

COROLLARY 2.6. *Let $\mathscr{L}$ be a weakly modular lattice. Then*

$$a \equiv b(\beta) \Leftrightarrow \{([m, n] \subset [a \wedge b, a \vee b] \ \& \ n/m \in \mathsf{A}(\mathscr{L})) \Rightarrow m = n\} .$$

Observe that the description of β as given in Corollary 2.6 is a direct consequence of the proof of [7, Theorem III. 4. 10].

REMARK. If $\mathscr{L}$ is a relatively complemented lattice, then [5,

p. 355] $y/x \approx_w b/a$ implies $y/x \approx b'/a'$ where $[a', b'] \subset [a, b]$. We say that b/a has a *close allele* d/c if there exists a quotient d/c such that $b/a \approx d/c$ & $(b \leqq c$ or $a \geqq d)$. Thus, in such a lattice $b/a \in \mathsf{A}$ if and only if b/a has a close allele and the congruences β and γ of $\mathscr{L}$ can be characterized in terms of close alleles.

PROPOSITION 2.7. *Let $\mathscr{L}$ be a lattice and let θ be a congruence relation of $\mathscr{L}$. Then the quotient lattice $\mathscr{L}/\theta$ is distributive if and only if $\theta \supset \gamma$.*

Proof. If $\mathscr{L}/\theta$ is distributive and if $a \equiv b(\gamma)$, then there are elements x_i such that

$$a \wedge b = x_0 \leqq x_1 \leqq \cdots \leqq x_s = a \vee b$$

and such that $x_{i+1}/x_i \in \mathsf{A}(\mathscr{L})$ for every $i = 0, 1, \cdots, s-1$. But then $[x_{i+1}]/[x_i] \in \mathsf{A}(\mathscr{L}/\theta)$ and by [7, Lemma III. 2.7] we have $[x_{i+1}] = [x_i]$. Thus $[a] = [b]$, i.e., $a \equiv b(\theta)$ and so $\gamma \subset \theta$.

To prove the converse, suppose $\mathscr{L}/\gamma$ is not distributive; then there exists an allele $[b]/[a]$ in $\mathscr{L}/\gamma$ such that $[a] \neq [b]$. By Corollary 2.2 there exists $\grave{a} \in [a]$, $\grave{b} \in [b]$ such that $\grave{b}/\grave{a}$ is an allele in $\mathscr{L}$; hence $\grave{a} \equiv \grave{b}(\gamma)$ and so we get $[a] = [b]$, a contradiction.

The following is an application of Proposition 2.7.

COROLLARY 2.8. *If $\mathscr{L}$ is a lattice, then* Coref $\mathscr{L} = \mathscr{L}/\gamma$ *and* Ref $\mathscr{L} = \mathscr{L}/\beta$.

3. Solvability in classes of lattices. If a, b are two elements of an ortholattice $\mathscr{L} = (L, \vee, \wedge, ', 0, 1)$, then

$$\mathrm{com}_{\mathscr{L}}(a, b) = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$$

is called the *commutator* of a, b (cf. [8]). The $n-th$ *commutator sublattice* $\mathscr{G}_n$ of a generalized orthomodular lattice $\mathscr{G}$ is defined by induction in the following way: $\mathscr{G}_0 = \mathscr{G}$ and $\mathscr{G}_n (n \geqq 1)$ is by definition the p-ideal generated in $\mathscr{G}_{n-1}$ by all the commutators of the generalized orthomodular lattice $\mathscr{G}_{n-1}$. The lattice $\mathscr{G}$ is said to be *solvable* (in the sense of Marsden) if there exists $n \in \mathsf{N}$ such that $\mathscr{G}_n = \langle 0 \rangle$. Recall that [8, Theorem 9, p. 361] $\mathscr{G}$ is solvable if and only if it is distributive.

It is easily verified that if we define $\mathscr{G}^{(0)} = \mathscr{G}$ and if $\mathscr{G}^{(n)} (n \geqq 1)$ equals to the ideal of the generalized orthomodular lattice $\mathscr{G}$ generated by all the commutators of $\mathscr{G}^{(n-1)}$, then $\mathscr{G}_n = \mathscr{G}^{(n)}$ for every $n = 0, 1, \cdots$. A generalized orthomodular lattice $\mathscr{G}$ is said to be *solvable in a class* $\mathscr{C}$ of lattices if and only if there exists $n \in \mathsf{N}$ such that

$\mathscr{G}^{(n)}$ belongs to $\mathscr{C}$.

Since it is known [8, Theorem 9, p. 361] that $\mathscr{G}_1 = \mathscr{G}_2 = \cdots$, it is clear that in the definitions above we can suppose $n = 1$. We remark that it is a simple matter to verify that a lattice $\mathscr{G}$ is solvable in the class of distributive lattices if and only if it is solvable in the sense of Marsden.

The solvability of generalized orthomodular lattices in the class of modular lattices has been investigated in [2].

PROPOSITION 3.1. *Let $\mathscr{G}$ be a generalized orthomodular lattice. Then an element a of $\mathscr{G}$ belongs to $\mathscr{G}'$ if and only if $a \equiv 0(\gamma)$.*

Proof. By [8, Theorems 6 and 7, p. 360], the commutator sublattice $\mathscr{G}'$ is the kernel of the congruence Δ. The conclusion $\gamma = \Delta$ is now immediate from Proposition 2.7.

THEOREM 3.2. *The commutator sublattice $\mathscr{G}'$ of a generalized orthomodular lattice $\mathscr{G}$ is a dually distributive ideal.*

Proof. To prove the assertion, it is sufficient to show that if I_1, I_2 are ideals of $\mathscr{G}$, then necessarily $G' \cap (I_1 \vee I_2) \subset (G' \cap I_1) \vee (G' \cap I_2)$. Suppose $m \in G'$ and $m \leqq i_1 \vee i_2$, $i_s \in I_s (s = 0, 1)$. Let x' denote the orthocomplement of an element $x \in [0, i_1 \vee i_2]$ in the orthomodular lattice determined by the interval $[0, i_1 \vee i_2]$.

Set

$$k_1 = i_1 \wedge (i_1' \vee m \vee i_2) \wedge (i_1' \vee m \vee i_2') ,$$
$$k_2 = i_2 \wedge (i_2' \vee m \vee i_1) \wedge (i_2' \vee m \vee i_1')$$

so that $k_s = i_s \wedge c \in I_s$ where

$$c = (i_1' \vee m \vee i_2) \wedge (i_1' \vee m \vee i_2') \wedge (i_2' \vee m \vee i_1) .$$

Let $w = i_1 \wedge (i_1' \vee m)$ and $v = i_1 \wedge m$. Since $v \leqq m \in G'$, $v \in G'$. Now we have $w \wedge v' = i_1 \wedge \mathrm{com}\,(i_1, m) \in G'$ and so $w = v \vee (w \wedge v') \in G'$.

Let $w^+ = w' \wedge k_1$. Since i_1 commutes with i_1' and $i_1 \wedge m'$, it follows that $[i_1' \vee (i_1 \wedge m')] \wedge i_1 = i_1 \wedge m'$. Thus $w^+ \leqq \mathrm{com}\,(i_1' \vee m, i_2)$ and this yields $w^+ \in G'$. Therefore $k_1 = w \vee w^+ \in G' \cap I_1$ and, by symmetry, $k_2 \in G' \cap I_2$.

It is clear that $m \leqq (i_1 \vee i_2) \wedge c$ and that every i_s commutes with each element $(\cdots)$ of the definition of the element c. Therefore, i_1 and i_2 commute with c and this gives

$$m \leqq (i_1 \vee i_2) \wedge c = (i_1 \wedge c) \vee (i_2 \wedge c) = k_1 \vee k_2 .$$

This shows that $m \in (G' \cup I_1) \vee (G' \cap I_2)$, completing the proof.

COROLLARY 3.3. *The ideal $\mathscr{G}'$ of a generalized orthomodular lattice $\mathscr{G}$ is neutral.*

Proof. By Proposition 3.1 and Theorem III. 3.10 of [7], the ideal $\mathscr{G}'$ is standard. By Theorem 3.2 and Theorem III. 2.5 of [7] we see that $\mathscr{G}'$ is neutral.

For an element $g \in G$ let $[g]$ denote the corresponding element of the quotient lattice $\mathscr{G}/\beta$ where $\mathscr{G}$ is a generalized orthomodular lattice and let $K([g]) = (g] \cap G'$. If $g_1 \in [g]$ and $k \in K([g])$, then $k \equiv k \wedge g_1 \wedge g(\gamma)$. Since $k/k \wedge g_1 \wedge g \nearrow_w g/g_1 \wedge g$ and $g \equiv g_1 \wedge g(\beta)$, we have $k \equiv k \wedge g_1 \wedge g(\beta)$. By Theorem 2.5 this implies $k \leqq g_1 \wedge g \leqq g_1$. The set $K([g])$ is therefore well-defined.

LEMMA 3.4. *Under the convention made above we have*

(i) $[g] = \sup_{\mathscr{G}/\beta}\{[k] \mid k \in K([g])\}$;

(ii) $K([g_1 \wedge g_2]) = K([g_1]) \wedge K([g_2])$

$K([g_1 \vee g_2]) = K([g_1]) \vee K([g_2])$

where the symbols $\vee$, $\wedge$ *of the right-hand side denote the join and the meet in the ideal lattice* $\mathscr{T}$ *of the lattice* $\mathscr{G}'$;

(iii) *the mapping* $f\colon [g] \mapsto K([g])$ *is an isomorphism of the lattice* $\mathscr{G}/\beta$ *onto a sublattice of the lattice* $\mathscr{T}$.

Proof. (i) If $[h]$ is such that $[h] < [g]$ with $h < g$, then, by Corollary 2.6, there are v, w satisfying $h \leqq v < w \leqq g$ and $w/v \in \mathsf{A}$. Let z denote a complement of v in $[0, w]$. Then $v \equiv w(\gamma)$ implies $z \in G'$. If $[z] \leqq [h]$, then $[v] = [w]$ and so $v \equiv w(\beta \cap \gamma)$. Therefore, by Theorem 2.5, $v = w$, a contradiction. Since $z \in K([g])$, we see that $[h]$ is not an upper bound for the set $\{[k] \mid k \in K([g])\}$. Conclusion (i) results.

(ii) This is immediate by Theorem 3.2.

Assertion (iii) now follows directly from (i) and (ii) above.

THEOREM 3.5. *Let $\mathscr{C}$ be a primitive class of lattices which contains a lattice with more than one element. Then a generalized orthomodular lattice is a lattice of $\mathscr{C}$ if and only if its commutator sublattice belongs to $\mathscr{C}$.*

COROLLARY 3.6. *A generalized orthomodular lattice is solvable in a primitive class $\mathscr{C}$ which contains a lattice with more than one element if and only if it belongs to the class $\mathscr{C}$.*

Proof. If $\mathscr{G}$ belongs to $\mathscr{C}$, then its sublattice $\mathscr{G}'$ belongs also to $\mathscr{C}$.

Conversely, suppose $\mathscr{G}' \in \mathscr{C}$. By [6, Lemma 8, p. 34], the ideal lattice $\mathscr{T}$ of $\mathscr{G}'$ also belongs to the class $\mathscr{C}$. Since $\mathscr{C}$ is primitive, every isomorphic image of a sublattice of $\mathscr{T}$ belongs to $\mathscr{C}$. It, therefore, follows from Lemma 3.4 (iii) that ref $\mathscr{G} \in \mathscr{C}$. It is now immediate that $\mathscr{G}$ belongs to $\mathscr{C}$.

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