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A NECESSARY AND SUFFICIENT CONDITION FOR UNIQUENESS OF SOLUTIONS TO TWO POINT BOUNDARY VALUE PROBLEMS

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In this paper it is shown that the uniqueness of solutions to two point boundary value problems in which one end point is held fixed is equivalent to the existence of a family of Liapunov functions.

T. Yoshizawa [6] and H. Okamura [5] demonstrated that the uniqueness of solutions to initial value problems was equivalent to existence of a Liapunov function. J. Kato and A. Strauss [4] and S. Bernfeld [1] provided necessary and sufficient conditions for the existence of solutions to initial value problems on $[t_0, \infty)$ with the use of Liapunov functions. With regard to boundary value problems J. H. George and W. G. Sutton [2] obtained a sufficient condition in terms of a Liapunov function for the existence and uniqueness of solutions to two point boundary value problems. In this paper we shall employ a variation of the Okamura function to obtain a generalization of the latter result for a certain class of two point boundary value problems.

1. **Preliminaries.** In this section we state the definition of a Liapunov function and establish a theorem which will be used in the next section. We shall consider the second order differential equation:

$$x'' = f(t, x, x') , \tag{1}$$

where f is a real-valued function defined and continuous on $[a, b] \times R^2$. It will be assumed that initial value problems associated with (1) exist, are unique, and that solutions are defined throughout $[a, b]$. In particular we shall be concerned with the uniqueness of solutions to (1) satisfying

$$x(t_1) = y_1 \qquad x(t_2) = y_2 \tag{2}$$

where $a \leqq t_1 < t_2 \leqq b$ and $y_1, y_2 \in R$. If $x_0(t)$ is any solution of (1) satisfying (2) for some points t_1, and t_2, then by setting $x(t) = y(t) + x_0(t)$ we obtain

$$y'(t) = F(t, y(t), y'(t)) , \tag{3}$$

where $F(t, y(t), y'(t)) = f(t, y(t) + x_0(t), y'(t) + x_0'(t)) - f(t, x_0(t), x_0'(t))$.

Thus equation (1) has a unique solution satisfying (2) if and only if $y(t) = 0$ is the only solution of (3) such that $y(t_1) = y(t_2) = 0$. Thus we shall restrict our attention to the differential equation

$$x'' = F(t, x, x') , \tag{4}$$

where $F(t, 0, 0) = 0$ and the boundary conditions

$$x(t_1) = x(t_2) = 0 . \tag{5}$$

The principal tool employed in this paper will be Liapunov functions.

DEFINITION. A Liapunov function for (4) is a real valued function V defined on $D = [a, b] \times S$ where S is a closed subset of R^2 and $(0, 0) \in S$, such that

$$V(t, 0, x_2) = 0 \tag{6}$$

$$V(t, x_1, x_2) > 0 \quad \text{if} \quad x_1 \neq 0 \tag{7}$$

$$V(t, x_1, x_2) \text{ is nondecreasing along solution curves of (4)} \tag{8}$$

By condition (8) we shall mean that if $x(t)$ is a solution of (4), then $V(t_1, x(t_1), x'(t_1)) \leqq V(t_2, x(t_2), x'(t_2))$ for all points t_1 and t_2, $t_1 < t_2$, such that $(t_i, x(t_i), x'(t_i)) \in D$.

If V is a real valued function satisfying (6) and (7), then the following theorem provides a sufficient condition for V to be a Liapunov function.

THEOREM 1. *Suppose V is continuous and satisfies a Lipschitz condition locally with respect to x_1 and x_2 in D, and*

$$V(t, x_1, x_2) = \frac{\lim}{h \to 0^+} + \frac{1}{h}[V(t + h, x_1 + hx_2, x_2 + hF(t, x_1, x_2)) - V(t, x_1, x_2)] \geqq 0$$

for t, x_1, x_2 in the interior of D. Then $V(t, x_1, x_2)$ is nondecreasing along solution curves of (4).

Proof. Yoshizawa [6].

The following theorem gives a sufficient condition for the uniqueness of solutions of (4) satisfying (5).

THEOREM 2. *Suppose $V(t, x_1, x_2)$ is a Liapunov function for* (4) *defined on $[a, b] \times R^2$. Then for any t_1 and t_2, $a \leqq t_1 < t_2 \leqq b$, there exists at most one solution to* (4) *satisfying* (5).

Proof. Employing a stronger definition of a Liapunov function George and Sutton [2] have given a proof of this theorem. We include a proof for referral at a later time. Suppose $y(t)$ is a nonzero solution of (4) satisfying (5). Then there exists a $t_0 \in (t_1, t_2)$ such that $y(t_0) \neq 0$. Thus $V(t_0, y(t_0), y'(t_0)) > 0$. However $y(t_2) = 0$ implies that $V(t_2, y(t_2), y'(t_2)) = 0$. This is a contradiction to the assumption that $V(t, x_1, x_2)$ is nondecreasing along the solution curves of (4).

Two known conditions insuring the uniqueness of solutions of (4) satisfying (5) are consequences of Theorem 2. Hartman [3, page 433] proved that if $(x'(t))^2 + x(t)F(t, x(t), x'(t)) > 0$ wherever $x(t) \neq 0$ and $x'(t)x''(t) = 0$, for all solutions $x(t)$ of (4) then $x(t) = 0$ is the only solution of (4) satisfying (5). It was noted by George and Sutton that $V(t, x_1, x_2) = (x_1)^2$ satisfied their definition of a Liapunov function and therefore by Theorem 2 the result of Hartman follows. The same choice for $V(t, x_1, x_2)$ will satisfy our definition. It is also well known that if $F(t, x_1, x_2)$ is continuous and strictly increasing in x_1 for fixed (t, x_2) then $x(t) = 0$ is the only solution of (4) satisfying (5). This result also follows from Theorem 2 by choosing

$$V(t, x_1, x_2) = \int_a^t |F(s, x_1, x_2) - F(s, 0, x_2)|\, ds .$$

2. Necessary and sufficient condition.

In this section we will further restrict our attention to the boundary value problem

$$x'' = F(t, x, x') \tag{9}$$

$$x(a) = 0 \qquad x(\gamma) = 0 , \tag{10}$$

where $F(t, 0, 0) = 0$ and $a < \gamma \leqq b$. Thus we shall fix $t_1 = a$. We now proceed to derive a necessary and sufficient condition for the uniqueness of solutions of (9) satisfying (10).

For each $M > 0$ let D_M denote the subset of $[a, b] \times R^2$ defined by $D_M = \{(t, \alpha, \beta): 2|\alpha/(t-a)| \leqq M,\ 2|\alpha/(t-b)| \leqq M,\ 2|\beta| \leqq M\} \cup \{(a, 0, \beta): 2|\beta| \leqq M\} \cup \{(b, 0, \beta): 2|\beta| \leqq M\}$. For each $(t_0, \alpha, \beta) \in D_M$, $a<t_0<b$, let $X^M_{(t_0,\alpha,\beta)}$ denote the set of all continuously differentiable functions $x(t)$ defined on $[a, b]$ whose second derivative exists and is continuous for all except at most one point of $[a, b]$ and which satisfy $|x'(t)| \leqq M$ for all $t \in [a, b]$, $x(a) = x(b) = 0$, $x(t_0) = \alpha$, and $x'(t_0) = \beta$. The restrictions on (t_0, α, β) in the definition of D_M insure that $X^M_{(t_0,\alpha,\beta)}$ is not empty.

LEMMA 1. *Suppose $a < t_0 < b$. There exists a solution to* (9) *satisfying* $x(a) = x(b) = 0$, $x(t_0) = \alpha$, *and* $x'(t_0) = \beta$ *if and only if*

$$\text{(11)} \qquad \underset{x(t) \in X^M_{(t_0, \alpha, \beta)}}{\text{infimum}} \int_a^b | x''(t) - F(t, x(t), x'(t)) | \, dt = 0$$

for some M.

Proof. If there exists a solution to (9) satisfying the above conditions then for some $M > 0$, $x(t) \in X^M_{(t_0, \alpha, \beta)}$. For this M, or any larger M, the above infimum will be zero. Conversely suppose the above infimum is zero for some $M > 0$. Let $\{x_k(t)\}$ be a sequence of functions in $X^M_{(t_0, \alpha, \beta)}$ such that

$$\text{(12)} \qquad \lim_{k \to \infty} \int_a^b | x''_k(t) - F(t, x_k(t), x'_k(t)) | \, dt = 0 \, .$$

Letting $y_k(t) = \int_a^t (x''_k(s) - F(s, x_k(s), x'_k(s)))ds$ we see by (12) that $y_k(t)$ converges to zero uniformly on $[a, b]$. Let $z_k(t) = x_k(t) - \int_a^t y_k(s)ds$. Then $z'_k(t) = x'_k(t) - y_k(t) = x'_k(a) + \int_a^t F(s, x_k(s), x'_k(s))ds$ and for $a \leqq t_1 < t_2 \leqq b$

$$| z'_k(t_1) - z'_k(t_2) | \leqq \int_{t_1}^{t_2} | F(s, x_k(s), x'_k(s)) | \, ds \leqq K_M \, | t_2 - t_1 | \, ,$$

where $K_M = \max_{(t, x_1, x_2) \in D_M} | F(t, x_1, x_2) |$. Also

$$| z'_k(t) | \leqq M + K_M(b - a) \, .$$

Therefore, there exists a uniformly convergent subsequence of $\{z'_k(t)\}$ which we shall again denote by $\{z'_k(t)\}$. In a similar manner the sequence of functions $\{z_k(t)\}$ can easily be shown to be equicontinuous and uniformly bounded. Thus we again obtain a subsequence which we denote by $\{z_k(t)\}$, such that $\{z_k(t)\}$ and $\{z'_k(t)\}$ converge uniformly on $[a, b]$. Denote the limit function by $z(t)$. Since $z_k(a) = 0$ for all k, we have that $z(a) = 0$. Also $y_k(t)$ converges to 0 uniformly on $[a, b]$ implies that $x_k(t)$ converges uniformly to $z(t)$ and $x'_k(t)$ converges uniformly to $z'(t)$ on $[a, b]$.

Thus $z(b) = 0$, $z(t_0) = \alpha$,
$z'(t_0) = \beta$, and

$$z''(t) = F(t, z(t), z'(t)) \, .$$

Thus $z(t)$ is a solution with the desired properties.

For each $M > 0$ we define a real valued function V_M with domain D_M by

$$(13)\qquad V_M(t, x_1, x_2) = \begin{cases} \underset{x(t)\in X^M_{(t,x_1,x_2)}}{\text{infimum}} \int_a^b |x'(t) - F(t, x(t), x'(t))|\, dt, & x_1 \neq 0 \\ 0 & , x_1 = 0 . \end{cases}$$

THEOREM 3. *Suppose that there exists at most one solution to* (9) *satisfying* (10) *for every* γ, $a < \gamma \leqq b$. *Then for each* $M > 0$, V_M *satisfies the following conditions*:

$$(14)\qquad V_M(t, 0, x_2) = 0$$

$$(15)\qquad V_M(t, x_1, x_2) > 0 \qquad \text{if } x_1 \neq 0$$

(16) $V_M(t, x_1, x_2)$ *is nondecreasing along solution curves* $x(t)$ *of* (9) *which satisfy* $x(a) = 0$ *and* $(t, x(t), x'(t)) \in D_M$ *for all* $t \in [a, b]$.

Proof. V_M clearly satisfies (14). Suppose $x_1 \neq 0$ and $V_M(t, x_1, x_2) = 0$. Then by Lemma 1 there exists a solution $x(t)$ of (9) such that $x(a) = x(b) = 0$ and $x(t) = x_1$. This contradicts the uniqueness assumption. Thus V_M satisfies condition (15). Let $x(t)$ be a solution of (9) such that $(t, x(t), x'(t)) \in D_M$ for all $t \in [a, b]$ and $x(a) = 0$. Let $a \leqq t_1 < t_2 \leqq b$. If $t_1 = a$ or $t_2 = b$ then it follows trivially from the uniqueness of solutions of (9) satisfying (10), the uniqueness of initial value problems associated with (9), and properties (14) and (15) that $V_M(t_1, x(t_1), x'(t_1)) \leqq V_M(t_2, x(t_2), x'(t_2))$. Thus assume that $a < t_1 < t_2 < b$. Again from uniqueness of solutions of (9) satisfying (10) it follows that $x(t_1) \neq 0$ and $x(t_2) \neq 0$. For each $y(t) \in X^M_{(t_1, x(t_1), x'(t_1))}$ the function

$$z(t) = \begin{cases} x(t) & a \leqq t \leqq t_1 \\ y(t) & t_1 < t \leqq b \end{cases}$$

is again an element of $X^M_{(t_1, x(t_1), x'(t_1))}$. Therefore

$$V_M(t_1, x(t_1), x'(t_1)) = \underset{x(t)\in X^M_{(t_1,x(t_1),x'(t_1))}}{\text{infimum}} \int_{t_1}^b |x''(t) - F(t, x(t), x'(t))|\, dt$$

and in a similar manner

$$V_M(t_2, x(t_2), x'(t_2)) = \underset{x(t)\in X^M_{(t_2,x(t_2),x'(t_2))}}{\text{infimum}} \int_{t_2}^b |x''(t) - F(t, x(t), x'(t))|\, dt .$$

Let $\{x_k(t)\}$ be a sequence in $X^M_{(t_2, x(t_2), x'(t_2))}$ such that

$$V_M(t_2, x(t_2), x'(t_2)) = \lim_{k\to\infty} \int_{t_2}^b |x_k''(t) - F(t, x_k(t), x_k'(t))|\, dt .$$

Then if

$$y_k(t) = \begin{cases} x(t) & a \leqq t \leqq t_2 \\ x_k(t) & t_2 < t \leqq b \end{cases}$$

we have

$$\begin{aligned} V_M(t_1, x(t_1), x'(t_1)) &\leqq \lim_{k\to\infty} \int_{t_1}^{b} |y_k''(t) - F(t, y_k(t), y_k'(t))| \, dt \\ &= \lim_{k\to\infty} \int_{t_2}^{b} |x_k''(t) - F(t, x_k(t), x_k'(t))| \, dt \\ &= V(t_2, x(t_2), x'(t_2)) \, . \end{aligned}$$

Thus V_M satisfies (16).

On the other hand if there exists a family of subsets $[a, b] \times S_M = D_M$, S_M closed, of $[a, b] \times R^2$ such that every solution $x(t)$ of (9) satisfies $(t, x(t), x'(t)) \in D_M$ for some M and a family of real valued functions V_M defined on D_M and satisfying (14), (15), and (16), then the only solution of (9) satisfying (10) for any $\gamma \in (a, b]$ is identically zero on $[a, b]$. The proof of this is exactly the same as the proof of Theorem 2. Note that condition (16) is weaker than condition (8).

THEOREM 4. *There exists at most one solution of* (9) *satisfying* (10) *for all* $\gamma \in (a, b]$ *if and only if there exists a family of subsets* $D_M = [a, b] \times S_M$ *of* $[a, b] \times R^2$ *such that every solution* $x(t)$ *of* (9) *satisfies* $(t, x(t), x'(t)) \in D_M$ *for some* M *and a family of real valued functions* V_M *defined on* D_M *satisfying condition* (14), (15), *and* (16).

Note that a similar theorem to Theorem 4 can be proved if the right end point is fixed.

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