PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 1, 1975

A SHEAF THEORETIC REPRESENTATION OF RINGS WITH BOOLEAN ORTHOGONALITIES

PATRICK N. STEWART

It is shown that certain associative rings with Boolean orthogonalities are isomorphic to rings of global sections.

Let A be a ring and $\perp$ a relation on A. For each subset S of A define

$$S^{\perp} = \{x \in A \mid x \perp s \quad \text{for} \quad \text{all} \quad s \in S\} \quad \text{and} \quad S^{\perp\perp} = (S^{\perp})^{\perp}.$$

When $S = \{s\}$ we write $s^{\perp}$ and $s^{\perp\perp}$ instead of $\{s\}^{\perp}$ and $\{s\}^{\perp\perp}$. Subsets of A of the form $S^{\perp}$ are *polars*. The relation $\perp$ is a *Boolean orthogonality* if all polars are two-sided ideals and if, for all $x, y \in A$,

1. $x \perp y \rightarrow y \perp x$, 2. $x \perp x \rightarrow x = 0$, and
3. $x^{\perp\perp} \cap y^{\perp\perp} = (0) \rightarrow x \perp y$.

The set of polars is a Boolean algebra (see [3]) with meet and join defined by

$$B \wedge C = B \cap C \quad \text{and} \quad B \vee C = (B^{\perp} \wedge C^{\perp})^{\perp}.$$

Boolean orthogonalities have been studied by Davis [3], Cornish [1] and by Cornish and Stewart [2].

Throughout this paper we shall assume that A is an associative ring with an identity and with a Boolean orthogonality $\perp$. We shall also assume that the following *finiteness condition* is satisfied:

for any two elements $x, y \in A$ there is a finite set $F \subseteq A$ such that $x^{\perp\perp} \wedge y^{\perp\perp} = F^{\perp\perp}$.

Notice that if $F = \{f_1, \cdots, f_n\}$, then $F^{\perp} = f_1^{\perp} \wedge \cdots \wedge f_n^{\perp}$ and $F^{\perp\perp} = f_1^{\perp\perp} \vee \cdots \vee f_n^{\perp\perp}$.

An ideal I of A is a $\perp$-*ideal* if $F^{\perp\perp} \subseteq I$ for every finite set $F \subseteq I$, and I is $\perp$-*prime* if $I \neq A$ and whenever the intersection of two polars B and C is contained in I, either $B \subseteq I$ or $C \subseteq I$.

LEMMA. *Assume that P is either a $\perp$-prime ideal or $P = A$, that I is a $\perp$-ideal and that $x \in A \setminus I$ is such that $x^{\perp\perp} \wedge a^{\perp\perp} \subseteq I$ implies that $a \in P$. Then there is a $\perp$-prime $\perp$-ideal Q such that $I \subseteq Q \subseteq P$ and $x \notin Q$.*

Proof. Using Zorn's Lemma select a $\perp$-ideal $Q \supseteq I$ maximal with respect to the property "$x \notin Q$ and $x^{\perp\perp} \wedge a^{\perp\perp} \subseteq Q$ implies that $a \in P$". Clearly $I \subseteq Q \subseteq P$.

Suppose that B and C are polars neither of which is contained in Q. Choose $b \in B \setminus Q$ and $c \in C \setminus Q$. Then

$$B' = \cup \{F^{\perp\perp} \mid F \text{ is a finite subset of } \{b\} \cup Q\}$$

and

$$C' = \cup \{G^{\perp\perp} \mid G \text{ is a finite subset of } \{c\} \cup Q\}$$

are $\perp$-ideals which properly contain Q. By the maximality of Q either $x^{\perp\perp} \subseteq B'$ or $x^{\perp\perp} \wedge b_1^{\perp\perp} \subseteq B'$ for some $b_1 \in A \setminus P$, and $x^{\perp\perp} \subseteq C'$ or $x^{\perp\perp} \wedge c_1^{\perp\perp} \subseteq C'$ for some $c_1 \in A \setminus P$. Thus we obtain finite sets $\{b, f_1, \cdots, f_n\} \subseteq \{b\} \cup Q$ and $\{c, g_1, \cdots, g_m\} \subseteq \{c\} \cup Q$ such that one of $x^{\perp\perp}$, $x^{\perp\perp} \wedge b_1^{\perp\perp}$, $x^{\perp\perp} \wedge c_1^{\perp\perp}$ or $x^{\perp\perp} \wedge b_1^{\perp\perp} \wedge c_1^{\perp\perp}$ is contained in

$$\begin{aligned} &\{b, f_1, \cdots, f_n\}^{\perp\perp} \wedge \{c, g_1, \cdots, g_m\}^{\perp\perp} \\ &= (b^{\perp\perp} \vee f_1^{\perp\perp} \vee \cdots \vee f_n^{\perp\perp}) \wedge (c^{\perp\perp} \vee g_1^{\perp\perp} \vee \cdots \vee g_m^{\perp\perp}) \\ &= (b^{\perp\perp} \wedge c^{\perp\perp}) \vee H^{\perp\perp} \end{aligned}$$

where H is a finite subset of Q (we have used the distributivity of the Boolean algebra of polars and also the finiteness condition). If $b^{\perp\perp} \wedge c^{\perp\perp} \subseteq Q$, then $x^{\perp\perp} \subseteq Q$ or $x^{\perp\perp} \wedge d^{\perp\perp} \subseteq Q$ for some $d \in A \setminus P$ both of which contradict the choice of Q. Thus $B \cap C \not\subseteq Q$ and we conclude that Q is $\perp$-prime.

For the remainder of this paper $\bar{X}$ will be fixed set of $\perp$-prime ideals which contains all $\perp$-prime $\perp$-ideals and which is *full* (that is, if I is a sum of polars and $I \neq A$, then $I \subseteq P$ for some $P \in \bar{X}$).

PROPOSITION 2. (Cornish [1]). *For each $P \in \bar{X}$,*

$$\{x \in A \mid x^{\perp} \not\subseteq P\} = \cap \{R \in \bar{X} \mid R \subseteq P\} = \cap \{Q \in \bar{X} \mid Q \subseteq P$$

and Q is a $\perp$-prime $\perp$-ideal$\}$.

Proof. Suppose that $x^{\perp} \not\subseteq P$ and R is a $\perp$-prime ideal contained in P. Then $x^{\perp\perp} \wedge x^{\perp} = (0) \subseteq R$ and so $x \in R$.

If $x^{\perp} \subseteq P$, then by Lemma 1 (take $I = x^{\perp}$) there is a $\perp$-prime $\perp$-ideal $Q \subseteq P$ such that $x \notin Q$. This establishes the result.

The set described in the proposition will be denoted by O_P. We note that $O_P = P$ if and only if P is minimal in $\underline{\bar{X}}$.

Let $P \in \underline{\bar{X}}$. The set O_P, being an intersection of $\perp$-ideals, is itself a $\perp$-ideal. Define a relation (also denoted by $\perp$) on A/O_P by

$$(x + O_P) \perp (y + O_P) \leftrightarrow x^{\perp\perp} \wedge y^{\perp\perp} \subseteq O_P.$$

This relation is well-defined because if $x_1 = x + a$ and $y_1 = y + b$ where $a, b \in O_P$, then

$$x_1^{\perp\perp} \wedge y_1^{\perp\perp} = (x + a)^{\perp\perp} \wedge (y + b)^{\perp\perp} \subseteq (x^{\perp\perp} \vee a^{\perp\perp}) \wedge (y^{\perp\perp} \vee b^{\perp\perp})$$

and so $x_1^{\perp\perp} \wedge y_1^{\perp\perp} \subseteq (x^{\perp\perp} \wedge y^{\perp\perp}) \vee F^{\perp\perp}$ where F is a finite subset of O_P. It is routine to check that

$$x^{\perp} + O_P \subseteq (x + O_P)^{\perp} \quad \text{and} \quad x^{\perp\perp} + O_P \subseteq (x + O_P)^{\perp\perp}$$

for each $x \in A$, and that the relation $\perp$ is a Boolean orthogonality on A/O_P.

PROPOSITION 3. *For each $P \in \underline{\bar{X}}$, $\bar{P} = P/O_P$ is a $\perp$-prime ideal of A/O_P which contains all proper polars of A/O_P.*

Proof. Let $\bar{B}$ and $\bar{C}$ be polars in A/O_P such that $\bar{B} \cap \bar{C} \subseteq \bar{P}$. Suppose that $\bar{B} \not\subseteq \bar{P}$. Then there is an element $b \in A$ such that $b + O_P \in \bar{B} \backslash \bar{P}$. Let $c + O_P \in \bar{C}$. Then

$$(b^{\perp\perp} + O_P) \cap (c^{\perp\perp} + O_P) \subseteq (b + O_P)^{\perp\perp} \cap (c + O_P)^{\perp\perp} \subseteq \bar{B} \cap \bar{C} \subseteq \bar{P}$$

and so $b^{\perp\perp} \cap c^{\perp\perp} \subseteq P$. Since $b \notin P$ we conclude that $c \in P$ and so $\bar{C} \subseteq \bar{P}$. Thus $\bar{P}$ is $\perp$-prime.

Suppose that $a^{\perp\perp} \wedge b^{\perp\perp} \subseteq O_P$. Then there is a finite set $\{f_1, \cdots, f_n\} \subseteq O_P$ such that

$$a^{\perp\perp} \wedge b^{\perp\perp} = \{f_1, \cdots, f_n\}^{\perp\perp} = f_1^{\perp\perp} \vee \cdots \vee f_n^{\perp\perp}.$$

For each $i = 1, \cdots, n$, $f_i \in O_P$ and so $f_i^{\perp} \not\subseteq P$. Thus $f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} \not\subseteq P$. Also, $b^{\perp\perp} \wedge f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} \subseteq a^{\perp}$ because $a^{\perp\perp} \wedge b^{\perp\perp} \wedge f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} = (0)$. If $a \notin O_p$, then $a^{\perp} \subseteq P$ and so, since $f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} \not\subseteq P$, $b^{\perp\perp} \subseteq P$. Thus $\bar{P}$ contains $(a + O_P)^{\perp}$ for all $a \notin O_P$. It follows that $\bar{P}$ contains all proper polars of A/O_P.

Let S be the disjoint union of the factor rings A/O_P. The relation (also denoted by $\perp$) on the product

$$\Pi\{A/O_P \mid P \in \bar{X}\}$$
$$= \{f \colon \bar{X} \to S \mid f(P) \in A/O_P \quad \text{for all} \quad P \in \bar{X}\}$$

defined by

$$f \perp g \leftrightarrow f(P) \perp g(P) \quad \text{in} \quad A/O_P \quad \text{for all} \quad P \in \bar{X}$$

is a Boolean orthogonality. Each $a \in A$ determines a function $\hat{a} \in \Pi\{A/O_P \mid P \in \bar{X}\}$ defined by $\hat{a}(P) = a + O_P$. It follows from Lemma 1 that $\cap\{P \mid P$ is a $\perp$-prime $\perp$-ideal$\} = (0)$ and so $\cap\{O_P \mid P \in \bar{X}\} = (0)$. Thus we obtain the usual embedding

$$A \xrightarrow{\wedge} \hat{A} \subseteq \Pi\{A/O_P \mid P \in \bar{X}\}.$$

This embedding respects orthogonalities; that is, $a \perp b$ in A if and only if $\hat{a} \perp \hat{b}$ in the product.

We define a topology on $\bar{X}$ by declaring the basic open sets to be the subsets of the form

$$\bar{X}(a) = \{P \in \bar{X} \mid a^{\perp\perp} \not\subseteq P\}.$$

Notice that $\bar{X}(a) \cap \bar{X}(b) \supseteq \bar{X}(c)$ for all $c \in a^{\perp\perp} \wedge b^{\perp\perp}$ and so these sets do qualify as a topological base.

Suppose that $\{\bar{X}(a) \mid a \in C\}$ is a cover of $\bar{X}$ consisting of basic open sets. Then $\Sigma\{a^{\perp\perp} \mid a \in C\} = A$ because $\bar{X}$ is full. Since A has an identity there is a finite set $F \subseteq C$ such that $\Sigma\{a^{\perp\perp} \mid a \in F\} = A$. Thus $\{\bar{X}(a) \mid a \in F\}$ covers $\bar{X}$ and so $\bar{X}$ is quasi-compact.

Give S the topology generated by sets of the form $\hat{a}[U] = \{a + O_P \mid P \in U\}$ where U is open in $\bar{X}$ and $a \in A$. We obtain a sheaf of rings $(S, \pi, \bar{X})$ where $\pi \colon S \to \bar{X}$ is the projection onto $\bar{X}$.

Let $\Gamma = \{f \mid f \in \Pi\{A/O_P \mid P \in \bar{X}\}$ is continuous$\}$ be the ring of global sections. The following observation shows that $\hat{A} \subseteq \Gamma$: for all $x, y \in A$, $\{P \in \bar{X} \mid x - y \in O_P\}$ is open in $\bar{X}$. To see this notice that if $x - y \in O_Q$, then $Q \in \bar{X}(u) \subseteq \{P \in \bar{X} \mid x - y \in O_P\}$ where u is any element in $(x - y)^{\perp}\backslash Q$.

THEOREM 4. $\hat{A} = \Gamma$.

Proof. Let $f \in \Gamma$. Since $\bar{X}$ is quasi-compact there are finite sets $\{a_1, \cdots, a_n\}$ and $\{v_1, \cdots, v_n\}$ such that $\bar{X} = \bar{X}(a_1) \cup \cdots \cup \bar{X}(a_n)$ and $f(P) = v_i + O_P$ for all $P \in \bar{X}(a_i)$.

Notice that $v_i - v_j \in \cap\{O_P \mid P \in \bar{X}(a_i) \cap \bar{X}(a_j)\}$, so $(v_i - v_j)^{\perp} \subseteq$

$Q \in \bar{X}$ implies that $a_i^{\perp\perp} \wedge a_j^{\perp\perp} \subseteq Q$. It follows from Lemma 1 (take $P = A$ and $I = (v_i - v_j)^{\perp}$ for each $x \notin (v_i - v_j)^{\perp}$) that

$$(v_i - v_j)^{\perp} = \cap\{Q \mid (v_i - v_j)^{\perp} \subseteq Q \in \bar{X}\}$$

and so $a_i^{\perp\perp} \wedge a_j^{\perp\perp} \subseteq (v_i - v_j)^{\perp}$. Thus $(v_i - v_j)^{\perp\perp} \wedge a_i^{\perp\perp} \subseteq a_j^{\perp}$.

Since $\bar{X} = \bar{X}(a_1) \cup \cdots \cup \bar{X}(a_n)$ and $\bar{X}$ is full, $a_1^{\perp\perp} + \cdots + a_n^{\perp\perp} = A$. Choose $u_i \in a_i^{\perp\perp}$ such that $1 = u_1 + \cdots + u_n$ and let $v = u_1v_1 + \cdots + u_nv_n$. Then

$$v - v_j = u_1(v_1 - v_j) + \cdots + u_n(v_n - v_j) \in a_j^{\perp} \subseteq O_P$$

for all $P \in \bar{X}(a_j)$. Thus $f(P) = v_j + O_P = v + O_P$ for all $P \in \bar{X}(a_j)$ and so $f = \hat{v} \in \hat{A}$.

f-rings (Keimal [4]). Let A be an f-ring with identity. The relation defined by $x \perp y \leftrightarrow |x| \wedge |y| = 0$ is a Boolean orthogonality and $x^{\perp\perp} \wedge y^{\perp\perp} = (|x| \wedge |y|)^{\perp\perp}$. Let $\bar{X}$ be the set of irreducible ℓ-ideals. Then $\bar{X}$ is full because polars are ℓ-ideals and sums of ℓ-ideals are again ℓ-ideals. Also, all $\perp$-prime $\perp$-ideals are irreducible ℓ-ideals and so A is isomorphic to the f-ring of all global sections of the sheaf $(S, \pi, \bar{X})$.

Reduced rings (Koh [5]). Let A be a ring with identity and no nonzero nilpotent elements. The relation defined by $x \perp y \leftrightarrow xy = 0$ is a Boolean orthogonality and $x^{\perp\perp} \wedge y^{\perp\perp} = (xy)^{\perp\perp}$. Let $\bar{X}$ be the set of all prime ideals of A. Clearly $\bar{X}$ is full. Also, all $\perp$-prime $\perp$-ideals are completely prime and so A is isomorphic to the ring of global sections of the sheaf $(S, \pi, \bar{X})$. Each stalk A/O_P is reduced (Proposition 2) and the prime ideal P/O_P contains all zero divisors (Proposition 3).

Semiprime rings. Let A be a semiprime ring with identity. The relation defined by $x \perp y \leftrightarrow (x)(y) = (0)$ is a Boolean orthogonality. However, the finiteness condition may not be satisfied as the following example shows.

Let R be a semiprime ring with identity, R' the ring of 3×3 matrices with entries from R,

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define $\bar{x}$ and $\bar{y}$ in $P = \Pi\{R_n \mid R_n = R' \text{ for } n = 1, 2, \cdots\}$ by

$$\bar{x}(n) = \begin{cases} x & \text{if} \quad n \equiv 1 \pmod 2 \\ 0 & \text{if} \quad n \not\equiv 1 \pmod 2 \end{cases}$$
$$\bar{y}(n) = \begin{cases} y & \text{if} \quad n \equiv 1 \pmod 3 \\ 0 & \text{if} \quad n \not\equiv 1 \pmod 3). \end{cases}$$

Notice that $\bar{x}\bar{y} = \bar{y}\bar{x} = \bar{x}^2 = \bar{y}^2 = 0$. Let E be the subring of P which is generated by the identity of P, $\bar{x}$, $\bar{y}$ and

$$\Sigma\{R_n \mid R_n = R' \quad \text{for} \quad n = 1, 2, \cdots\}. \quad \text{Then}$$

$$\bar{x}^{\perp\perp} = \{f \in E \mid f(n) = 0 \quad \text{for} \quad n \not\equiv 1 \quad (\text{mod } 2)\},$$

$$\bar{y}^{\perp\perp} = \{f \in E \mid f(n) = 0 \quad \text{for} \quad n \not\equiv 1 \quad (\text{mod } 3)\},$$

and so

$$\bar{x}^{\perp\perp} \wedge \bar{y}^{\perp\perp} = \{f \in E \mid f(n) = 0 \quad \text{for} \quad n \not\equiv 1 \quad (\text{mod } 6)\}.$$

If $\bar{x}^{\perp\perp} \wedge \bar{y}^{\perp\perp} = \{f_1, \cdots, f_n\}^{\perp\perp}$, then at least one of the f_i must satisfy $f_i(n) \neq 0$ for infinitely many positive integers n. But then there are integers α, β and γ such that $f_i(n) = (\alpha + \beta\bar{x} + \gamma\bar{y})(n)$ for all but a finite number of positive integers n. This is incompatible with the requirement that $f_i(n) = 0$ for $n \not\equiv 1$ (mod 6).

When the finiteness condition is satisfied (for instance, when A satisfies the maximum condition on annihilators), A is isomorphic to the ring of all global sections of the sheaf $(S, \pi, \bar{X})$ where $\bar{X}$ is the set of prime ideals of A. Each stalk A/O_P is semiprime (Proposition 1) and the prime ideal P/O_P contains all two-sided annihilator ideals of A/O_P (Proposition 2).

References

1. W. H. Cornish, *Boolean orthogonalities and minimal primes*, preprint.

2. W. H. Cornish and P. N. Stewart, *Direct and subdirect decompositions of universal algebras with a Boolean orthogonality*, preprint.

3. G. Davis, *Rings with orthogonality relations*, Bull. Austral. Math. Soc., **4** (1971), 163–178.

4. K. Keimal, *The representation of lattice ordered groups and rings by sections in sheaves*, Lecture Notes in Math. 248, 1–98, Springer, Berlin-Heidelberg-New York, 1971.

5. K. Koh, *On functional representations of a ring without nilpotent elements*, Canad. Math. Bull., **14** (1971), 349–352.

Received April 30, 1974. Part of the research for this paper was done at the 1973 Canadian Mathematical Congress Summer Research Institute in Sherbrooke, Quebec, Canada.

DALHOUSIE UNIVERSITY