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ON THE SEMISIMPLICITY OF GROUP RINGS OF SOME LOCALLY FINITE GROUPS

D. S. PASSMAN

We consider the semisimplicity problem for group rings of some locally finite groups. In particular we study locally solvable groups and linear groups in the mixed characteristic case. While the results here are by no means definitive, we hope the techniques constitute a first step in the complete solution.

Our notation follows that of [2] and [4] and all groups considered are assumed to be locally finite unless otherwise stated. If K is a field of characteristic 0 then in this case $K[G]$ is trivially seen to be semisimple. Thus we assume throughout that $p > 0$ is a fixed prime and that K is a fixed field of characteristic p.

1. Group ring lemmas.

The following few results are basic for handling nil ideals in group rings.

LEMMA 1.1. *Let*

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \in G$, $x_i \neq 1$ and let $x \in G$. Then there exists n, i such that x^{p^n} is conjugate to $(x_i x)^{p^n}$ in G. In particular if σ is a set of primes and if x is a σ-element then $x_i x$ is a $\sigma \cup \{p\}$-element.

Proof. We have $\alpha x \in JK[G]$ so αx is nilpotent and hence $(\alpha x)^{p^n} = 0$ for some n. Thus by Lemma 3.4 of [2]

$$0 = (\alpha x)^{p^n} = x^{p^n} + \sum a_i^{p^n} (x_i x)^{p^n} + \beta$$

with $\beta \in [K[G], K[G]]$, the commutator subspace. Since the sum of the coefficients in β over any conjugacy class is zero it then follows that the x^{p^n} term must be partially cancelled by some conjugate of $(x_i x)^{p^n}$ for some i. Hence x^{p^n} is conjugate to $(x_i x)^{p^n}$ and the result follows.

LEMMA 1.2. *Let P be a normal p-subgroup of G, let π_P: $K[G] \to K[P]$ denote the natural projection and suppose that*

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \in G$, $x_i \neq 1$ *satisfies* $\pi_P(\alpha) \notin JK[P]$. *If* $x \in G$ *then there exists* n, i *such that* $x_i \notin P$ *and* x^{p^n} *is conjugate modulo* P *to* $(x_i x)^{p^n}$. *In particular if* σ *is a set of primes and if* x *is a* σ*-element then* $x_i x$ *is a* $\sigma \cup \{p\}$*-element.*

Proof. Let $-: K[G] \to K[G/P]$ be the natural homomorphism and observe that the kernel of this map is precisely $JK[P] \cdot K[G]$ since P is a p-group. Then $\pi_P(\alpha)$ is by assumption a nonzero scalar, say b, and

$$b^{-1}\bar{\alpha} = \bar{1} + \sum{}' (b^{-1}a_i)\bar{x}_i \in JK[\bar{G}]$$

where the sum Σ' is over all $x_i \notin P$. Thus Lemma 1.1 applied to the group $\bar{G}$ implies that for some n, i we have $\bar{x}^{p^n}$ conjugate in $\bar{G}$ to $\overline{x_i x}^{p^n}$. Since P is a p-group this clearly yields the result.

LEMMA 1.3. *Let* $G = NH$ *be finite with* $N \triangleleft G$ *and* $H \cap N = \langle 1 \rangle$. *If* $JK[G] \cap K[H] \neq 0$ *then every* p'*-conjugacy class of* N *is normalized by an element of* H *of order* p.

Proof. By assumption we may choose

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \in H$, $x_i \neq 1$. If $x \in N$ is a p'-element then by Lemma 1.1 there exists n, i with x^{p^n} conjugate to $(x_i x)^{p^n}$. If $g \in G$ with $g^{-1}(x^{p^n})g = (x_i x)^{p^n}$ then we see that x^{p^n} is centralized by $g(x_i x)g^{-1}$. Hence since x is a p'-element, $\langle x \rangle = \langle x^{p^n} \rangle$ so x is centralized by $g(x_i x)g^{-1}$.

Write $g(x_i x)g^{-1} = yh$ with $y \in N$, $h \in H$. Then since $N \triangleleft G$ we have modulo N

$$h^{p^n} \equiv (yh)^{p^n} = g(x_i x)^{p^n} g^{-1} = x^{p^n} \equiv 1$$

so $h^{p^n} \in H \cap N = \langle 1 \rangle$ and h is a p-element of H. Furthermore $h \neq 1$ since $yh = g(x_i x)g^{-1} \notin N$. Finally $x^{yh} = x$ shows that h normalizes the N-conjugacy class of x and the lemma is proved.

LEMMA 1.4. *Let* G *have two finite subgroups* N *and* H. *Suppose* $N_0 \triangleleft N$ *with* N/N_0 *an abelian* p'*-group and suppose that* H *normalizes both* N *and* N_0. *If* $H \cap N = \langle 1 \rangle$ *and* $JK[G] \cap K[H] \neq 0$ *then*

$$N/N_0 = \bigcup_h \mathbf{C}_{N/N_0}(h)$$

where h runs through all elements of H of order p.

Proof. By Lemma 16.9 of [2] we may assume that $G = NH$. If $\bar{x} \in N/N_0$ then since N/N_0 is a p'-group there exists $x \in N$, a p'-element, with $\bar{x} = xN_0/N_0$. Now by the preceding lemma there exists $h \in H$ of order p which normalizes the N-conjugacy class of x and hence the N/N_0-conjugacy class of $\bar{x}$. Finally since N/N_0 is abelian, h centralizes $\bar{x}$.

The following is a partial converse.

LEMMA 1.5. *Let $G = NH$ be finite with $N \lhd G$. Suppose that $N = \bigcup_h \mathbf{C}_N(h)$ where h runs through all elements of H of order p. Then $JK[G] \cap K[H] \neq 0$.*

Proof. Set $\alpha = \hat{H} = \Sigma_{h \in H} h$. We show that $\alpha \in JK[G]$ and in fact we show that $K[G]\alpha$ is a left ideal of square zero. Since $h\alpha = \alpha$ for $h \in H$, this ideal has as a spanning set elements of the form $x\alpha$ with $x \in N$ and it suffices to show that for all such $x, \alpha x \alpha = 0$.

Given $x \in N$ by assumption there exists $y \in H$ of order p which centralizes it. If $Y = \langle y \rangle$ then $\alpha = \hat{H} = \hat{Y}\beta$ where β is a sum of right coset representatives for Y in H. Since x and y commute and $|Y| = p$ we then have

$$\begin{aligned} \alpha x \alpha &= \alpha x \hat{Y}\beta = \alpha \hat{Y} \cdot x\beta \\ &= |Y| \alpha \cdot x\beta = 0 \end{aligned}$$

and the result follows.

In locally finite groups the concept of locally finite index is trivial but the following does seem to be of interest. Let N be a subgroup of G. We say that N is *almost normal* in G if for every finite subgroup H of G we have $[\langle N, H \rangle : N] < \infty$. Clearly every normal subgroup of G is almost normal and indeed we have

LEMMA 1.6. *Let N be a subgroup of G. Then N is almost normal in G if and only if every finite subgroup H of G normalizes some normal subgroup of N of finite index.*

Proof. Let H be a finite subgroup of G. If N is almost normal in G then $[\langle N, H \rangle : N] < \infty$ and both H and N normalize the core of N in $\langle N, H \rangle$.

Conversely suppose H normalizes N_0 with $N_0 \lhd N$ and of finite index. Then $N_0 \lhd \langle N, H\rangle$ and $\langle N, H\rangle / N_0$ is a locally finite group generated by the finite groups N/N_0 and N_0H/N_0. Thus $[\langle N, H\rangle : N] \leqq [\langle N, H\rangle : N_0] < \infty$.

Recall that if H is a subgroup of G then

$$\mathbf{D}_G(H) = \{x \in G \mid [H : \mathbf{C}_H(x)] < \infty\}$$

is the almost centralizer of H in G. Thus in particular $\mathbf{D}_G(G) = \Delta(G)$ in the f.c. subgroup of G.

LEMMA 1.7. *Let N be an almost normal subgroup of G. Then $D = \mathbf{D}_G(N)$ is normal in G. Furthermore if $JK[N]$ is nilpotent then D carries the radical of G, that is*

$$JK[G] = JK[D] \cdot K[G].$$

Proof. Let H be an finite subgroup of G. Then by assumption N has finite index in $M = \langle N, H\rangle$. Thus clearly

$$D \cap M = \mathbf{D}_M(N) = \Delta(M) \lhd M$$

and it follows easily that $D \lhd G$.

Now suppose further that $JK[N]$ is nilpotent. Since $D \lhd G$ and G is locally finite we have

$$\pi_D(JK[G]) \cdot K[G] \supseteq JK[G] \supseteq JK[D] \cdot K[G]$$

where $\pi_D : K[G] \to K[D]$ is the natural projection. Thus it suffices to show that the ideal $\pi_D(JK[G])$ of $K[D]$ is nil. Let $\alpha \in JK[G]$ and take $H = \langle \operatorname{Supp} \alpha \rangle$ in the above. Then $\alpha \in JK[G] \cap K[M] \subseteq JK[M]$ by Lemma 16.9 of **[2]**. Also $[M : N] < \infty$ and $JK[N]$ is nilpotent so $JK[M]$ is nilpotent by Lemma 16.8 of **[2]**. Hence Theorem 20.2 of **[2]** yields $JK[M] = JK[\Delta(M)] \cdot K[M]$ so $\pi_{\Delta(M)}(\alpha)$ is nilpotent. Finally $\Delta(M) = D \cap M$ so $\pi_D(\alpha) = \pi_{\Delta(M)}(\alpha)$ is nilpotent and the result follows.

We remark that not every subgroup of a locally finite group is almost normal. For example let N be a infinite locally finite group and let $H \neq \langle 1\rangle$ be finite. Then $G = H \wr N$ is locally finite but $[\langle H, N\rangle : N] = [G : N] = \infty$.

2. Locally solvable groups. The next result is a key lemma in the study of Sylow intersections in solvable groups (see **[1]**, for example).

LEMMA 2.1. *Let P be a finite p-group which acts faithfully on a finite abelian p'-group Q. If either P is abelian or both $|P|$ and $|Q|$ are odd, then there exists $x \in Q$ with $\mathbf{C}_P(x) = \langle 1 \rangle$.*

Proof. We proceed by induction on $|Q|$. Suppose $Q = Q_1 \times Q_2$ and each factor is nontrivial and P-invariant. Then there exist $x_i \in Q_i$ with $\mathbf{C}_P(x_i) = \mathbf{C}_P(Q_i)$ so if $x = x_1x_2$ then $\mathbf{C}_P(x) = \mathbf{C}_P(Q_1) \cap \mathbf{C}_P(Q_2) = \langle 1 \rangle$. Thus we may assume that Q is indecomposable as a P-module and hence Q is a q-group for some $q \neq p$. Also P acts faithfully on $\Omega_1(Q)$ so we may take Q to be elementary abelian and then P acts irreducibly on Q. If P is abelian then by Schur's lemma P acts semiregularly on Q. Hence for all $x \in Q - \{1\}$, $\mathbf{C}_P(x) = \langle 1 \rangle$.

We now assume that both $|P|$ and $|Q|$ are odd and prove that Q contains at least two orbits under the action of P of elements x with $\mathbf{C}_P(x) = \langle 1 \rangle$. First if P is cyclic then P acts semiregularly on $Q^{\#} = Q - \{1\}$. The number of such orbits is then $(|Q| - 1)/|P|$, a nonzero even number since both $|P|$ and $|Q| \neq 1$ are odd.

Now suppose P is not cyclic so, since $p > 2$, P has a normal abelian (p, p)-subgroup U. If $H = \mathbf{C}_P(U)$ then $H \lhd P$, $[P: H] = p$ and $P = \langle H, y \rangle$ for some element $y \in P$. If L is a noncentral (in P) subgroup of U of order p and if $V = \mathbf{C}_Q(L)$ then

$$Q = V \times V^y \times V^{y^2} \times \cdots \times V^{y^{p-1}}$$

is a direct product of H-submodules of Q. This all follows from Schur's lemma since U cannot act semiregularly. If N is the kernel of the action of H on V then by induction there exist two H-orbits $A, B \subseteq V^{\#}$ with the property that $x \in A, B$ implies that $\mathbf{C}_H(x) = N$.

Consider the two subsets of Q given by

$$S = A \times B^y \times B^{y^2} \times \cdots \times B^{y^{p-1}}$$

$$T = A \times A^y \times B^{y^2} \times \cdots \times B^{y^{p-1}}.$$

If $x \in S, T$ then clearly $\mathbf{C}_H(x) = \bigcap N^{y^i} = \langle 1 \rangle$. Then also $\mathbf{C}_P(x) = \langle 1 \rangle$ since $hy \in \mathbf{C}_P(x)$ for some $h \in H$ would imply using $p > 3$ that A and B are the same H-orbit. Finally it is clear from $P = \langle H, y \rangle$ that no element of S can be P-conjugate to an element of T. Thus Q does indeed have at least two such orbits of elements x with $\mathbf{C}_P(x) = \langle 1 \rangle$ and the result follows.

We remark that the above lemma is false in many instances if the prime 2 is present. Indeed the following three examples are typical of what occurs.

First let $p = 2$ and suppose $q = 2^n - 1$ is a Mersenne prime. Then the dihedral group P of order 2^{n+1} acts faithfully on Q, an abelian group of type (q, q). If x, y are distinct noncentral involutions of P then clearly $|\mathbf{C}_Q(x)| = q$ and $\mathbf{C}_Q(x) \cap \mathbf{C}_Q(y) = \langle 1 \rangle$ since the cyclic subgroup of P of index 2 acts semiregularly. Thus since P has 2^n noncentral involutions x we have

$$\left| \bigcup_x \mathbf{C}_Q(x)^{\#} \right| = 2^n(q-1) = (q+1)(q-1) = |Q^{\#}|$$

and every element of $Q^{\#}$ is fixed by some involution of P.

Now let $p = 2$ and suppose $q = 2^n + 1$ is a Fermat prime. If P_0 is cyclic of order $2^n = q - 1$ then P_0 acts faithfully and transitively on $V^{\#}$ where $V \cong Z_q$ is cyclic of order q. Thus $P = P_0 \wr Z_2$ acts faithfully on $Q = V_1 \times V_2$, a direct product of two copies of V. Write $P = \langle P_1, P_2, x \rangle$ where P_i is cyclic of order $q - 1$ and acts transitively on $V_i^{\#}$ and where x interchanges V_1 and V_2. If $v = (v_1, v_2) \in Q$ and say $v_i = 1$ then $\mathbf{C}_P(v) \supseteq P_j$ for $j \neq i$. On the other hand if $v_i \neq 1$ for $i = 1, 2$ then by transitivity there exists $y_i \in P_i$ with $v_i^{y_i} = v_j$ $(j \neq i)$, viewed as elements of V, so that $y_1 y_2 x$ centralizes v.

Finally let $q = 2$ and let $p = 2^n - 1$ be a Mersenne prime. Then Z_p acts faithfully and transitively on $V^{\#}$ where V is elementary abelian of order 2^n and hence $P = Z_P \wr Z_P$ acts faithfully on $Q = V_1 \times V_2 \times \cdots \times V_p$ a direct product of p copies of V. As in the preceding example the transitivity of Z_p on $V^{\#}$ implies easily that every element of Q has a nontrivial centralizer in P.

As an indication of the basically different behavior with respect to semisimplicity of odd and even order finite solvable groups we prove the following.

PROPOSITION 2.2. *Let G be a finite solvable group and let P be a p-subgroup of G. Suppose that either P is abelian or $|G|$ is odd. Then $JK[G] \cap K[P] \neq 0$ if and only if $P \cap \mathbf{O}_p(G) \neq \langle 1 \rangle$.*

Proof. Suppose first that $L = P \cap \mathbf{O}_p(G) \neq \langle 1 \rangle$. Then for the augmentation ideal $\omega(K[L]) \subseteq K[P]$ we have

$$0 \neq \omega(K[L]) \subseteq \omega(K[\mathbf{O}_p(G)]) \subseteq JK[G]$$

so $JK[G] \cap K[P] \neq 0$.

Conversely suppose that $P \cap \mathbf{O}_p(G) = \langle 1 \rangle$ and define $N \lhd G$ by $N \supseteq \mathbf{O}_p(G)$ and $N/\mathbf{O}_p(G) = \mathrm{Fit}(G/\mathbf{O}_p(G))$. By Fitting's theorem P acts faithfully on $N/\mathbf{O}_p(G)$ and hence on N/N_0, the Frattini quotient of the nilpotent p'-group $N/\mathbf{O}_p(G)$. Now according to Lemma 1.4 we must have

$$N/N_0 = \bigcup_h \mathbf{C}_{N/N_0}(h)$$

for all $h \in P^{\#}$ but since either P is abelian or $|G|$ is odd this violates Lemma 2.1. The result follows.

On the other hand if $G = QP$ for any of the three examples given above then G is solvable, $P \cap \mathbf{O}_p(G) = \langle 1 \rangle$ since P acts faithfully on Q and $JK[G] \cap K[P] \neq 0$ by Lemma 1.5.

LEMMA 2.3. *Let G be a finite group with subgroups H, H_1 and H_2.*

(i) *Suppose that for all $g \in G$, $H_1^g \cap H_2$ contains an element of order p. Then there exists an element $x \in H_1$ of order p with $[G : \mathbf{C}(x)] \leqq |H_1| \cdot |H_2|$.*

(ii) *Suppose G acts transitively as permutations on Ω and that for each $\alpha \in \Omega$, H contains an element of order p fixing α. Then there exists an element $x \in H$ of order p with $[G : \mathbf{C}(x)] \leqq |H| \cdot |G_\alpha|$.*

Proof. We consider (i). Let X be the set of elements of H_1 of order p and let Y be those of H_2. Then by assumption for each $g \in G$ there exist $x \in X$, $y \in Y$ with $x^g = y$. Thus g belongs to a certain right coset of $\mathbf{C}(x)$ depending on x and y. We therefore have

$$G = \bigcup_{x,y} \mathbf{C}(x) g_{x,y}$$

and hence for some $x \in X$, $[G : \mathbf{C}(x)] \leqq |X| \cdot |Y|$. Since $X \subseteq H_1$, $Y \subseteq H_2$ this part follows.

Finally for (ii) we merely apply (i) with $H_1 = H$, $H_2 = G_\alpha$. For each $g \in G$ we have by assumption an element of order p in

$$H \cap G_{\alpha g^{-1}} = H \cap (G_\alpha)^{g^{-1}}$$

so there is an element of order p in $H^g \cap G_\alpha = H_1^g \cap H_2$.

LEMMA 2.4. *Let G be a locally finite group with $\mathbf{O}_q(G) = \langle 1 \rangle$ for all primes q. If H is a finite subgroup of G then there exists a subgroup G^* of G with $G^* \supseteq H$ and such that G^* is the ascending union of the finite groups $H \subseteq G_1 \subseteq G_2 \subseteq \cdots$. Furthermore for $i > j$, $G_j \cap \mathrm{Fit}(G_i) = \langle 1 \rangle$.*

Proof. We first find such a sequence of groups G_i with $G_i \cap \mathrm{Fit}(G_{i+1}) = \langle 1 \rangle$. Set $G_1 = H$ and suppose we have found $G_1, G_2, \cdots, G_n$. Now $\mathbf{O}_q(G) = \langle 1 \rangle$ for all primes so for each $x \in G_n$, $x \neq 1$ the normal closure $\langle x \rangle^G$ is not locally nilpotent. Thus there exists a finite group L with $\langle x \rangle^L$ not nilpotent. We merely let G_{n+1} be the group generated by G_n and those finitely many L's, one for each $x \in G_n$ $x \neq 1$. Clearly $G_n \cap \mathrm{Fit}(G_{n+1}) = \langle 1 \rangle$.

Finally let $i > j$ so $i \geqq j + 1$. Then

$$\begin{aligned} G_j \cap \mathrm{Fit}(G_i) &= G_j \cap (G_{j+1} \cap \mathrm{Fit}(G_i)) \\ &\subseteq G_j \cap \mathrm{Fit}(G_{j+1}) = \langle 1 \rangle \end{aligned}$$

and the lemma is proved with $G^* = \bigcup G_i$.

We now come to our main result on locally solvable groups. The oddness hypothesis is obviously too restrictive here and the conclusion is not strong enough. Never-the-less we do show that $JK[G] \neq 0$ implies the existence of some nontrivial global structure on G, certainly a first step towards the complete solution.

THEOREM 2.5. *Let K be a field of characteristic $p > 0$ and let G be a locally finite, locally solvable group. Suppose that either all p-subgroups of G are abelian or that G is a $2'$-group. Then $JK[G] \neq 0$ implies $\mathbf{O}_q(G) \neq \langle 1 \rangle$ for some prime q.*

Proof. We assume that $\mathbf{O}_q(G) = \langle 1 \rangle$ for all primes q and show that $JK[G] = 0$. Suppose by way of contradiction that $JK[G] \neq 0$ and let $\alpha \in JK[G]$ with $1 \in \mathrm{Supp}\,\alpha$. Set $H = \langle \mathrm{Supp}\,\alpha \rangle$ and apply Lemma 2.4. By Lemma 16.9 of **[2]** $\alpha \in JK[G^*]$ so we may assume that $G = G^* = \bigcup G_i$ since clearly $\mathbf{O}_q(G^*) = \langle 1 \rangle$ for all q. Set $F_i = \mathrm{Fit}(G_i)$ and write $F_i = P_i \times Q_i$ where $P_i = \mathbf{O}_p(F_i)$ and $Q_i = \mathbf{O}_{p'}(F_i)$.

Let $Q = \langle Q_1, Q_2, \cdots \rangle$. Since Q_i normalizes Q_j for $j \geqq i$, Q is clearly a p'-group. This group can best be visualized as the acending union of the n-fold semidirect products $Q_n Q_{n-1} \cdots Q_1$. Now G_i normalizes $\langle Q_i, Q_{i+1}, \cdots \rangle$ a normal subgroup of Q of finite index so since $G = \bigcup G_i$ we conclude from Lemma 1.6 that Q is almost normal in G. Furthermore Q is a p'-group so $JK[Q] = 0$ and hence by Lemma 1.7 $D = \mathbf{D}_G(Q)$ carries the radical, that is $JK[G] =$

$JK[D]\cdot K[G]$. Thus $\pi_D(JK[G])\subseteq JK[G]$ so replacing α by $\pi_D(\alpha)\neq 0$ if necessary we may assume that $H\subseteq D$. Now $H\subseteq G_1$ so H normalizes all Q_i and since $G_i\cap Q_j=\langle 1\rangle$ for $j>i$ it follows easily that for any $h\in H$

$$\mathbf{C}_Q(h)=\langle \mathbf{C}_{Q_1}(h),\mathbf{C}_{Q_2}(h),\cdots\rangle$$

and

$$[Q:\mathbf{C}_Q(h)]=\Pi_i[Q_i:\mathbf{C}_{Q_i}(h)].$$

Thus since $H\subseteq D$ we have $[Q:\mathbf{C}_Q(h)]<\infty$ and it follows that h centralizes all Q_i after awhile and hence since H is finite, H centralizes all Q_i for i sufficiently large.

Now set $R_n=\langle P_1,P_2,\cdots,P_n\rangle$ so that R_n is a p-subgroup of G_n. We also define $S_{n+1}/P_{n+1}=\mathrm{Fit}(G_{n+1}/P_{n+1})$. Then S_{n+1}/P_{n+1} is a nilpotent p'-group and we let $\bar{S}_{n+1}=(S_{n+1}/P_{n+1})/\Phi(S_{n+1}/P_{n+1})$ be its Frattini quotient. Observe that $H\subseteq G_1$ implies that H normalizes R_n and that R_nH acts on $\bar{S}_{n+1}$. We will use this action to show that for some element $h\in H^\#$ we have $[R_n:\mathbf{C}_{R_n}(h)]\leqq|H|^2$.

Now R_n is a p-subgroup of G_n so $R_n\cap P_{n+1}=\langle 1\rangle$ and hence by Fitting's theorem, since G_{n+1}/P_{n+1} is solvable, we see that R_n acts faithfully on S_{n+1}/P_{n+1}. Hence R_n also acts faithfully on $\bar{S}_{n+1}$. If H does not act faithfully on $\bar{S}_{n+1}$ and if $h\in H^\#$ acts trivially then $(R_n,h)\subseteq R_n$ act trivially so h centralizes R_n and $[R_n:\mathbf{C}_{R_n}(h)]=1\leqq |H|^2$. Thus we may assume that H acts faithfully on $\bar{S}_{n+1}$ and therefore that $H\cap S_{n+1}=\langle 1\rangle$ since S_{n+1} acts trivially on $\bar{S}_{n+1}$.

By assumption either R_n is abelian or both R_n and S_{n+1} have odd order. Hence we conclude from Lemma 2.1 that there exists $x\in\bar{S}_{n+1}$ with $\mathbf{C}_{R_n}(x)=\langle 1\rangle$. We consider the action of $L=R_nH$ on the L-orbit Ω of x. By the above $\mathbf{C}_L(x)\cap R_n=\langle 1\rangle$ so $|\mathbf{C}_L(x)|\leqq|H|$ for this particular $x\in\bar{S}_{n+1}$. Furthermore since $H\cap S_{n+1}=\langle 1\rangle$ Lemma 1.4 implies that every element of $\bar{S}_{n+1}$ is centralized by some element of H of order p. Thus by Lemma 2.3 (ii) there exists $h\in H^\#$ with

$$[L:\mathbf{C}_L(h)]\leqq|H|\cdot|\mathbf{C}_L(x)|\leqq|H|^2.$$

Since $[R_n:\mathbf{C}_{R_n}(h)]\leqq[L:\mathbf{C}_L(h)]$ this fact follows.

Let $P=\langle P_1,P_2,\cdots\rangle$. Then since P is the ascending union of the groups R_n and since H is finite, it follows from the above that there exists $h\in H^\#$ with $[P:\mathbf{C}_P(h)]\leqq|H|^2$. Again we have

$$\mathbf{C}_P(h)=\langle \mathbf{C}_{P_1}(h),\ \mathbf{C}_{P_2}(h),\cdots\rangle$$

and

$$|H|^2 \geqq [P : \mathbf{C}_P(h)] = \Pi_i [P_i : \mathbf{C}_{P_i}(h)]$$

so h centralizes all P_j with j sufficiently large. Since h also centralizes all Q_j with j sufficiently large, it follows that h centralizes F_j for some $j > 1$. But G_j is solvable and $h \in G_j$, $h \notin F_j$ so we have a contradiction by Fitting's theorem. This completes the proof.

We remark finally on locally solvable groups which are not necessarily locally finite. If G is such a group and if H is a finitely generated subgroup of G, then H is of course a finitely generated solvable group. Thus by a theorem of Zalesskii **[7]** (or see **[4]** Theorem 4.2) $JK[H] = NK[H]$ and hence by Lemma 4.1 of **[4]** we have $JK[G] = N^*K[G]$. Now by Theorem 1.6 of **[4]**

$$N^*K[G] = JK[\Lambda^+(G)] \cdot K[G]$$

where $\Lambda^+(G)$ is a certain locally finite characteristic subgroup of G. Clearly $\Lambda^+(G)$ is locally finite and locally solvable so Theorem 2.5 applied to $\Lambda^+(G)$ yields results on $JK[G]$.

3. Linear group reductions.

3. Linear group reductions. We now begin our work on locally finite linear groups over fields of finite characteristic $q \neq p$. The cases $q = 0$ and $q = p$ have already been considered in **[3]** and **[4]**. In the following, unless otherwise indicated, q will be a fixed prime different from p and all groups will be locally finite linear groups in characteristic q. The first lemma is well known. We let $GL_n(q^\infty)$ denote the general linear group over $GF(q^\infty)$, the algebraic closure of $GF(q)$.

LEMMA 3.1. *Let G be an irreducible subgroup of $GL_n(F)$ with F algebraically closed. Then G is conjugate in $GL_n(F)$ to a subgroup of $GL_n(q^\infty)$.*

Proof. Since F is algebraically closed we have $F \supseteq GF(q^\infty)$ and since G acts irreducibly the linear span FG is the whole matrix ring F_n.

Since $FG = F_n$ choose $x_1, x_2, \cdots, x_m \in G$ which form a basis for the matrix ring F_n. Then $H = \langle x_1, x_2, \cdots, x_m \rangle$ is a finite subgroup of G and the embedding of H in F_n is clearly an absolutely irreducible representation for H in characteristic q. Now H is finite so all such representations are realizeable over $GF(q^\infty)$ and hence there exists a nonsingular matrix $s \in GL_n(F)$ with $s^{-1}Hs \subseteq GL_n(q^\infty)$. Replacing G by $s^{-1}Gs$ we may clearly assume that $H \subseteq GL_n(q^\infty)$.

We now proceed as in the proof of Burnside's lemma. Let tr denote the usual matrix trace so that tr defines a nondegenerate bilinear form on F_n. Hence the matrix $[\operatorname{tr} x_i x_j]$ is nonsingular. Now let $x \in G$. Since the x_i's span F_n we have

$$x = \sum_i a_i x_i$$

for suitable $a_i \in F$. Hence multiplying by x_j and taking traces yields

$$\operatorname{tr} x x_j = \sum_i a_i \operatorname{tr} x_i x_j \qquad j = 1, 2, \cdots, m.$$

Observe that xx_j and x_ix_j are elements of G. Thus they are periodic matrices and have traces contained in $GF(q^\infty)$. Therefore the above is a set of m equations over $GF(q^\infty)$ in the m unknowns $a_1, a_2, \cdots, a_m$ with nonzero determinant. The solution is therefore in $GF(q^\infty)$ so $a_i \in GF(q^\infty)$ for all i and hence $x \in GL_n(q^\infty)$.

In view of earlier work on linear groups it is reasonable to expect that $\mathbf{O}_p(G) = \langle 1 \rangle$ implies $JK[G]$ nilpotent. Thus the following few lemmas are relevant.

LEMMA 3.2. *Let $G \subseteq GL_n(F)$ with $\mathbf{O}_p(G) = \langle 1 \rangle$. Suppose that $G_0 = G \cap SL_n(F)$ and $JK[G_0]$ is nilpotent. Then $JK[G]$ is nilpotent*

Proof. Now $G_0 \lhd G$ and $JK[G_0]$ is nilpotent so by Lemma 1.7, $JK[G] = JK[D] \cdot K[G]$ where $D = \mathbf{D}_G(G_0)$. It therefore suffices to show that $JK[D]$ is nilpotent.

Now $D_0 = D \cap G_0 = \Delta(G_0)$ and since D is a linear group, Lemma 1.2 (i) of [3] implies that D_0 has a subgroup Z of finite index which is central in D. Since $D/D_0 \subseteq GL_n(F)/SL_n(F)$ we have D/D_0 abelian. This implies that $H = \mathbf{C}_D(D_0/Z)$ is a nilpotent normal subgroup of D of finite index. Now $D \lhd G$ so $\mathbf{O}_p(D) = \langle 1 \rangle$ and hence $\mathbf{O}_p(H) = \langle 1 \rangle$. But H is nilpotent so H is a p'-group and $JK[H] = 0$. Finally $[D\colon H] < \infty$ so we conclude from Lemma 16.8 of [2] that $JK[D]$ is nilpotent.

LEMMA 3.3. *Let $G \subseteq GL_n(F)$ and suppose we know that for all $H \lhd G$ if $\dim FH < \dim FG$ then $JK[H]$ is nilpotent. Let $M \lhd G$ with $\dim FM < \dim FG$. Then either $JK[G]$ is nilpotent or $[M\colon M \cap \mathbf{Z}(G)] < \infty$.*

Proof. Since $\dim FM < \dim FG$ we know that $JK[M]$ is nilpotent. Hence by Lemma 1.7, $D = \mathbf{D}_G(M)$ carries the radical. Now $D \lhd G$ and if $\dim FD < \dim FG$ then $JK[D]$ would be nilpotent and hence so would $JK[G]$. On the other hand if $\dim FD = \dim FG$ then $G \subseteq FD$ so by Lemma 1.2 (i) of **[3]** M has a subgroup of finite index central in D and hence in G. Thus $[M: M \cap \mathbf{Z}(G)] < \infty$.

Let r be a prime. By a Sylow r-subgroup of G we mean a maximal r-subgroup. Thus by definition, every r-subgroup is certainly contained in a Sylow r-subgroup of G. Now suppose G is a locally finite linear group. Then by a theorem of Platonov (see **[6]** Theorem 9.10), for each prime r, the Sylow r-subgroups of G are conjugate in G. We will use this result implicitly in the remainder of this paper. Furthermore we have

LEMMA 3.4. *Let $G \subseteq GL_n(F)$ and let P be a Sylow p-subgroup of G. Then P contains a normal abelian divisible subgroup A of finite index, Moreover if F is algebraically closed then A can be diagonalized.*

Since the existence of subgroups of finite index is frequently annoying the following is useful. We use the subgroup $\mathcal{S}(G)$ as defined in **[5]** §5.

LEMMA 3.5. *Let $G \subseteq GL_n(F)$. Then G has a characteristic subgroup G_0 such that G/G_0 is a p' by finite group and such that G_0 has no proper subgroups of finite index. Moreover if $JK[G_0]$ is nilpotent then so is $JK[G]$ and if $\mathcal{S}(G_0)$ carries $JK[G_0]$ then $\mathcal{S}(G)$ carries $JK[G]$.*

Proof. For any group G let $R(G)$ be the intersection of all its normal subgroups of finite index and let $S(G)$ be given by $S(G)/R(G) = \mathbf{O}_{p'}(G/R(G))$. Then clearly $R(G)$ and $S(G)$ are characteristic subgroups of G. We show first that $G \subseteq GL_n(F)$ implies $[G: S(G)] < \infty$.

Let P and A be given as in Lemma 3.4. If H is a normal subgroup of G of finite index then $H \supseteq A$ since A has no subgroup of finite index. Since G has Sylow theorems it follows that P maps onto a Sylow p-subgroup of G/H so $|G/H|_p \leqq [P: A]$. Now choose $H \lhd G$ of finite index so that $|G/H|_p$ is as large as possible. Then $G \supseteq H \supseteq R(G)$ and $H/R(G)$ is residually finite so it follows that $H/R(G)$ is a p'-group. Hence $S(G) \supseteq H$ and $[G: S(G)] < \infty$.

Define $G_0 = S(G)^p$, the group generated by all p-elements of $S(G)$, or equivalently $G_0 = \mathbf{O}^{p'}(S(G))$. Clearly G_0 is characteristic in G and G/G_0 is p' by finite. We show now that G_0 has no proper subgroups of finite index. Let $S = S(G_0)$. Then S is a characteristic subgroup of

G_0 of finite index and hence $S \lhd G$. We consider the group $\bar{G} = G/S$. If $\bar{C}$ is the centralizer of $\bar{G}_0$ in $\overline{S(G)}$ then certainly $\overline{S(G)}/\bar{C}$ is finite. Also $\bar{C}/(\bar{C} \cap \bar{G}_0) \subseteq \overline{S(G)}/\bar{G}_0$ is a p'-group so $\bar{C}$ has a finite central Sylow p-subgroup and we conclude that $[\bar{C}: \mathbf{O}_{p'}(\bar{C})] < \infty$. Therefore $\mathbf{O}_{p'}(\bar{C})$ is a normal subgroup of $\bar{G}$ of finite index contained in $\overline{S(G)}$ so clearly $\overline{S(G)} \supseteq \mathbf{O}_{p'}(\bar{C}) \supseteq \overline{R(G)}$ since $R(G) \supseteq G_0 \supseteq S$. Hence by definition of $S(G)$ we have that $\overline{S(G)}/\mathbf{O}_{p'}(\bar{C})$ is a p'-group so $\overline{S(G)}$ is a p'-group and by definition of $S = S(G_0)$ we have $\bar{G}_0 = \langle 1 \rangle$ and $G_0 = S(G_0)$. Thus $G_0/R(G_0)$ is a p'-group. Since $G_0 = S(G)^p$ is generated by p-elements this yields $G_0 = R(G_0)$ and G_0 has no proper subgroups of finite index.

Suppose now that $JK[G_0]$ is nilpotent. Since $S(G)/G_0$ is a p'-group G_0 carries the radical of $S(G)$ and hence $JK[S(G)]$ is nilpotent. Thus by Lemma 16.8 of **[2]**, $JK[G]$ is nilpotent. Finally suppose $\mathcal{S}(G_0)$ carries the radical of G_0. Again G_0 carries $JK[S(G)]$ so $\mathcal{S}(G_0)$ carries the radical of $S(G)$. Since $\mathcal{S}(H)$ is generated by p-elements for any group H it follows easily that $\mathcal{S}(S(G)) = \mathcal{S}(G_0)$. Corollary 5.5 of **[5]** now yields the result.

4. Finite Sylow p-subgroups. Our linear group techniques differ sharply accordingly as the Sylow p-subgroup of G is finite or infinite. In this section we consider the finite case. The following lemma is proved in **[3]** in a slightly different form. It also follows easily from topological considerations.

LEMMA 4.1. *Let $G \subseteq GL_n(F)$ and let $T_1, T_2, \cdots, T_r$ be a finite number of affine subspaces of F_n with $G \subseteq \bigcup T_i$. Then G has a subgroup H of finite index with $H \subseteq T_i$ for some i.*

Proof. We proceed as in Lemma 2.1 of **[3]** with S deleted and with the T_i's affine subspaces. The latter causes no difficulty. At the end of that proof we deduce that G permutes transitively by right multiplication certain affine subspaces $M_1, M_2, \cdots, M_m$. Since $M_1 \cap G \neq \phi$ some M_i contains the identity. If H is the stabilizer of this M_i then $[G: H] < \infty$ and $M_iH \subseteq M_i$ yields $H \subseteq M_i \subseteq T_i$.

LEMMA 4.2. *Let $G \subseteq GL_n(F)$, let $y_1, y_2, \cdots, y_r \in F_n$ be a finite number of matrices and let $\{T_{ij}\}$ for $i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$ be a finite number of affine subspaces of F_n. Suppose that for each $x \in G$ there exists i, j with $x^{-1}y_ix \in T_{ij}$. Then G has a subgroup H of finite index such that for some fixed i, j and all $h \in H$, $h^{-1}y_ih \in T_{ij}$.*

Proof. Observe that G acts on F_n by conjugation and that this yields a homomorphism of G into $E = \mathrm{End}_F(F_n) \cong F_{n^2}$. Let the image of G be denoted by $\bar{G}$ so that $\bar{G}$ is contained in the appropriate general linear group. Furthermore for $y \in F_n$ and $e \in E$ we let y^e denote the image of y under e. Thus clearly for $x \in G$, $y \in F_n$ we have $y^{\bar{x}} = x^{-1}yx = y^x$.

For each i, j let

$$M_{ij} = \{e \in E \mid y_i^e \in T_{ij}\}.$$

Since T_{ij} is an affine subspace of F_n it follows easily that M_{ij} is an affine subspace of E. Moreover by assumption $\bar{G} \subseteq \bigcup_{ij} M_{ij}$. Hence by Lemma 4.1 $\bar{G}$ has a subgroup $\bar{H}$ of finite index with $\bar{H} \subseteq M_{ij}$ for some i, j. If H is the complete inverse image of $\bar{H}$ in G then H has the required properties.

If $G \subseteq GL_n(F)$ we let $P_0 = P_0(G)$ be the Sylow p-subgroup of the set of scalar matrices contained in G. Thus P_0 is isomorphic to a subgroup of the multiplicative group $F - \{0\} = F^\circ$. Observe that P_0 is independent of the choice of basis which gives rise to $GL_n(F)$. In other words if $s \in GL_n(F)$ and if G is replaced by $s^{-1}Gs$ then $P_0(s^{-1}Gs) = P_0(G)$. As usual we let π_{P_0}: $K[G] \to K[P_0]$ denote the natural projection and tr: $F_n \to F$ the ordinary matrix trace. The main result of this section is as follows.

PROPOSITION 4.3. *Let $G \subseteq GL_n(F)$ and let*

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \neq 1$ and with $\pi_{P_0}(\alpha) \notin JK[P_0]$. Suppose that the Sylow p-subgroups of G are finite and that Q is a Sylow q-subgroup of G. Then there exist $x_i \in \mathrm{Supp}\,\alpha$, a nonscalar group element, $H \subseteq G$ a subgroup of finite index and $\tilde{Q} \subseteq Q$ a subgroup of finite index such that

$$\mathrm{tr}\, x_i^h (1 - y) = 0$$

for all $h \in H$, $y \in \tilde{Q}$.

Proof. Since G has only finitely many conjugacy classes of p-elements it follows that there are only finitely many possibilities for $\mathrm{tr}\, x$ if x is a p-element. Say these values are $\mu_1, \mu_2, \cdots, \mu_t \in F$. Furthermore if x is a $\{p, q\}$-element of G then writing $x = x_p x_q$ as a product of its p and q parts with $(x_q)^{q^m} = 1$ we have since char $F = q$

$$(\operatorname{tr} x)^{q^m} = \operatorname{tr} x^{q^m_x} = \operatorname{tr} x_p^{q^m} = (\operatorname{tr} x_p)^{q^m}.$$

Thus $\operatorname{tr} x = \operatorname{tr} x_p = \mu_i$ for some i.

Let $x \in G$. Then

$$\alpha^x = 1 + \sum a_i x_i^x \in JK[G]$$

and $\pi_{P_0}(\alpha^x) = \pi_{P_0}(\alpha) \notin JK[P_0]$. If $y \in Q$ then y is of course a q-element so by Lemma 1.2 we deduce that for some $i, x_i^x \notin P_0$ and $x_i^x y$ is a $\{p, q\}$-element. Observe that this implies that x_i is not a scalar matrix since the scalars contain no elements of order q. Hence we have shown that given $x \in G$, $y \in Q$ there exist i, j with $x_i \in \operatorname{Supp} \alpha$ a nonscalar matrix and with $\operatorname{tr} x_i^x y = \mu_j$.

For fixed x and for those nonscalar x_i's let

$$M_{ij} = \{\gamma \in F_n \mid \operatorname{tr} x_i^x \gamma = \mu_j\}.$$

Then M_{ij} is clearly an affine subspace of F_n and we have $Q \subseteq \bigcup M_{ij}$. Thus by Lemma 4.1 Q has a subgroup Q_x of finite index such that for some subscript $i = f(x)$ we have $Q_x \subseteq M_{ij}$ for some j. That is, $\operatorname{tr} x_i^x y = \mu_j$ for all $y \in Q_x$. Note that $1 \in Q_x$ so $\mu_j = \operatorname{tr} x_i^x$ and the above becomes

$$\operatorname{tr} x_i^x (1 - y) = 0$$

for all $y \in Q_x$. Note also that by choice x_i is not a scalar matrix.

Now for each nonscalar x_i define S_i to be the subspace of F_n given by

$$S_i = \langle x_i^x \mid f(x) = i \rangle.$$

Observe then that for each $x \in G$ there exists i, namely $i = f(x)$, with $x_i^x \in S_i$. Thus by Lemma 4.2 G has a subgroup H of finite index such that for some $i, x_i^h \in S_i$ for all $h \in H$. Say this occurs for the nonscalar matrix x_1.

Let $\{x_1^{w_k}\}$ be a finite spanning set for S_1 with $w_k \in G$ and with $f(w_k) = 1$. If $\tilde{Q} = \bigcap_k Q_{w_k}$ then $[Q : \tilde{Q}] < \infty$ and for all $y \in \tilde{Q}$

$$\operatorname{tr} x_1^{w_k} (1 - y) = 0$$

for all k. Thus for all $s_1 \in S_1$ we have $\operatorname{tr} s_1(1 - y) = 0$ and since S_1 contains all H-conjugates of x_1 the result follows.

Observe that in the above if Q is finite then the conclusion is decidedly uninteresting. Namely we could then have $\tilde{Q} = \langle 1 \rangle$ so certainly $\operatorname{tr} x_i^h(1-y) = 0$ for $y \in \tilde{Q}$. Fortunately in this case we can apply the following well known theorem of Brauer and Feit (see **[6]** Corollary 9.7).

PROPOSITION 4.4. (Brauer-Feit). *Let G be a locally finite subgroup of $GL_n(F)$ with F a field of characteristic $q > 0$. If the Sylow q-subgroups of G are finite then G has a normal abelian subgroup of finite index.*

This is of course a modular analog of Jordan's theorem for complex linear groups. Furthermore there is a bound for the index depending upon n and the size of the Sylow q-subgroups.

5. Infinite Sylow p-subgroups.

We now consider the case of infinite Sylow p-subgroups. This will require a close look at p^nth roots of unity.

LEMMA 5.1. *Let $f > 1$ be an integer and assume that $p \mid f-1$ and that $4 \mid f-1$ if $p = 2$. Then for all integers $a \geqq 1$*

$$|f^a - 1|_p = |a|_p \cdot |f-1|_p.$$

Proof. This is standard. We first consider some special cases. Suppose $p \nmid a$. Since $f \equiv 1\,(p)$ we have

$$\frac{f^a - 1}{f-1} = 1 + f + \cdots + f^{a-1} \equiv a\,(p)$$

so $|f^a - 1|_p = |f-1|_p = |a|_p\,|f-1|_p$.

Now let $a = p$. If $p = 2$ then $4 \mid f-1$ so $f = 1 + 4k$ and

$$\frac{f^a - 1}{f-1} = 1 + f = 2 + 4k.$$

Thus $|f^a - 1|_2 = 2|f-1|_2$. On the other hand for $p > 2$ we have $p \mid f-1$ so $f \equiv 1 + pk\,(p^2)$. Thus $f^i \equiv 1 + ipk\,(p^2)$ and

$$\begin{aligned}\frac{f^a - 1}{f-1} = 1 + f + \cdots + f^{p-1} &\equiv p + p^2\frac{(p-1)}{2}k\,(p^2)\\ &\equiv p\,(p^2).\end{aligned}$$

Therefore we have $|f^a - 1|_p = p\,|f-1|_p = |a|_p \cdot |f-1|_p$.

The result follows easily by induction on a. Namely if $a = bc$ is a proper factorization then

$$\begin{aligned}|f^a - 1|_p = |(f^b)^c - 1|_p &= |c|_p\,|f^b - 1|_p \\ &= |c|_p\,|b|_p\,|f-1|_p = |a|_p \cdot |f-1|_p.\end{aligned}$$

LEMMA 5.2. *Let $GF(f)$ be a finite field and suppose that $p \mid f-1$ and that $4 \mid f-1$ if $p = 2$. Let η generate the Sylow p-subgroup of the multiplicative group $GF(f)^\circ$. Then for all n the polynomial $x^{p^n} - \eta \in GF(f)[x]$ is irreducible.*

Proof. Note that $o(\eta) = p^m = |f-1|_p$. Thus if δ is a root of $x^{p^n} - \eta$ then $o(\delta) = p^{m+n}$. If $\delta \in GF(f^a)$ then $p^{m+n} \leqq |f^a - 1|_p = |a|_p \cdot |f-1|_p$ by Lemma 5.1 so $p^n \leqq |a|_p$ and $p^n \leqq a$. Thus $x^{p^n} - \eta$ must be irreducible.

We now assume that $G \subseteq GL_n(q^\infty)$ and that G has an infinite Sylow p-subgroup P as described in Lemma 3.4. Furthermore by considering a conjugate of G in $GL_n(q^\infty)$ if necessary we may assume that the maximal abelian divisible subgroup A is diagonalized. If $[P: A] = p^a$ then we define the field $F_0 = F_0(G)$ by $F_0 = GF(q)[\mathscr{E}]$ where $o(\mathscr{E}) = p^{a+2}$. Observe that if F is any finite field containing F_0 then clearly $|F^\circ|_p \geqq p^2$ and hence Lemma 5.2 with $F = GF(f)$ will always apply. We fix the choice of F_0.

Let $k = k(A)$ denote the maximal number of distinct eigenvalues of any element of A. Clearly $1 \leqq k(A) \leqq n$. If $k(A) = 1$ then A consists of scalar matrices and is essentially trivial for our purposes. Thus our interest is in $k(A) \geqq 2$.

Let F be a finite subfield of $GF(q^\infty)$. By an F-functional $l\colon GF(q^\infty)_n \to GF(q^\infty)$ we mean a linear functional of the form

$$l([x_{ij}]) = \sum_i^n f_i x_{ii}$$

with $f_i \in F$, some $f_i = 0$ so that not all diagonal entries occur and some $f_i \neq 0$ so that this is not the zero form. The following lemma is the crux of our argument.

LEMMA 5.3. *Let $G \subseteq GL_n(q^\infty)$ and let P be a Sylow p-subgroup of G as described above and with A diagonal. Let*

$$\alpha = 1 + \sum k_i x_i \in JK[G]$$

with $x_i \neq 1$ and $\pi_{P_0}(\alpha) \notin JK[P_0]$. Define

$$F = F(\alpha) = F_0[\operatorname{tr} x_i \mid x_i \in \operatorname{Supp} \alpha].$$

(i) *If $k(A) \geqq 3$ then there exists an F-functional l and a nonscalar x_i with $l(x_i) \in F$.*

(ii) *If $k(A) = 2$ and $G \subseteq SL_n(q^\infty)$ then there exists an F-functional l and a nonscalar x_i such that $l(x_i)^n \in F$.*

Proof. Since $k(A) \geqq 2$ we have $A \neq \langle 1 \rangle$ and we can choose $y \in A$, $y \neq 1$ to have the maximal number $k = k(A)$ of distinct eigenvalues. Since any root of y in A has at least as many distinct eigenvalues as y does, by taking a suitable root if necessary, we may assume that $o(y) > n^2$. This will only be needed for (ii).

Let L be the finite subfield of $GF(q^\infty)$ generated by F_0 and all the entries of all the matrices x_i. Clearly $L \supseteq F$. Let $|L^\circ|_p = p^h$ and choose $x \in A$ with $x^{p^h} = y$. Since A is diagonal we have $x = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Now x is a p-element so by Lemma 1.2 there exists $x_i \notin P_0$ such that $x_i x$ is a p-element, say $i = 1$. This clearly implies that x_1 is not scalar. Write

$$x_1 = \begin{pmatrix} w_1 & & * \\ & w_2 & \\ & \ddots & \\ * & & w_n \end{pmatrix}$$

so all $w_i \in L$ and since $x_1 x$ is a p-element it is conjugate to some element $z \in P$. Note that z may not be diagonal but let its eigenvalues be $\mu_1, \mu_2, \cdots, \mu_n$. Thus we have

$$\sum w_i \lambda_i = \operatorname{tr} x_1 x = \operatorname{tr} z = \sum \mu_i. \tag{1}$$

Choose p-element $\lambda \in GF(q^\infty)$ of sufficiently large order so that $\lambda_i, \mu_i \in \langle \lambda \rangle$ and $o(\lambda) \geqq p^h$. Say $o(\lambda) = p^{h+m}$ and write

$$\lambda_i = \lambda^{b'_i} = \lambda^{p^m b_i + c_i} \qquad 0 \leqq c_i < p^m$$

$$\mu_i = \lambda^{d'_i} = \lambda^{p^m d_i + e_i} \qquad 0 \leqq e_i < p^m.$$

Then (1) yields

$$\sum_1^n (w_i\lambda^{p^m b_i})\lambda^{c_i} = \sum_1^n (\lambda^{p^m d_i})\lambda^{e_i} \tag{2}$$

Note that $o(\lambda^{p^m}) = p^h$ so $\lambda^{p^m} \in L$ and thus λ is a root of the polynomial in $L[t]$ given by

$$\sum_1^n (w_i\lambda^{p^m b_i})t^{c_i} - \sum_1^n (\lambda^{p^m d_i})t^{e_i}.$$

Furthermore since λ^{p^m} in fact generates the p-part of L° Lemma 5.2 implies that the minimal equation for λ over L has degree p^m. Since all c_i and e_i satisfy $0 \leqq c_i,\ e_i < p^m$ we deduce therefore that this polynomial must vanish identically. Hence

$$\sum_1^n (w_i\lambda^{p^m b_i})t^{c_i} = \sum_1^n (\lambda^{p^m d_i})t^{e_i}. \tag{3}$$

We first consider the left hand side (lhs) of (3). If $c_i = c_j$ then $\lambda_i/\lambda_j = \lambda^{p^m(b_i-b_j)}$ so $(\lambda_i/\lambda_j)^{p^h} = 1$ and $\lambda_i^{p^h} = \lambda_j^{p^h}$. Note that by definition of x

$$y = x^{p^h} = \mathrm{diag}(\lambda_1^{p^h}, \lambda_2^{p^h}, \cdots, \lambda_n^{p^h})$$

so we see that x has at least as many distinct c_i's as y has distinct eigenvalues. Now certainly x has at least as many distinct eigenvalues as it has distinct c_i's. Finally $y \in A$ was chosen to have the maximal number k of distinct eigenvalues. All this implies that x has precisely k distinct eigenvalues and that these have distinct c_i's. For convenience let us assume that the rows and columns are so labeled that $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct. Then the lhs of (3) looks like

$$\mathrm{lhs} = \sum_1^k (\sigma_i\lambda^{p^m b_i})t^{c_i}$$

where each σ_i is the appropriate sum of those w_j's such that $\lambda_j = \lambda_i$.

Now if some $\sigma_i = 0$ then

$$l(x_1) = \sum_{\lambda_j = \lambda_i} w_j = 0$$

is an appropriate F-functional with value $0 \in F$ since $k \geqq 2$. Thus we may assume that $\sigma_i \neq 0$ for all i. This implies that the lhs of (3) contains precisely k terms.

We now consider the right hand side (rhs) of (3). By (3) and the above there must be at least k distinct e_i's. Since $o(\lambda) = p^{m+h}$ we have $(\mu_i/\mu_j)^{p^h} = \lambda^{(e_i - e_j)p^h}$ and hence $\mu_i^{p^h} = \mu_j^{p^h}$ if and only if $e_i = e_j$. Now $L \supseteq F_0$ so $h \geqq a$ by definition of F_0. Hence $z^{p^a} \in A$ implies $z^{p^h} \in A$. Observe that z^{p^h} has eigenvalues $\mu_1^{p^h}, \mu_2^{p^h}, \cdots, \mu_n^{p^h}$ so by definition of $k(A)$ there are at most k distinct $\mu_i^{p^h}$'s and hence at most k distinct e_i's. This therefore implies that there are precisely k distinct e_i's say $e_1, e_2, \cdots, e_k$.

Now observe that since $h \geqq a$, $\mu_i^{p^a} = \mu_j^{p^a}$ implies $\mu_i^{p^h} = \mu_j^{p^h}$ and hence $e_i = e_j$. Thus there are at least k distinct $\mu_i^{p^a}$'s. But $z^{p^a} \in A$ so by definition of k there are at most k distinct $\mu_i^{p^a}$'s. Therefore we deduce that $e_i = e_j$ implies that $\mu_i^{p^a} = \mu_j^{p^a}$ so $(\mu_i/\mu_j)^{p^a} = 1$ and $\mu_i/\mu_j \in F_0$ by definition of F_0. Finally by grouping together all the terms of the rhs with the same e_i we have

$$\text{rhs} = \sum_1^k (\tau_i \lambda^{p^m d_i}) t^{e_i}$$

with $\tau_i \in F_0$ since τ_i is the sum of all those terms μ_j/μ_i with $e_j = e_i$.

We now have the equal polynomials in t

$$\sum_1^k (\sigma_i \lambda^{p^m b_i}) t^{c_i} = \sum_1^k (\tau_i \lambda^{p^m d_i}) t^{e_i}$$

with the c_i's distinct, the e_i's distinct and all $\sigma_i \neq 0$. Thus the terms on the right and left sides must match one for one and by renumbering the right side if necessary we deduce that for $i = 1, 2, \cdots, k$

$$\text{(4)} \qquad \begin{aligned} c_i &= e_i \\ \sigma_i &= \tau_i \lambda^{p^m(d_i - b_i)} \qquad \tau_i \in F_0. \end{aligned}$$

Let $\mu = \lambda^{p^m}$ so that $o(\mu) = p^h$ and μ generates the p-part of L°. Then $\sigma_i = \tau_i \mu^{d_i - b_i}$. Now clearly $L \supseteq F \supseteq F_0$ so if $|F^\circ|_p = p^r$ we have $h \geqq r$ and say $h = r + s$. Write

$$\sigma_i = \tau_i \mu^{d_i - b_i} = \tau_i \mu^{p^s u_i + v_i} \qquad 0 \leqq v_i < p^s.$$

Then $\tau_i \in F_0 \subseteq F$, $\mu^{p^s} \in F$, $\operatorname{tr} x_1 \in F$ so

$$\operatorname{tr} x_1 = \sigma_1 + \sigma_2 + \cdots + \sigma_k = \sum_1^k (\tau_i \mu^{p^s u_i}) \mu^{v_i}$$

and μ is a root of the polynomial in $F[t]$ given by

$$\operatorname{tr} x_1 - \sum_1^k (\tau_i \mu^{p^s u_i}) t^{v_i}.$$

But $o(\mu^{p^s}) = p^r = |F^\circ|_p$ so Lemma 5.2 implies that the minimal polynomial of μ over F has degree p^s. Since $0 \leqq v_i < p^s$ we deduce that

$$\sum_1^k (\tau_i \mu^{p^s u_i}) t^{v_i} = \operatorname{tr} x_1. \tag{5}$$

Suppose two distinct v_i's occur. Then let v be one such nonzero v_i. It follows that

$$\sum_{v_i = v} \tau_i \mu^{p^s u_i} = 0$$

so

$$l(x_1) = \sum_{v_i = v} \sigma_i = \Big(\sum_{v_i = v} \tau_i \mu^{p^s u_i} \Big) \mu^v = 0$$

is an appropriate F-functional for x_1 with value $0 \in F$.

Thus we may suppose that all $v_i = v$. If $\operatorname{tr} x_1 \neq 0$ then by (5) we must have $v = 0$ and hence since $k \geqq 2$

$$l(x_1) = \sigma_1 = \tau_1 \mu^{p^s u_1} \in F$$

is an appropriate functional. Also if $k > 2$ then

$$l(x_1) = \sigma_1 / \tau_1 \mu^{p^s u_1} - \sigma_2 / \tau_2 \mu^{p^s u_2} = 0$$

is an appropriate F-functional with value $0 \in F$.

There remains the case $k(A) = 2$, $\operatorname{tr} x_1 = 0$ and here we can assume $G \subseteq SL_n(q^\infty)$. Note that $\sigma_1 + \sigma_2 = \operatorname{tr} x_1 = 0$ and by (4) $\sigma_1 = \tau_1 \mu_1 / \lambda_1$ and $\sigma_2 = \tau_2 \mu_2 / \lambda_2$. Suppose that λ_1 occurs in x with multiplicity b so that λ_2 occurs with multiplicity $n - b$. Then

$$1 = \det x = \lambda_1^b \lambda_2^{n-b}.$$

Now suppose that $e_j = e_1$ for c values of j so $e_j = e_2$ for $n - c$ values of j. Since $e_i = e_j$ implies that μ_i/μ_j is a p^ath root of unity we then have

$$1 = \det z = \mu_1^c \mu_2^{n-c} \rho$$

where $\rho^{p^a} = 1$ and hence $\rho \in F_0 \subseteq F$.

By renumbering if necessary we may suppose that $o(\lambda_1) \geqq o(\lambda_2)$ so that $o(\lambda_1) = o(x)$. Since $\sigma_2 = -\sigma_1$ we have using $\mu_1^c \mu_2^{n-c} \in F$

$$\sigma_1^n = \sigma_1^c(-\sigma_2)^{n-c} = (\tau_1/\lambda_1)^c(-\tau_2/\lambda_2)^{n-c}\mu_1^c\mu_2^{n-c}$$

so

$$\sigma_1^n = \eta \lambda_1^{-c} \lambda_2^{c-n} \tag{6}$$

for some $\eta \in F$. Thus using $\lambda_2^{n-b} = \lambda_1^{-b}$ we have

$$\begin{aligned} \sigma_1^{n(n-b)} &= \eta^{n-b}\lambda_1^{-c(n-b)}\lambda_2^{(c-n)(n-b)} \\ &= \eta^{n-b}\lambda_1^{-c(n-b)}\lambda_1^{-b(c-n)} \\ &= \eta^{n-b}\lambda_1^{n(b-c)}. \end{aligned}$$

Note that $\sigma_1 \neq 0$ so $\eta \neq 0$ and $\sigma_1, \eta \in L$. Thus $\lambda_1^{n(b-c)}$ is a p-element of L so by definition of h, $o(\lambda_1^{n(b-c)}) \leqq p^h$ and $\lambda_1^{p^h n(b-c)} = 1$. Now $o(x) = o(\lambda_1)$ and $x^{p^h} = y$ so $y^{n(b-c)} = x^{p^h n(b-c)} = 1$. On the other hand y was chosen to have order larger than n^2. Since $1 \leqq b, c \leqq n-1$ this implies easily that $b = c$. Therefore (6) yields

$$\sigma_1^n = \eta\lambda_1^{-b}\lambda_2^{b-n} = \eta \in F$$

and $l(x_1) = \sigma_1$ is an appropriate F-functional with value an nth root of an element of F. This completes the proof.

The main result of this section is now an easy consequence.

PROPOSITION 5.4. *Given the assumptions of Lemma* 5.3, *there exists a subgroup H of finite index in G, an F-functional l and a nonscalar $x_i \in \operatorname{Supp}\alpha$ with*

$$l(x_i^h) = l(x_i)$$

for all $h \in H$.

Proof. We use the notation of Lemma 5.3. For each F-functional l and constant c with $c^n \in F$ let

$$M(l, c) = \{\gamma \in GF(q^\infty)_n \mid l(\gamma) = c\}.$$

Then $M(l, c)$ is an affine subspace of the matrix ring $GF(q^\infty)_n$. Furthermore since F is finite there are only finitely many of these.

Let $x \in G$. Then

$$\alpha^x = 1 + \sum k_i x_i^x \in JK[G]$$

with $x_i^x \neq 1$ and since P_0 is central, $\pi_{P_0}(\alpha^x) = \pi_{P_0}(\alpha) \notin JK[P_0]$. Moreover since $\operatorname{tr} x_i^x = \operatorname{tr} x_i$ we have $F(\alpha^x) = F(\alpha)$. Thus by Lemma 5.3 applied to α^x, there exists an F-functional l and nonscalar x_i^x (and hence x_i is nonscalar) so that $l(x_i^x) = c$ for some c with $c^n \in F$. In other words we have shown that for each $x \in G$ there exists a nonscalar $x_i \in \operatorname{Supp} \alpha$ with $x_i^x \in M(l, c)$ for some l, c. Thus by Lemma 4.2 G has a subgroup H of finite index such that for some fixed i, l, c we have $x_i^h \in M(l, c)$ for all $h \in H$. Finally since $1 \in H$, the definition of $M(l, c)$ yields

$$l(x_i^h) = c = l(x_i^1)$$

and the result follows.

6. Some linear groups.

The results of the preceding two sections lead us fairly naturally to the following definition. Let $G \subseteq GL_n(F)$. We say that G is a *large* subgroup of $GL_n(F)$ if for all nonscalar matrices $x \in G$ and all subgroups H of finite index in G, the F-linear span of the matrices $x^{h_1} - x^{h_2}$ for all $h_1, h_2 \in H$ consists precisely of all the matrices in F_n of trace 0. Let us write $S(x, H)$ for the above linear span of $x^{h_1} - x^{h_2}$ and $T(F_n)$ for the set of all matrices of trace 0. We have clearly

LEMMA 6.1. *Let $G \subseteq GL_n(F)$.*

(i) *If L is a field extension of F then G is large in $GL_n(L)$ if and only if it is large in $GL_n(F)$.*

(ii) *If $s \in GL_n(F)$ then G is large in $GL_n(F)$ if and only if $s^{-1}Gs$ is large.*

(iii) *Suppose G is large in $GL_n(F)$, $N \lhd G$ and $[G: H] < \infty$. Then H is large in $GL_n(F)$. Moreover either N consists of scalar matrices or N acts irreducibly.*

Our main result is as follows.

THEOREM 6.2. *Let K be a field of characteristic $p > 0$ and let G be a locally finite group. Suppose that $G \subseteq SL_n(F)$ is a large subgroup of $GL_n(F)$ where F is a field of finite characteristic $q \neq p$. If P_0 denotes the Sylow p-subgroup of the group of scalar matrices in G, then*

$$JK[G] = JK[P_0] \cdot K[G].$$

Proof. By Lemma 6.1 (i) we may assume that F is algebraically closed. If G consists of scalar matrices then, since $G \subseteq SL_n(F)$, G is in fact a finite abelian group so the result is clearly true here. Thus we may assume by Lemma 6.1 (iii) that G acts irreducibly. Now according to Lemma 3.1 G is conjugate to a subgroup of $GL_n(q^\infty)$. Therefore finally by Lemma 6.1 (i) (ii) we may assume that $F = GF(q^\infty)$ and clearly also that $n \geqq 2$.

Now we have

$$\pi_{P_0}(JK[G]) \cdot K[G] \supseteq JK[G] \supseteq JK[P_0] \cdot K[G].$$

Thus we need only show that $\pi_{P_0}(JK[G]) \subseteq JK[P_0]$. Suppose by way of contradiction that there exists $\beta \in JK[G]$ with $\pi_{P_0}(\beta) \not\in JK[P_0]$. Then certainly $\pi_{P_0}(\beta) \neq 0$ so we can choose $w \in \operatorname{Supp} \beta$, $w \in P_0$. If $\beta = aw + \cdots$ then clearly $\alpha = a^{-1}w^{-1}\beta \in JK[G]$, $\pi_{P_0}(\alpha) = a^{-1}w^{-1}\pi_{P_0}(\beta) \not\in JK[P_0]$ and

$$\alpha = 1 + \sum k_i x_i$$

with $x_i \neq 1$. There are now three cases to consider.

Suppose first that the Sylow p-subgroups of G are infinite and use the notation of Proposition 5.4. By replacing G by a conjugate if necessary we may assume that the divisible subgroup A of P is diagonal. Since P is infinite and $[P: A] < \infty$ we have that A is infinite. Furthermore $G \subseteq SL_n(q^\infty)$ so $k(A) \geqq 2$ since otherwise A would consist of scalars and have order at most n. Thus Proposition 5.4 applies and there exists a subgroup H of G of finite index, a functional l for some subfield of $GF(q^\infty)$ and a nonscalar $x_i \in \operatorname{Supp} \alpha$ with $l(x_i^h) = l(x_i)$ for all $h \in H$. Then for $h_1, h_2 \in H$ we have $l(x_i^{h_1} - x_i^{h_2}) = 0$ so l annihilates $S(x_i, H)$. Since G is large, l therefore annihilates $T(GF(q^\infty)_n)$ certainly a contradiction since $n \geqq 2$.

Now suppose that the Sylow p-subgroups of G are finite but the Sylow q-subgroups of G are infinite and use the notation of Proposition

4.3. Then there exists a nonscalar $x_i \in \operatorname{Supp} \alpha$, $H \subseteq G$ a subgroup of finite index and $\tilde{Q} \subseteq Q$ a subgroup of finite index such that

$$\operatorname{tr} x_i^h(1-y) = 0$$

for all $h \in H$, $y \in \tilde{Q}$. Thus for a fixed $y \in \tilde{Q}$ we have $\operatorname{tr} S(x_i, h) \times (1-y) = 0$ so since G is large $\operatorname{tr} T(GF(q^\infty)_n)(1-y) = 0$ and hence $1-y$ is a scalar matrix. But then y is a scalar q-element so $y = 1$ and $\tilde{Q} = \langle 1 \rangle$, a contradiction since we assumed Q is infinite.

Finally suppose that the Sylow q-subgroups of G are finite. Then by the Brauer-Feit result, Proposition 4.4, G has a normal abelian subgroup B of finite index. Since $n > 1$ B cannot be irreducible and hence by Lemma 6.1 (iii) B consists of scalar matrices and is central in G. Now for any $x \in G$ we have $S(x, B) = 0 \neq T(GF(q^\infty)_n)$ so G must consist of scalar matrices, a contradiction since $n > 1$ and G is irreducible. Thus $\pi_{P_0}(JK[G]) \subseteq JK[P_0]$ and the theorem is proved.

We remark that the assumption of largeness is not as restrictive as it might seem. For example if one wished to study linear groups inductively on the dimension of FG as in Lemma 3.3 then the limiting groups in which induction does not work might be expected to be large. We will see this below at least when $n = 2$.

In addition the assumption $G \subseteq SL_n(F)$ in the above is not very restrictive in view of Lemma 3.2. Finally we could of course neaten the definition of large by assuming that G has no proper subgroups of finite index. We could safely do this in view of Lemma 3.5. We now consider subgroups of $GL_2(F)$.

LEMMA 6.3. *Let $G \subseteq GL_2(F)$ with F algebraically closed and let M be a subspace of F_2. Suppose $T(F_2) > M > 0$ and $g^{-1}Mg = M$ for all $g \in G$. Then either M consists of scalar matrices* (which can only occur for $q = 2$) *or G has a subgroup of finite index which is reducible.*

Proof. Let $u_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then it is easy to see that $g^{-1}u_1g \in Fu_1$ implies that g is upper triangular and $g^{-1}u_2g \in Fu_2$ implies that g is diagonal. Hence if G normalizes either Fu_1 or Fu_2 then G is reducible. We observe in general that if $s \in GL_2(F)$ then G normalizes M if and only if $s^{-1}Gs$ normalizes $s^{-1}Ms$. Thus we can freely modify M by conjugation. Since $\dim T(F_2) = 3$ we have $\dim M = 1$ or 2.

Suppose first that $\dim M = 2$. Then it follows immediately that for some nonscalar matrix τ

$$M = \{\alpha \in T(F_2) \mid \operatorname{tr} \alpha\tau = 0\}.$$

If τ has distinct eigenvalues then by conjugating we may assume τ is diagonal and then M is the set of all matrices of the form $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$. Observe now that M contains precisely two 1-dimensional subspaces of singular matrices namely Fu_1 and Fu_3 where $u_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since conjugation preserves rank G must permute these two and G has a subgroup of index $\leqq 2$ which normalizes Fu_1 and is therefore reducible. If τ has distinct eigenvalues then by conjugating we may assume that $\tau = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$ and then M is the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. Since Fu_1 is then the unique subspace of M of singular matrices we see that G normalizes Fu_1 and is reducible.

Now let $\dim M = 1$ so that $M = F\tau$ with τ nonscalar. By conjugating we may assume that τ is diagonal or $\begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$. Observe that G normalizes $M + S$ where S is the set of scalar matrices. If τ is diagonal then $M + S$ consists of all the diagonal matrices. Thus $M + S$ contains precisely two subspaces of singular matrices one of which is Fu_2. It follows that a subgroup of G of index $\leqq 2$ normalizes Fu_2 and is therefore reducible. Finally if $\tau = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$ then $M + S$ consists of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ so Fu_1 is its unique subspace of singular matrices and the lemma is proved.

LEMMA 6.4. *Let $G \subseteq GL_2(F)$ with F algebraically closed. Then either G is large or G has a subgroup of finite index which is reducible.*

Proof. Suppose G is not large and choose $x \in G$ a nonscalar matrix and $H \subseteq G$ a subgroup of finite index with $S(x, H) \neq T(F_2)$. Then $T(F_2) > S(x, H) \supseteq 0$ and $S(x, H)$ is clearly normalized by H. Thus if $S(x, H)$ is not contained in the scalar matrices then by Lemma 6.3 applied to H we see that H has a reducible subgroup of finite index. Finally if $S(x, H)$ consists of scalar matrices then for all $h \in H$, $x^h = x + \lambda I$ for some $\lambda \in F$. Since $\det x^h = \det x$ there are at most two possible values for λ. Therefore $[H : \mathbf{C}_H(x)] \leqq 2$ and $\mathbf{C}_H(x)$ is reducible since its centralizer contains the nonscalar matrix x.

As an application we have for example

PROPOSITION 6.5. *Let G be a locally finite subgroup of $GL_2(F)$ with* $\operatorname{char} F = q > 0$. *Let K be a field of characteristic $p \neq q$ and suppose* $\mathbf{O}_p(G) = \langle 1 \rangle$. *Then $JK[G]$ is nilpotent.*

Proof. We may clearly assume that F is algebraically closed and by Lemma 3.2 we may assume that $G \subseteq SL_2(F)$. Hence if G is large in $GL_2(F)$ then Theorem 6.2 yields the result. On the other hand if G is not large then by Lemma 6.4 G has a normal subgroup H of finite index which is reducible. Then H has a normal Sylow q-subgroup Q with abelian quotient. If Q is finite then $\mathbf{C}_H(Q)$ is a normal nilpotent subgroup of G of finite index. Since $\mathbf{O}_p(G) = \langle 1 \rangle$, $\mathbf{C}_H(Q)$ is a p'-group so its group ring is semisimple. On the other hand if Q is infinite then we have easily here $D = \mathbf{D}_H(Q)$ centralizes Q. Thus again D is a p'-group and by Lemma 1.7 $JK[H] = JK[D] \cdot K[H] = 0$. Therefore in either case G has a subgroup of finite index with a semisimple group ring and Lemma 16.8 of **[2]** yields the result.

We remark that there is no real difficulty in dropping the $\mathbf{O}_p(G) = \langle 1 \rangle$ assumption in the above. The following is certainly not surprising.

LEMMA 6.6. *Let F be an infinite field. Then $SL_n(F)$ is large in $GL_n(F)$.*

Proof. First $PSL_n(F)$ is simple and infinite so $SL_n(F)$ has no proper subgroups of finite index. Let $G = SL_n(F)$ and let $x \in G$ be any nonscalar matrix. Now all vectors cannot be eigenvectors for x so choose v_1 so that $v_2 = xv_1 \notin Fv_1$. If we then extend v_1, v_2 to a basis of the space V being acted on, then by a conjugation in G we may assume x has the form

$$x = \begin{pmatrix} 0 \quad r \quad 0 \quad \cdots \quad 0 \\ \hline \\ * \end{pmatrix}$$

with $r \neq 0$. Since F is infinite choose $\alpha \in F$, $\alpha \neq 0$ with $\alpha^{2n} \neq 1$ and set

$$d_1 = \operatorname{diag}(\alpha^{-(n-1)}, \alpha, \alpha, \cdots, \alpha)$$
$$d_2 = \operatorname{diag}(\alpha, \alpha^{-(n-1)}, \alpha, \cdots, \alpha).$$

Then $d_1, d_2 \in G$ and $\alpha^{-(n-1)} \neq \alpha$ since $\alpha^n \neq 1$.

Now it is easy to see that $y = d_1^{-1} x d_1 - x$ looks like

$$y = \left(\begin{array}{c|cccc} 0 & s & 0 & \cdots & 0 \\ \hline * & & & 0 & \end{array}\right)$$

with $s \neq 0$ and then $z = d_2^{-1} y d_2 - y$ looks like

$$z = \left(\begin{array}{cc|c} 0 & a & \\ b & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

with $a \neq 0$. Moreover since $\alpha^{2n} \neq 1$

$$\alpha^n d_1^{-1} z d_1 - z = a(\alpha^{2n} - 1)e_{12}$$

so the matrix unit e_{12} is contained in $S(x, G)$.

Finally by an appropriate permutation of the basis effected by conjugation in G, e_{12} is conjugate to any ce_{ij} for some $c \neq 0$ and any $i \neq j$. So all e_{ij} with $i \neq j$ are in $S(x, G)$. Since e_{ij} is conjugate via $1 + e_{ji}$ to $(e_{ij} - e_{ji}) + (e_{ii} - e_{jj})$ we conclude that $e_{ii} - e_{jj} \in S(x, G)$. Thus $S(x, G) \supseteq T(F_n)$ and the lemma is proved.

In our last result we drop our assumption that groups are locally finite or that char $F = q$. However the only new results here concern those particular cases.

PROPOSITION 6.7. *Let K be a field of characteristic $p > 0$ and let F be an infinite field of any characteristic.*

(i) *If $G = SL_n(F)$ or $GL_n(F)$ and if $P_0 = P_0(G)$ denotes the Sylow p-subgroup of the scalar matrices in G then*

$$JK[G] = JK[P_0] \cdot K[G].$$

(ii) *If $G = PSL_n(F)$ then $JK[G] = 0$.*

Proof. Suppose first that $F \not\subseteq GF(q^\infty)$ for some prime q possibly equal to p. Then the groups $G = SL_n(F)$, $GL_n(F)$ and $PSL_n(F)$ are not locally finite. By Theorems 4.4, 4.8 and 1.6 of **[4]**, $JK[G] = JK[\Lambda^+(G)] \cdot K[G]$ where $\Lambda^+(G)$ is a certain locally finite characteristic subgroup of G. The result now follows easily in this case.

Now let $F \subseteq GF(p^\infty)$ and apply Theorem 4.4 of **[4]** and Theorem 20.3 of **[2]**. It then follows immediately that for $G = PSL_n(F)$ we have $JK[G] = 0$. On the other hand if $G = SL_n(F)$ or $GL_n(F)$ then clearly $P_0 = \mathbf{O}_p(G)$ and G/P_0 has no finite normal subgroup whose order is divisible by p. Thus in this case we have easily $JK[G] = JK[P_0] \cdot K[G]$.

Finally let $F \subseteq GF(q^\infty)$ with $q \neq p$. By Lemma 6.6, $G = SL_n(F)$ is large in $GL_n(F)$ and hence by Theorem 6.2, $JK[G] = JK[P_0] \cdot K[G]$. In particular $JK[G]$ is nilpotent. Now let $G =$

$GL_n(G)$ and set $D = \mathbf{D}_G(SL_n(F))$. Then by the above and Lemma 1.7, D carries the radical of G. Since $SL_n(F)$ has no proper subgroups of finite index we have clearly $D = \mathbf{C}_G(SL_n(F))$ is the set of scalar matrices in G so the result follows here. In addition since the center of $SL_n(F)$ is finite we see that $K[PSL_n(F)]$ is a direct summand of the semisimple algebra $K[SL_n(F)/P_0]$. Thus $K[PSL_n(F)]$ is semisimple and the proposition is proved.

On the other hand, as was pointed out by A. E. Zalesskii, other types of classical groups are not large in general. For example let G be the orthogonal group with respect to transpose t so that

$$G = \{x \in GL_n(F) \mid x^t x = 1\}.$$

If $n \geqq 2$ then G contains the nonscalar symmetric matrix

$$x = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right)$$

where I is the $(n-2) \times (n-2)$ identity matrix. Since G normalizes the set of symmetric matrices we have clearly $S(x, G) \neq T(F_n)$ here and thus G is not large.

Added in proof. A. E. Zalesskii has suggested the following nice paraphrase of Lemma 1.5. The proof is essentially the same.

LEMMA 1.5′. *Let H be a finite subgroup of G and suppose that for all $x \in G$, $H \cap H^x$ contains an element of order p. Then $JK[G] \cap K[H] \neq 0$.*

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UNIVERSITY OF WISCONSIN, MADISON