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JORDAN *-HOMOMORPHISMS BETWEEN REDUCED BANACH *-ALGEBRAS

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A number of known results on Jordan *-homomorphism between B^*-algebras are generalized to Jordan *-homomorphisms between reduced Banach *-algebras. However the main results presented here are new even for maps between B^*-algebras. We state these results briefly. For any *-algebra $\mathfrak{A}$, let $\mathfrak{A}_{qU}$ be the set of quasi-unitary elements. Let $\mathfrak{A}$ and $\mathfrak{B}$ be reduced Banach *-algebras (= A^*-algebras). Let $\varphi: \mathfrak{A} \rightarrow \mathfrak{B}$ be a linear map. Then φ is a Jordan *-homomorphism if and only if $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$. If φ is bijective these conditions are equivalent to φ being a weakly positive isometry with respect to the Gelfand-Naimark norms of $\mathfrak{A}$ and $\mathfrak{B}$.

The main results of this note are contained in Theorems 3 and 4. Theorem 1 is merely a restatement of results in **[11]**, and Theorem 2 contains a generalization to the context of reduced Banach *-algebras of results previously known for B^*-algebras. Several of these results have been recently used by the author to characterize *-homomorphisms **[13]**. Further comments on the results, and their history, will be given when they are stated. First we introduce our terminology and notation. Any terms not explained here are used in the sense defined in C. E. Rickart's book **[17]**.

We use **C**, **R**, and **N** to denote the sets of complex numbers, real numbers, and natural numbers respectively. We use λ^* to denote the complex conjugate of $\lambda \in \mathbf{C}$. All algebras have complex scalars.

Any associative algebra $\mathfrak{A}$ can be made into a Jordan algebra by defining a product

$$a \circ b = 2^{-1}(ab + ba) \qquad \forall\ a, b \in \mathfrak{A}.$$

A linear map $\varphi: \mathfrak{A} \rightarrow \mathfrak{B}$ is called a Jordan homomorphism if it preserves the Jordan structure of the algebra. Thus a linear map $\varphi: \mathfrak{A} \rightarrow \mathfrak{B}$ is a Jordan homomorphism iff

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a) \qquad \forall\ a, b \in \mathfrak{A}.$$

It is easy to check that this condition can be replaced by

$$\varphi(a^2) = \varphi(a)^2 \qquad \forall\ a \in \mathfrak{A}.$$

The terms Jordan algebra and Jordan homomorphism derive from a generalization of the formalism of quantum mechanics due to P. Jordan **[6]** which was further discussed by P. Jordan, J. von Neumann, and E. Wigner **[7]**. The term Jordan homomorphism seems to have been used first in two fundamental papers by N. Jacobson and C. E. Rickart **[4, 5]**. Under other names, Jordan homomorphims had been considered earlier in purely algebraic contexts.

If $\mathfrak{A}$ and $\mathfrak{B}$ are *-algebras, a linear map $\varphi: \mathfrak{A} \to \mathfrak{B}$ is called a *-map if it preserves (i.e., commutes with) the involutions. A Jordan *-homomorphism between *-algebras is simply a Jordan homomorphism which is also a *-map. Jordan *-homomorphisms between B^*-algebras preserve the quantum mechanical structure of the algebras. They have been called C^*-homomorphisms by R. V. Kadison **[8]** and others.

For any *-algebra $\mathfrak{A}$ the set of hermitian elements is denoted by $\mathfrak{A}_H$. It is trivial to check that a linear *-map $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a Jordan *-homomorphism if and only if it satisfies

$$\varphi(h^2) = \varphi(h)^2 \qquad \forall\, h \in \mathfrak{A}_H.$$

This is the condition we will use.

In any algebra we denote an identity element by 1. A linear map φ between algebras with identity elements is called unital if $\varphi(1) = 1$.

A map $\varphi: \mathfrak{A} \to \mathfrak{B}$ between *-algebras is called weakly positive if it satisfies

$$\varphi(h^2) \in \mathfrak{B}_+ \qquad \forall\, h \in \mathfrak{A}_H.$$

Here $\mathfrak{B}_+$ is the set $\{\sum_{j=1}^n b_j^* b_j : b_j \in \mathfrak{B}\}$. One of the important differences between reduced Banach *-algebras and B^*-algebras is the failure of the equality $\{h^2 : h \in \mathfrak{B}_H\} = \mathfrak{B}_+$ in the former case. This complicates calculations with Jordan *-homomorphisms. In particular Jordan *-homomorphisms between Banach *-algebras are weakly positive but not usually positive.

One of the fundamental properties of Banach *-algebras (we do not require the involution to be continuous) is that they have a universal *-representation which includes (in a certain weak sense) all other *-representations. The norm carried back from this *-representation is the largest submultiplicative pseudo-norm on the Banach *-algebra which satisfies the B^*-condition ($\|a^*a\| = \|a\|^2$). It is called the Gelfand-Naimark pseudo-norm, and is denoted by γ. The Gelfand-Naimark pseudo-norm on a *-algebra $\mathfrak{A}$ can also be described by

$$\gamma(a) = \sup\{\|T_a\| : T \text{ is is *-representation of } \mathfrak{A}\}.$$

Hence it is clear that the *-ideal of elements in $\mathfrak{A}$ whch are represented by zero in all *-representations of $\mathfrak{A}$ (which is called the reducing ideal, and is denoted by $\mathfrak{A}_R$) is given by

$$\mathfrak{A}_R = \{a : \gamma(a) = 0\}.$$

If $\mathfrak{A}_R = \{0\}$ the *-algebra $\mathfrak{A}$ is said to be reduced. Clearly γ is a norm rather than just a pseudo-norm if and only if the *-algebra is reduced. We use the terms "γ-isometry", "γ-contraction" and "γ unit ball" to abbreviate "isometry relative to the Gelfand-Naimark pseudo-norms", etc. A Banach *-algebra is a B^*-algebra if and only if its complete norm equals γ.

For any *-algebra $\mathfrak{A}$ a state is a positive linear functional ω such that there is a *-representation T of $\mathfrak{A}$ and a topologically cyclic unit vector x in the Hilbert space on which T acts satisfying

$$\omega(a) = (T_a x, x) \qquad \forall\ a \in \mathfrak{A}.$$

The Gelfand-Naimark pseudo-norm can be described in terms of states:

$$\gamma(a) = \sup\{\omega(a^*a)^{\frac{1}{2}} : \omega \text{ is a state of } \mathfrak{A}\}.$$

Conversely in a Banach *-algebra $\mathfrak{A}$ with an identity element states can be described in terms of the Gelfand-Naimark pseudo-norm:

$$\{\text{States on } \mathfrak{A}\} = \{\text{linear functionals } \omega \text{ on } \mathfrak{A} \text{ such that}$$
$$\omega(1) = 1 = \|\omega\|_\gamma\}$$

where $\|\omega\|_\gamma = \sup\{|\omega(a)| : a \text{ belongs to the } \gamma\text{-unit ball}\}$. For a reduced *-algebra $\mathfrak{A}$, there are enough states to separate points, and in particular an element $h \in \mathfrak{A}$ is hermitian if and only if $\omega(h)$ is real for each state on $\mathfrak{A}$.

An element u of the γ-unit ball of $\mathfrak{A}$ is called a vertex if the set of linear functionals ω on $\mathfrak{A}$ such that $\omega(u) = 1 = \|\omega\|_\gamma$ separates points of $\mathfrak{A}$. In the course of proving Theorem 2 we will extend a result of H. F. Bahnenblust and S. Karlin [1] to show that an element in a reduced Banach *-algebra $\mathfrak{A}$ is a vertex of the γ-unit ball if and only if it is unitary. We denote the set of unitary elements in a *-algebra $\mathfrak{A}$ by $\mathfrak{A}_U = \{u \in \mathfrak{A} : u^*u = uu^* = 1\}$. The set $\{v \in \mathfrak{A} : v^*v = vv^* = v + v^*\}$ of quasi-unitary elements is denoted by $\mathfrak{A}_{qU}$. For a *-algebra with an identity element the involutive map $v \to 1 - v$ carries the set of quasi-unitary elements onto the set of unitary elements and *visa-versa*. The set of quasi-unitary elements is a group under quasi-multiplication and the involution is the (quasi-) inverse map in this group. The next theorem, which is one of our major tools, explains the importance of quasi-unitary elements.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach *-algebra. For each $a \in \mathfrak{A}$,*

$$\gamma(a) = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : a = \sum_{j=1}^{n} \lambda_j v_j,\ 0 = \sum_{j=1}^{n} \lambda_j \text{ where } n \in \mathbf{N},\right.$$
$$\left. \lambda_j \in \mathbf{C},\ v_j \in \mathfrak{A}_{qU}\right\}.$$

If $\mathfrak{A}$ has an identity element then for each $a \in \mathfrak{A}$,

$$\gamma(a) = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : a = \sum_{j=1}^{n} \lambda_j u_j \text{ where } n \in \mathbf{N},\ \lambda_j \in \mathbf{C},\right.$$
$$\left. u_j \in \mathfrak{A}_U\right\}.$$

*Hence if $\mathfrak{A}$ and $\mathfrak{B}$ are Banach *-algebras and $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a linear map satisying either $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$ or* (when $\mathfrak{A}$ has an identity element) *$\varphi(\mathfrak{A}_U) \subseteq \mathfrak{B}_U$ then φ is a γ-contraction.*

Proof. See **[11]**, especially the remark at the bottom of page 63.

We remark that if $\mathfrak{A}$ and $\mathfrak{B}$ are Banach *-algebras, $\mathfrak{B}$ is reduced, and $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a γ-contraction then φ is continuous with respect to the complete norms of $\mathfrak{A}$ and $\mathfrak{B}$. This follows from a standard application of the closed graph theorem since γ is always continuous with respect to the complete norm.

Next we extend some results known previously for B^*-algebras. In applying condition (b) of this theorem the following remark is sometimes useful. If $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a Jordan homomorphism, $\mathfrak{A}$ has an identity element, and $\mathfrak{B}$ is a topological algebra, which is the closure of the algebra generated by $\varphi(\mathfrak{A})$, then φ is unital **[14,** 0.10.3]. It is easy to prove, starting from (b), that $\mathrm{Ker}(\psi)$ is a closed *-ideal [**14,** 0.10.8]. This is also an immediate consequence of Theorem 3(c) below.

THEOREM 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be reduced Banach *-algebras with identity elements. Let $\psi: \mathfrak{A} \to \mathfrak{B}$ be a linear map. Then the following are equivalent.*

(a) $\psi(\mathfrak{A}_U) \subseteq \mathfrak{B}_U$.

(b) *There is a unitary element* $u \in \mathfrak{B}$ *and a unital Jordan *-homomorphism* $\varphi: \mathfrak{A} \to \mathfrak{B}$ *satisfying*

$$\psi(a) = u\varphi(a) \qquad \forall \ a \in \mathfrak{A}.$$

If ψ *is a bijection these conditions are also equivalent to*:

(c) ψ *is a* γ*-isometry.*

REMARK. We could prove this theorem by extending ψ to a map between B^*-algebras and then quoting known theorems. Instead we will indicate how to modify and piece together various known proofs to cover the present situation. In the process we give a proof for the B^*-algebra case which we believe is easier than any proof which has previously been written down in one place. We begin by modifying a proof due to A. L. T. Paterson **[15]** to prove (a) implies (b). In the B^*-algebra case this result is due to B. Russo and H. A. Dye **[18]**. The implication (b) $\Rightarrow$ (a) is easy algebra which is essentially an observation of N. Jacobson and C. E. Rickart **[4]**. When ψ is a bijection the implication ((a) and (b)) $\Rightarrow$ (c) follows from Theorem 1. In the B^*-algebra case the implication is due to B. Russo and H. A. Dye **[18]** and now has an easy proof due to L. A. Harris **[3]**. We use a result of P. Miles **[10]** and modify an argument due to H. F. Bohnenblust and S. Karlin **[1]** to show (c) $\Rightarrow$ (a). In the B^*-algebra case the implication (c) $\Rightarrow$ (a) is due to R. V. Kadison **[8]**.

Proof. Suppose ψ satisfies (a). Denote $\psi(1)$ by u and define $\varphi: \mathfrak{A} \to \mathfrak{B}$ by $\varphi(a) = u^*\psi(a)$ for each $a \in \mathfrak{A}$. Then it is enough to show that φ is a Jordan *-homomorphism.

First we show that φ is a linear *-map. It is obviously linear and it is a γ-contraction by Theorem 1. Let ω be an arbitrary state of $\mathfrak{B}$. Then $\omega(1) = 1$ and $|\omega(b)| \leqq \gamma_{\mathfrak{B}}(b)$ holds for all $b \in \mathfrak{B}$. Thus $\varphi^*(\omega)(1) = \omega\varphi(1) = \omega(1) = 1$ and $|\varphi^*(\omega)(a)| = |\omega(\varphi(a))| \leqq \gamma_{\mathfrak{B}}(\varphi(a)) \leqq \gamma_{\mathfrak{A}}(a)$ hold for all $a \in \mathfrak{A}$. Hence $\varphi^*(\omega)$ is a state of $\mathfrak{A}$. Therefore $\omega(\varphi(h)) = \varphi^*(\omega)(h)$ is real for all $h \in \mathfrak{A}_H$. Since $\mathfrak{B}$ is reduced and ω was an arbitrary state, this implies $\varphi(h)$ is hermitian. Thus φ is a *-map.

Next we show that $\varphi(h^2) = \varphi(h)^2$ for all $h \in \mathfrak{A}_H$. The involution in $\mathfrak{A}$ is norm continous since $\mathfrak{A}$ is reduced. Hence e^{ith} is a unitary element of $\mathfrak{A}$ for each $t \in \mathbf{R}$, and $h \in \mathfrak{A}_H$. Hence $\varphi(e^{ith})$ is unitary so $\varphi(e^{ith})\varphi(e^{-ith}) = \varphi(e^{ith})\varphi((e^{ith})^*) = \varphi(e^{ith})\varphi(e^{ith})^* = 1$. Expanding the first few terms of this identity shows

$$\|1 - [1 + it\varphi(h) - 2^{-1}t^2\varphi(h^2)][1 - it\varphi(h) - 2^{-1}t^2\varphi(h^2)]\| = O(t^3)$$

as t approaches zero. We conclude

$$t^2\|\varphi(h)^2 - \varphi(h^2)\| = O(t^3)$$

which implies $\varphi(h^2) = \varphi(h)^2$. This implies φ is a Jordan *-homomorphism. Hence (a) implies (b).

Now suppose (b) holds. In order to prove (a) it is obviously sufficient to show $\varphi(\mathfrak{A}_U) \subseteq \mathfrak{B}_U$ holds. For any unitary element $w \in \mathfrak{A}_U$ let $h, k \in \mathfrak{A}_H$ satisfy $w = h + ik$. Then h and k commute and $h^2 + k^2 = 1$. Hence $\varphi(h)^2 + \varphi(k)^2 = \varphi(h^2 + k^2) = \varphi(1) = 1$. Thus $\varphi(w) = \varphi(h) + i\varphi(k)$ is unitary if $\varphi(h)$ and $\varphi(k)$ commute. However a calculation shows $0 = \varphi((hk - kh)^2) = (\varphi(h)\varphi(k) - \varphi(k)\varphi(h))^2$ (cf. **[4]**). Since a skew hermitian element in a reduced *-algebra (such as $\mathfrak{B}$) is zero if its square is zero, $\varphi(h)$ and $\varphi(k)$ commute. Hence (b) implies (a).

Now suppose ψ is a bijection. If (b) holds, the map φ is a bijection and hence a Jordan *-isomorphism. Thus both ψ and ψ^{-1} satisfy (a) so $\psi(\mathfrak{A}_U) = \mathfrak{B}_U$. Hence by Theorem 1 ψ is a γ-isometry. Therefore (b) implies (c).

Assume that ψ is a γ-isometry. We will show that an element u in a reduced Banach *-algebra is a vertex of the γ-unit ball if and only if it is unitary. Since an isometry obviously preserves vertices it will follow that $\psi(\mathfrak{A}_U) = \mathfrak{B}_U$.

P. Miles **[10]**, generalizing a result of R. V. Kadison **[8]**, shows that for any *-algebra $\mathfrak{A}$ and any (not necessarily complete) B^*-norm γ on $\mathfrak{A}$, an element $v \in \mathfrak{A}$ is an extreme point of the γ-unit ball if and only if it satisfies

$$(1 - v^*v)\mathfrak{A}(1 - vv^*) = \{0\}.$$

If v satisfies this condition it is a partial isometry since $(v - vv^*v)^*(v - vv^*v) = (1 - v^*v)v^*(1 - vv^*)v = 0$ holds. Thus any γ-vertex is at least a partial isometry.

Choose a faithful, γ-isometric *-representation T of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. H. F. Bohnenblust and S. Karlin **[1**, Theorem 11**]** show that for every partial isometry $v \in \mathfrak{A}$ the set of linear functionals ω on $\mathfrak{A}$ satisfying $\omega(v) = 1$ and $|\omega(a)| \leqq \gamma(a)$ for all $a \in \mathfrak{A}$ is the weak* closed convex hull of the set of linear maps of the form

$$a \to (T_a x,\ T_v x)$$

where x belongs to $\mathfrak{H}$ and $\|x\| = \|T_v x\| = 1$ holds. However all these linear functionals vanish on $1 - vv^*$. Thus if v is a γ-vertex then $vv^* = 1$. However if v is a γ-vertex then v^* is also a γ-vertex so $v^*v = 1$ also holds. Thus v is unitary. Hence (c) implies (a).

In the next theorem, condition (a) has not previously been considered. However it is natural in a number of contexts **[14]**. The equivalence of (b) and (c) is essentially due to R. V. Kadison **[9]** when the *-algebras are B^*-algebras. When the *-representation T of condition (c) is faithful, the condition says that φ is essentially the sum of a *-homomorphism and a *-anti-homomorphism.

THEOREM 3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be reduced Banach *-algebras. Then the following are equivalent for a linear map $\varphi: \mathfrak{A} \to \mathfrak{B}$.*

(a) $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$.

(b) *φ is a Jordan *-homomorphism.*

(c) *The von Neumann algebra $\mathfrak{B}'$ generated by any *-representation of the closed *-subalgebra of $\mathfrak{B}$ generated by $\varphi(\mathfrak{A})$ contains a central projection e satisfying*

$$\left.\begin{aligned} T_{\varphi(ab)}e &= T_{\varphi(a)\varphi(b)}e \\ T_{\varphi(ab)}(1-e) &= T_{\varphi(b)\varphi(a)}(1-e) \end{aligned}\right. \qquad \forall\, a, b \in \mathfrak{A}.$$

When these conditions hold φ is a weakly positive γ-contraction.

Proof. Assume (a) holds. Whether or not $\mathfrak{A}$ already has an identity element we adjoin a new one. That is, we construct the Banach *-algebra $\mathfrak{A}^1$ with $\mathbf{C} \oplus \mathfrak{A}$ as linear space, $(\lambda \oplus a)^* = \lambda^* \oplus a^*$ as involution, $(\lambda \oplus a)(\mu \oplus b) = \lambda\mu \oplus \lambda b + \mu a + ab$ as product and $\|\lambda \oplus a\| = |\lambda| + \|a\|$ as norm. Then $\mathfrak{A}^1$ is still reduced since

$$T^1_{\lambda \oplus a} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + T_a \end{pmatrix}$$

is a faithful *-representation of $\mathfrak{A}^1$ on $\mathfrak{H} \oplus \mathfrak{H}$ when T is a faithful *-representation of $\mathfrak{A}$ on $\mathfrak{H}$. Construct $\mathfrak{B}^1$ similarly. Define $\varphi^1: \mathfrak{A}^1 \to \mathfrak{B}^1$ by $\varphi^1(\lambda \oplus a) = \lambda \oplus \varphi(a)$. It is easy to check that an element in $\mathfrak{A}^1$ is unitary if and only if it has the form $\zeta(1 \oplus (-v))$ for some quasi-unitary element v in $\mathfrak{A}$ and some complex number ζ of norm one. Since a similar statement holds for $\mathfrak{B}, \varphi^1(\mathfrak{A}^1_U) \subseteq \mathfrak{B}^1_U$ holds. Hence Theorem 2 shows φ^1 (which is obviously unital) is a

Jordan *-homomorphism. Its restriction φ is also a Jordan *-homomorphism. Thus (a) implies (b).

A trivial modification of the proof that (b) implies (a) in Theorem 2 shows that (b) implies (a) here also. (Notice that this proof does not use the hypothesis that $\mathfrak{A}$ is reduced.)

When T is chosen faithful, condition (c) implies (b) in a trivial way. Now assume (a) and hence (b) hold. Replace $\mathfrak{B}$ by the closed subalgebra generated by $\varphi(\mathfrak{A})$. (It is a *-subalgebra.) Extend φ to $\varphi^1: \mathfrak{A}^1 \to \mathfrak{B}^1$ as in the proof that (a) implies (b). Then $\varphi^1(\mathfrak{A}^1_U) \subseteq \mathfrak{B}^1_U$ so φ^1 and hence φ are γ-contractions by Theorem 1 and Jordan *-homomorphisms by Theorem 2. Let $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ be the B^*-enveloping algebras of $\mathfrak{A}^1$ and $\mathfrak{B}^1$ respectively. These are simply the completions of the incomplete normed algebras $(\mathfrak{A}^1, \gamma_{\mathfrak{A}^1})$ and $(\mathfrak{B}^1, \gamma_{\mathfrak{B}^1})$. The γ-contraction φ can be extended by continuity to $\bar{\varphi}: \bar{\mathfrak{A}} \to \bar{\mathfrak{B}}$. Obviously $\bar{\varphi}$ is still a Jordan *-homomorphism and a contraction. Thus we extend $\bar{\varphi}$ once more to its double adjoint map $\bar{\varphi}^{**}: \bar{\mathfrak{A}}^{**} \to \bar{\mathfrak{B}}^{**}$ which is just the extension of $\bar{\varphi}$ by continuity in the $\bar{\mathfrak{A}}^*$-topology. The double dual spaces $\bar{\mathfrak{A}}^{**}$ and $\bar{\mathfrak{B}}^{**}$ are the W^*-enveloping algebras of $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ under *-algebra operations inherited from their interpretation as the weak closures of the universal representations of $\mathfrak{A}$ and $\mathfrak{B}$ [**2**, §12; or **14**, §4.5]. A Lemma of R. V. Kadison [**9**, Lemma 2.4] shows that $\bar{\varphi}^{**}$ is again a Jordan *-homomorphism. Arguments of R. V. Kadison [**8**, Theorem 10; 9, Theorem 2.6] based on a fundamental result of N. Jacobson and C. E. Rickart [**4**] show that $\bar{\varphi}^{**}$, and hence φ, have the asserted form with $\mathfrak{B}' = \bar{\mathfrak{B}}^{**}$. (In [**8**] and [**9**] Kadison actually assumes φ is surjective, but this is not necessary for the proof.) Now if T is any *-representation of $\bar{\mathfrak{A}}$ then there is an extension $\bar{T}$ of T to be a *-representation of $\bar{\mathfrak{A}}^{**}$ on the same Hilbert space so that the image of $\bar{\mathfrak{A}}^{**}$ under $\bar{T}$ is the von Neumann algebra generated by $T_{\bar{\mathfrak{A}}}$, and also generated by $T_{\mathfrak{A}}$. Since every *-representation of $\mathfrak{A}$ is the restriction of a *-representation T of $\bar{\mathfrak{A}}$, this proves (c).

We have already shown that φ is a γ-contraction. A Jordan *-homomorphism such as φ satisfies $\varphi(h^2) = \varphi(h)^2 \in \mathfrak{B}_+$ so it is obviously weakly positive.

For B^*-algebras the equivalence of conditions (b) and (c) in the next theorem is implicit in [**8**].

THEOREM 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach *-algebras with $\mathfrak{A}$ or $\mathfrak{B}$ reduced. Let $\varphi: \mathfrak{A} \to \mathfrak{B}$ be a linear bijection. Then the following are equivalent.*

(a) $\varphi(\mathfrak{A}_{qU}) = \mathfrak{B}_{qU}$.

(b) *φ is a weakly positive γ-isometry.*

(c) *φ is a Jordan *-isomorphism.*

If $\mathfrak{A}$ and $\mathfrak{B}$ have identity elements then these conditions are also equivalent to:

(d) *$\varphi(\mathfrak{A}_U) = \mathfrak{B}_U$ and φ is weakly positive.*

(e) *φ is a unital Jordan *-isomorphism.*

Proof. Suppose φ satisfies (a). Theorem 1 shows that both φ and φ^{-1} are γ-contractions. Thus φ is a γ-isometry. Hence both $\mathfrak{A}$ and $\mathfrak{B}$ are reduced. Now Theorem 3 applies and shows that φ (or φ^{-1}) is a bijective Jordan *-homomorphism and hence a Jordan *-isomorphism. Therefore φ is weakly positive. Thus (a) implies (b) and (c).

Suppose φ satisfies (c). If $\mathfrak{B}$ is reduced, the proof that (b) implies (a) in Theorem 3 shows that $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$ holds. In this case φ is a γ-contraction as well as a bijection so $a \in \mathfrak{A}_R$ implies $\gamma_{\mathfrak{B}}(\varphi(a)) \leqq \gamma_{\mathfrak{A}}(a) = 0$ which in turn implies $\varphi(a)$, and hence a, are zero. Thus $\mathfrak{A}$ is also reduced. By symmetry it follows also that $\mathfrak{B}$ is reduced if $\mathfrak{A}$ is reduced. Hence Theorem 3 shows that $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$ and $\varphi^{-1}(\mathfrak{B}_{qU}) \subseteq \mathfrak{A}_{qU}$ both hold. This verifies condition (a).

Suppose (b) holds. Extend φ by continuity to an isometry $\bar{\varphi}: \bar{\mathfrak{A}} \to \bar{\mathfrak{B}}$ where $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are the B^*-enveloping algebras of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. The set of positive elements in a B^*-algebra such as $\bar{\mathfrak{B}}$ is closed. Hence by continuity $\bar{\varphi}(h^2) \in \bar{\mathfrak{B}}_+$ for any $h \in \bar{\mathfrak{A}}_H$. Since $\bar{\mathfrak{A}}$ is a B^*-algebra $\bar{\varphi}$ is positive.

We now extend $\bar{\varphi}$ again by taking its double dual map $\bar{\varphi}^{**}: \bar{\mathfrak{A}}^{**} \to \bar{\mathfrak{B}}^{**}$. The double dual space $\bar{\mathfrak{A}}^{**}(\bar{\mathfrak{B}}^{**})$ is natural identified with the closure in the weak operator topology of the universal *-representation of $\bar{\mathfrak{A}}(\bar{\mathfrak{B}})$. From this interpretation it is clear that $\bar{\varphi}^{**}$ is again a positive map. However Theorem 2 shows that $\bar{\varphi}^{**}(1)$ is unitary. It is also positive since $\bar{\varphi}^{**}$ is positive. Hence $\bar{\varphi}^{**}(1)$ is 1. Then Theorem 2 shows that $\bar{\varphi}^{**}$, and hence its restriction φ, are Jordan *-homomorphisms. Thus (b) implies (c).

Theorem 2 shows the equivalence of (b) and (d). If (d) holds then Theorem 2 shows $\varphi(1)$ is a positive unitary element and hence $\varphi(1) = 1$. Thus (d) and (e) are equivalent.

The reader of this paper will be interested in two recent papers by A. L. T. Paterson and A. M. Sinclair [16] and by K. Ylinen [19] which deal with Jordan *-homomorphisms between B^*-algebras without identity elements. All of their results can be reformulated as theorems about reduced Banach *-algebras. Except for those already given, the reformulations which the author has been able to prove are unpleasantly technical. It appears to be unknown whether the statement of Theorem 1 in **[16]** remains valid when "C^*-algebra" is simply replaced by "reduced Banach *-algebra".

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