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THE RANGE OF A NORMAL DERIVATION

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The inner derivation δ_A implemented by an element A of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on the separable Hilbert space $\mathcal{H}$ is the map $X \to AX - XA$ $(X \in \mathcal{B}(\mathcal{H}))$. The main result of this paper is that when A is normal, range inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ is equivalent to the condition that $B = f(A)$ where $\Lambda(z, w) = (f(z) - f(w))(z - w)^{-1}$ (taken as 0 when $z = w$) has the property that $\Lambda(z, w)t(z, w)$ is a trace class kernel on $L^2(\mu)$ whenever $t(z, w)$ is such a kernel. Here μ is the dominating scalar valued spectral measure of A constructed in multiplicity theory. In order that a Borel function f satisfy this condition it is necessary that f be equal almost everywhere to a Lipschitz function with derivative in $\sigma(A)$ at each limit point of $\sigma(A)$ and it is sufficient (for A self-adjoint) that $f \in C^{(3)}(\mathbf{R})$.

For such operators B there is also a factorization $\delta_B = \delta_A\tau = \tau\delta_A$ by an ultraweakly continuous linear map τ from $\mathcal{B}(\mathcal{H})$ into itself satisfying $\tau(A_1'XA_2') = A_1'\tau(X)A_2'$ for $X \in \mathcal{B}(\mathcal{H})$ and A_1', A_2' commuting with A.

When $\mathcal{H}$ is finite dimensional $(X, Y) = \operatorname{trace}(XY^*)$ is an inner product and $\mathcal{B}(\mathcal{H}) = \mathcal{R}(\delta_A) \oplus \{A^*\}'$ is the orthogonal direct sum of the range of δ_A and $\{A^*\}' = \{Y \in \mathcal{B}(\mathcal{H}) : YA^* = A^*Y\}$, the commutant of the adjoint of A. This simple fact suggests that $\mathcal{R}(\delta_A)$ is a natural subspace, like the commutant, associated with A. The orthogonal decomposition also shows that range inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ holds for a pair of operators if and only if $B \in \{A\}''$, or equivalently, if and only if B is a polynomial in A. In this case $\delta_B = \delta_A\tau = \tau\delta_A$ with τ as above.

When $\mathcal{H}$ is infinite dimensional the situation is less clear. We do not know whether $\mathcal{R}(\delta_A) \cap \{A^*\}' = 0$ in general. The sum $\mathcal{R}(\delta_A) + \{A^*\}'$ is always weakly dense in $\mathcal{B}(\mathcal{H})$ but is rarely norm closed; in fact for A normal it is closed if and only if A has a finite specturm **[1]**.

The condition $B \in \{A\}''$ is neither sufficient for $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ (even if A is positive and compact **[19]**), nor necessary [Yang Ho, private communication].

If $A \in \mathcal{B}(\mathcal{H})$ and $B = f(A)$, where f is analytic in a neighborhood of the specturm of A, then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ but range inclusion does not imply $B = f(A)$ for some analytic f **[20]**. Finally, if $\{A\}'$ contains no nonzero trace class operator then the norm closure of $\mathcal{R}(\delta_A)$ contains the ideal of compact operators **[22]**. In this case there are operators $B \notin \{A\}''$ with $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)^-$. There are normal operators A with this property (multiplication by x in $L^2(0, 1)$ for example) and compact operators **[16]** but none that is both normal and compact. (See the remark following (2.1) below).

As byproducts of our study of the range of a normal derivation we obtain improvements of the results of **[19, 20]** and a simpler proof of the theorem of **[1]** mentioned above. Our results also yield solutions to some asymptotic commutativity problems raised in **[3]**.

1. Auxiliary results. If E, F, G are Banach spaces, $S \in \mathcal{B}(E, G)$ and $T \in \mathcal{B}(F, G)$, then the closed graph theorem implies that $\mathcal{R}(S) \subset \mathcal{R}(T)$ if and only if there is $R \in \mathcal{B}(E, F/\mathrm{Ker}(T))$ with $S = \tilde{T}R$, where $\tilde{T}$ is the element of $\mathcal{B}(F/\mathrm{Ker}(T), G)$ associated with T. In particular, range inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ of derivations on $\mathcal{B}(\mathcal{H})$ amounts to a factorization $\delta_B = \tilde{\delta}_A \sigma$ with σ a bounded linear operator from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})/\{A\}'$. Our first goal is to show that if A is normal this trivial factorization can be sharpened to: $\delta_B = \delta_A \sigma$ for some ultraweakly continuous linear operator σ from $\mathcal{B}(\mathcal{H})$ into *itself*. For this and later applications we need some simple facts about range inclusion in general.

LEMMA (1.1). *If $S, T \in \mathcal{B}(E, F)$ the following are equivalent*:

(1) *There exists a constant c such that $\|Sx\| \leqq c\|Tx\|$ for all $x \in E$.*

(2) *There exists a constant c such that for each $f \in F^*$ there is a $g \in F^*$ with $\|g\| \leqq c\|f\|$ and $S^*f = T^*g$.*

(3) $\mathcal{R}(S^*) \subset \mathcal{R}(T^*)$.

Proof. Suppose that (1) holds, and fix $f \in F^*$. Then $Tx \to \langle Sx, f\rangle$ extends to a bounded linear functional on F by the Hahn-Banach theorem and therefore there is a vector $g \in F^*$ with $\|g\| \leqq c\|f\|$ such that $\langle Sx, f\rangle = \langle Tx, g\rangle$ for all $x \in E$. Hence $S^*f = T^*g$ so that (2) is satisfied.

Clearly (2) implies (3). Suppose that (3) holds. If $f \in F^*$ then $S^*f = T^*g$ for some $g \in F^*$ and therefore $|\langle Sx, f\rangle| = |\langle x, T^*g\rangle| \leqq \|g\|\|Tx\|$ for each $x \in E$. The uniform boundedness theorem implies that there is a constant c such that $|\langle Sx, f\rangle| \leqq c\|Tx\|\|f\|$ for all $x \in E$ and all $f \in F^*$. Then $\|Sx\| \leqq c\|Tx\|$ so that (1) holds.

COROLLARY (1.2). *If $S, T \in \mathcal{B}(E, F)$ then $\mathcal{R}(S^{**}) \subset \mathcal{R}(T^{**})$ if and only if $\mathcal{R}(S) \subset \mathcal{R}(T^{**})$ where $\mathcal{R}(S)$ is identified with its canonical image in F^{**}.*

Proof. Since S^{**} is an extension of S the necessity is trivial. Suppose $\mathcal{R}(S) \subset \mathcal{R}(T^{**})$. If $x \in E$ then there is $\xi \in E^{**}$ with

$Sx = T^{**}\xi$ and hence $|\langle x, S^*f\rangle| = |\langle T^{**}\xi, f\rangle| \leqq \|\xi\| \|T^*f\|$ for $f \in F^*$. Hence by the uniform boundedness theorem there is a constant c such that $|\langle x, S^*f\rangle| \leqq c\|x\| \|T^*f\|$ for $x \in E$ and $f \in F^*$. Consequently $\|S^*f\| \leqq c\|T^*f\|$ and so $\mathcal{R}(S^{**}) \subset \mathcal{R}(T^{**})$ by Lemma (1.1).

COROLLARY (1.3). *If $S, T \in \mathcal{B}(E, F)$ these are equivalent*:
(1) $\mathcal{R}(S^*) \subset \mathcal{R}(T^*)$
(2) $\mathcal{R}(S^*) \subset \mathcal{R}(T^{***})$
(3) $\mathcal{R}(S^{***}) \subset \mathcal{R}(T^{***})$.

Proof. Conditions (2) and (3) are equivalent by Corollary (1.2). Also (1) trivially implies (2). Suppose (2) holds. If $f \in F^*$ then $S^*f = T^{***}\xi$ for some $\xi \in F^{***}$. Now each such ψ has the form $\xi = \xi_0 + \xi_1$ where $\xi_0 \in F^0$ and ξ_1 F^*. Also T^{***} extends T^* and maps F^0 into E^0 and so $S^*f - T^*\xi_1 = T^{***}\xi_0 \in E^* \cap E^0 = \{0\}$. Thus $S^*f = T^*\xi_1 \in \mathcal{R}(T^*)$. Therefore (2) implies (1).

In the next result and in several subsequent arguments we shall make use of the duality relations between the Banach space $\mathcal{K} = \mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$, equipped with the usual sup norm, and the Banach space $\mathcal{T} = \mathcal{T}(\mathcal{H})$ of trace class operators on $\mathcal{H}$, equipped with the trace norm $\| \;\|_{\mathcal{T}}$. Recall **[4, 14]** that $\mathcal{T}$ may be isometrically identified with the conjugate space of $\mathcal{K}$ and that $\mathcal{B}(\mathcal{H})$ is the conjugate space of $\mathcal{T}$. The canonical bilinear form here is $\langle X, T\rangle = \text{trace}(XT) = \text{trace}(TX)$ for $T \in \mathcal{T}$ and X belonging to either $\mathcal{B}(\mathcal{H})$ or to $\mathcal{K}$. Finally, the ultraweak topology on $\mathcal{B}(\mathcal{H})$ is the weak* topology $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{T})$.

COROLLARY (1.4). *These are equivalent for $A, B \in \mathcal{B}(\mathcal{H})$*:
(1) *There exists a constant c such that $\|\delta_B(X)\| \leqq c\|\delta_A(X)\|$ for all $X \in \mathcal{B}(\mathcal{H})$.*
(2) *There exists a constant c such that for each $T \in \mathcal{T}(\mathcal{H})$ there is $S \in \mathcal{T}(\mathcal{H})$ with $\|S\|_{\mathcal{T}} \leqq c\|T\|_{\mathcal{T}}$ 1nd $\delta_B(T) = \delta_A(S)$.*
(3) $\delta_B^*(\mathcal{B}(\mathcal{H})^*) \subset \delta_A^*(\mathcal{B}(\mathcal{H})^*)$.
(4) $\delta_B(\mathcal{T}(\mathcal{H})) \subset \delta_A(\mathcal{T}(\mathcal{H}))$.

Proof. Conditions (3) and (4) are equivalent by Corollary (1.3) with $S = (\delta_{\mathcal{B}} | \mathcal{K})$ and $T = (\delta_A | \mathcal{K})$. Also (1) and (3) and (2) and (4) are equivalent by Lemma (1.1).

COROLLARY (1.5). *These are equivalent for $A, B \in \mathcal{B}(\mathcal{H})$*:
(1) $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$.
(2) $\delta_B(\mathcal{K}) \subset \mathcal{R}(\delta_A)$.
(3) *There exists a bounded linear map σ from $\mathcal{B}(\mathcal{H})$ into*

$\mathcal{B}(\mathcal{H})/\{A\}'$ such that $\delta_B = \tilde{\delta}_A\sigma$ where $\tilde{\delta}_A : \mathcal{B}(\mathcal{H})/\{A\}' \to \mathcal{R}(\delta_A)$ is the canonical map associated with δ_A.

(4) *There exists a constant c such that $\|\delta_B(T)\|_{\mathcal{T}} \leqq c\,\|\delta_A(T)\|_{\mathcal{T}}$ for all $T \in \mathcal{T}(\mathcal{H})$.*

Proof. Conditions (1) and (4) are equivalent by Lemma (1.1) applied to $S = (\delta_B \mid \mathcal{T})$ and $T = (\delta_A \mid \mathcal{T})$. Conditions (1) and (2) are equivalent by Corollary (1.2) applied to the restrictions of δ_A and δ_B to $\mathcal{K}$, and (1) and (3) are equivalent by the remark preceding Lemma (1.1).

2. Normal derivations. In this section we show that if A is a normal operator on a Hilbert space $\mathcal{H}$ (assumed to be separable here and in the remainder of the paper) then range inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ holds only for operators $B \in \{A\}''$. We use the fact that there is a projection P of norm one from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ with the property $P(A'_1XA'_2) = A'_1P(X)A'_2$ for $X \in \mathcal{B}(\mathcal{H})$ and A'_1, A'_2 in $\{A\}'$. In fact any projection of norm one onto the commutant has this commutativity property **[17]**. The existence of such projections is a standard fact in the theory of von Neumann algebras **[2**; Chapter 2]. A simple way to obtain one is to choose a unitary operator V with $\{V\}' = \{A\}'$ and set $P(X) = \operatorname{glim}_{n\to\infty} V^{*n}XV^n$ where glim is a fixed generalized limit on l^∞ and the equality is in the weak (inner product) sense. (See **[23]**)

The commutativity property of P immediately implies $\mathcal{R}(\delta_A) \subset \mathcal{R}(1-P)$ but one does not have equality here in general since for A normal, $\mathcal{R}(\delta_A) + \{A\}'$ is norm closed only in the trivial case in which the spectrum of A is finite **[1]**. The following fact is sufficient for our needs here:

LEMMA (2.1). *$\mathcal{R}(\delta_A)$ and $\mathcal{R}(1-P)$ have same ultraweak closures. Hence if $A\xi = \lambda\xi$ and $A\eta = \lambda\eta$ for $\xi, \eta \in \mathcal{H}$ then $((1-P)(X)\xi, \eta) = 0$.*

Proof. We have $\mathcal{R}(\delta_A) \subset \mathcal{R}(1-P)$ so that by considering annihilators in $\mathcal{T}(\mathcal{H})$ it suffices to show that $\langle (1-P)(X), T\rangle = 0$ for each trace class operator T that commutes with A. Now for such a T we have the polar factorization $T = U(T^*T)^{\frac{1}{2}}$ where U is a partial isometry and both factors belong to $\{A\}'$. Since $P(XU) = P(X)U$ it suffices to consider the case $T \geqq 0$. In fact, by the spectral theorem we need only show that $\langle (1-P)(X), E\rangle = 0$ for $X \in \mathcal{B}(\mathcal{H})$ and $E \in \{A\}'$ a projection of finite rank. Now for the projection P constructed in **[12]** this can easily be verified since $P(X)$ is obtained from the operators V^*XV with V unitary in $\{A\}''$. However we can also prove the assertion assuming only the existence of P as follows: With respect to the decomposition $\mathcal{H} = \mathcal{R}(E) \oplus \mathcal{R}(E)^\perp = \mathcal{H}_0 \oplus \mathcal{H}_1$ simple calculations with two by two operator matrices show that P induces a norm one projection p from

$\mathcal{B}(\mathcal{H}_0)$ onto the commutant of $A_0 = (A \mid \mathcal{H}_0)$ such that $\langle X - P(X), E\rangle =$ trace$(X_0 - p(X_0))$, where $X_0 = E(X \mid \mathcal{H}_0)$, and this last quantity is 0 because the formula $\mathcal{B}(\mathcal{H}_0) = \mathcal{R}(\delta_{A_0}) + \{A_0^*\}'$ shows that $\mathcal{R}(1-p) = \mathcal{R}(\delta_{A_0})$, that is, $X_0 - p(X_0)$ is a commutator.

The second assertion of the Lemma follows from the first by obserivng that the operator $\xi \otimes \eta$ defined by $(\xi \otimes \eta)(\zeta) = (\zeta, \eta)\xi$ is a trace class operator commuting with A so that $0 = \langle (1-P)(X)\rangle, \xi \otimes \eta\rangle = ((1-P)(X)\xi, \eta)$.

REMARK. A similar duality argument shows that if A is normal and compact and if $B \in \mathcal{B}(\mathcal{H})$, then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)^-$ if and only if $B - \lambda \in \{A\}'' \cap \mathcal{K}$ for some scalar λ.

The next result appears in [7; Theorem 3.2]:

LEMMA (2.2). *If A is a normal operator on $\mathcal{H}$ then*

$$\bigcap_{z \in C} \mathcal{R}(A - zI) = \{0\}.$$

Although we shall make no use of the fact, it is worth observing that (2.2) implies a stronger version of itself.

COROLLARY (2.3). *Let μ be a* (positive, regular) *Borel measure on* $\mathbf{C}$ *with compact support and let A be the operator $f(z) \to zf(z)$ on $L^2(\mu)$. If $f \in \mathcal{R}(A - zI)$ for μ almost every $z \in \mathbf{C}$ then $f = 0$.*

Proof. Let $\{K_n\}$ be a sequence of compact sets with $\lim \mu(K_n) = \|\mu\|$ and $f \in \mathcal{R}(A - zI)$ for all $z \in K_n$. If P_n is the spectral projection corresponding to K_n then $P_n f \in \mathcal{R}(P_n(A - \lambda I)P_n)$ for all $\lambda \in K_n$ and also for $\lambda \in \mathbf{C} \setminus K_n$ because $\sigma(P_n A \mid P_n H) \subset K_n$. Thus $P_n f = 0$ for all n and consequently $f = 0$.

We shall also need the following result of Korotkov. For a proof see **[10, 21]:**

LEMMA (2.4). *Let μ be a Borel measure on $\mathbf{C}$ of compact support. If $T \in \mathcal{B}(L^2(\mu))$ has $\mathcal{R}(T) \subset L^\infty(\mu)$ then T is a Hilbert-Schmidt operator with kernel $t \in L^2(\mu \times \mu)$ satisfying*

$$\text{ess sup}_z \int |t(z, w)|^2 \, d\mu(w) \leqq K^2$$

where K is the norm of T as an operator from L^2 to L^∞.

We come now to the main result of this section.

THEOREM (2.5). *If A is a normal operator on $\mathcal{H}$ then $\mathcal{R}(\delta_A)$ contains no nonzero left or right ideal of $\mathcal{B}(\mathcal{H})$. Hence if $B \in \mathcal{B}(\mathcal{H})$ and $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then $B \in \{A\}''$.*

Proof. Observe first that for any $S, T \in \mathcal{B}(\mathcal{H})$ the identity $X\delta_S(T') = \delta_S(XT') - \delta_S(X)T'$ implies that if $\mathcal{R}(\delta_S) \subset \mathcal{R}(\delta_T)$ then $\mathcal{R}(\delta_T)$ must contain the left (and dually, also the right) ideal of $\mathcal{B}(\mathcal{H})$ generated by $\delta_S(T')$ for each T' commuting with T. Hence if $\mathcal{R}(\delta_T)$ is known not to contain any left or right ideal ideals then $S \in \{T\}''$. Thus the second assertion of the theorem is a simple consequence of the first.

For $\xi, \eta \in \mathcal{H}$ let $\xi \otimes \eta$ denote the operator $\zeta \to (\zeta, \eta)\xi$. Every left ideal contains an ideal $\mathcal{H} \otimes \eta$ and so it is enough to show that $\mathcal{H} \otimes \eta \subset \mathcal{R}(\delta_A)$ implies $\eta = 0$. (The assertion for right ideals follows on taking adjoints.)

By restricting to the smallest reducing subspace of A that contains η we can suppose that A is the operator $f(z) \to zf(z)$ in $\mathcal{H} = L^2(\mu)$ for some regular Borel measure μ on $\mathbf{C}$ of compact support. Let P be a projection of norm 1 from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$.

For $\xi \in L^2(\mu)$ let $\gamma(\xi)$ be the unique element of $\mathcal{B}(\mathcal{H})$ with

$$\delta_A(\gamma(\xi)) = \xi \otimes \eta$$
$$P(\gamma(\xi)) = 0.$$

The operator γ is continuous by the closed graph theorem and if $M_h \in \mathcal{B}(\mathcal{H})$ is the multiplication operator on $\mathcal{H}$ induced by $h \in L^\infty(\mu)$ then $\gamma(M_h\xi) = M_h\gamma(\xi)$ because $\delta_A(M_h\gamma(\xi)) = M_h\delta_A(\gamma(\xi)) = M_h\xi \otimes \eta$ and $P(M_h\gamma(\xi)) = M_hP(\gamma(\xi)) = 0$. In particular, $\gamma(h) = M_h(\gamma(1))$ for $h \in L^\infty(\mu)$.

If $h \in L^\infty(\mu)$, $\xi \in L^2(\mu)$ then $\|M_h\gamma(1)\xi\| = \|\gamma(h)\xi\| \leqq \|\gamma\| \|h\|_2 \|\xi\|_2$ so that $h \to M_h\gamma(1)\xi = h \cdot \gamma(1)\xi$ is continuous in the L^2 norm, and therefore $\gamma(1)\xi \in L^\infty(\mu)$ with $\|\gamma(1)\xi\|_\infty \leqq \|\gamma\| \|\xi\|_2$. By Lemma (2.4) the operator $\gamma(1)$ is Hilbert-Schmidt with kernel $t(z, w)$ satisfying $\operatorname{ess\,sup} \int |t(z, w)|^2 d\mu(w) = K^2 < \infty$. Fix a vector $\xi \in L^2(\mu)$. Then there is a measurable set $E = E_\xi$ with $\mu(E') = 0$ and

$$\delta_A(\gamma(1))\xi(z) = \int (z - w)t(z, w)\xi(w)d\mu(w)$$

for $z \in E$. Since $\delta_A(\gamma(1))\xi = (1 \otimes \eta)\xi = (\xi, \eta)$, the Cauchy-Schwarz inequality gives

$$|(\xi, \eta)| \leqq \left(\int |(z - w)\xi(w)|^2 d\mu(w)\right)^{\frac{1}{2}} \left(\int |t(z, w)|^2 d\mu(w)\right)^{\frac{1}{2}}$$

for $z \in E$. Therefore $|(\xi, \eta)| \leqq K \|(A - zI)\xi\| = K \|(A - zI)^*\xi\|$ almost everywhere, and consequently, for all $z \in \sigma(A)$ by continuity of the right side of this inequality. It follows from this that $\eta \in \mathcal{R}(A - zI)$ for each $z \in \mathbf{C}$ (see the proof of Lemma (1.1)) and therefore $\eta = 0$ by Lemma (2.2).

COROLLARY (2.6). *Let A be a normal operator on $\mathcal{H}$. If $B \in \mathcal{B}(\mathcal{H})$ and $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then $\delta_B = \delta_A\sigma = \sigma\delta_A$ for some ultraweakly continuous linear operator σ from $\mathcal{B}(\mathcal{H})$ into itself with the property $\sigma(A_1'XA_2') = A_1'\sigma(X)A_2'$ for $X \in \mathcal{B}(\mathcal{H})$ and A_1', A_2' commuting with A.*

Proof. Suppose $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$. Then by (1.5) we can factor $\delta_B = \tilde{\delta}_A\tau_0$ for some $\tau_0 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\{A\}'$. Making use of a projection P of norm 1 from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ we can replace τ_0 by $\tau \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ to get $\delta_B = \delta_A\tau$. Thus for $X \in \mathcal{B}(\mathcal{H})$, $\tau(X)$ is the unique operator satisying $P(\tau(X)) = 0$, $\delta_A(\tau(X)) = \delta_B(X)$. Since $B \in \{A\}''$ it is easy to check that τ inherits the commutativity properties of P, that is $\tau(A_1'XA_2') = A_1'\cdot(X)A_2'$.

We now replace τ by an ultraweakly continuous σ with the desired properties by the following device: the map $(\tau \mid \mathcal{K})$ from $\mathcal{K}$ into $\mathcal{B}(\mathcal{H}) = \mathcal{K}^{**}$ has adjoint $(\tau \mid \mathcal{K})^*$ from $\mathcal{K}^{***}$ into $\mathcal{K}^*$. For $T \in \mathcal{T} = \mathcal{K}^* \subset \mathcal{K}^{***}$ we therefore obtain an operator $\alpha(T) = (\tau \mid \mathcal{K})^*(T)$ in $\mathcal{T}$. It is clear that $\alpha \in \mathcal{B}(\mathcal{T})$, that $(\delta_A \mid \mathcal{T})\alpha = \alpha(\delta_A \mid \mathcal{T}) = (\delta_B \mid \mathcal{T})$ and that $\delta_B = \alpha^*\delta_A = \delta_A\alpha^*$. Since $\alpha^* \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ agrees with τ on $\mathcal{K}$ we also have $\alpha^*(A_1'XA_2') = A_1'\alpha^*(X)A_2'$, first for $X \in \mathcal{K}$, then by ultraweak continuity of α^* and of multiplication by a fixed element of $\mathcal{B}(\mathcal{H})$, for all X in the ultraweak closure of $\mathcal{K}$ which is $\mathcal{B}(\mathcal{H})$. Thus $\sigma = \alpha^*$ satisfies the requirements of the Corollary.

3. On a separable space an operator B in the second commutant of a normal operator A must be a bounded Borel function of A. In this section we determine which such functions are admissible for range inclusion of the corresponding derivations. For this we need to develop some background information about Hadamard multipliers.

DEFINITION (3.1). A matrix (γ_{ij}) is a Hadamard multiplier of $\mathcal{B}(\mathcal{H})$ if there is an orthonormal basis $\{\xi_i\}$ of $\mathcal{H}$ such that for each $X \in \mathcal{B}(\mathcal{H})$ there is an operator $\Gamma(X) \in \mathcal{B}(\mathcal{H})$ with $(\Gamma(X)\xi_j, \xi_i) = \gamma_{ij}(X\xi_j, \xi_i)$ for all i, j.

Thus (γ_{ij}) is a H-multiplier if the Hadamard product of (γ_{ij}) and the matrix of any bounded operator is again the matrix of a bounded operator. I. Schur **[15]** gives several sufficient conditions that (γ_{ij}) be an H-multiplier, among them being that (γ_{ij}) is itself the matrix of an operator.

LEMMA (3.2). (1) *The condition that (γ_{ij}) be an H-multiplier is independent of the choice of orthonormal basis of $\mathcal{H}$.*

(2) *If* (γ_{ij}) *is an H-multiplier then* $X \to \Gamma(X)$ *defines a bounded linear operator on* $\mathcal{B}(\mathcal{H})$. *Moreover* Γ *is ultraweakly continuous and is the adjoint of the H-multiplier on* $\mathcal{T}(\mathcal{H})$ *induced by the transpose of* (γ_{ij}).

(3) *Let* $\{\Gamma_n\}$ *be a sequence of H-multipliers. If* $\sup_n \|\Gamma_n\| < \infty$ *and if* $(\Gamma_n(X)\xi_j, \xi_i) \to \alpha_{ij}(X\xi_j, \xi_i)$ *for each* $X \in \mathcal{B}(\mathcal{H})$ *and each i, j then* (α_{ij}) *is an H-multiplier.*

(4) *Let* $\alpha_{ij} = \begin{cases} 1 & \text{if} \quad i \leqq j \\ 0 & \text{if} \quad i > j \end{cases}$. *Then* (α_{ij}) *is not an H-multiplier on* $\mathcal{B}(\mathcal{H})$ *for* $\mathcal{H} = l^2(\mathbf{Z})$ *or for* $\mathcal{H} = l^2(\mathbf{Z}^+)$.

Proof. (1) If $\{\eta_i\}$ is another orthonormal basis of $\mathcal{H}$ and if $(\Gamma'(X)\eta_j, \eta_i) = \gamma_{ij}(X\eta_j, \eta_i)$ then $\Gamma'(X) = U^*\Gamma(UXU^*)U$ where U is the unitary operator defined by $U\eta_i = \xi_i$.

(2) That Γ is bounded on $\mathcal{B}(\mathcal{H})$ is a simple consequence of the closed graph theorem. The other two assertions of (2) are easy to verify.

(3) The hypotheses imply that for each $X \in \mathcal{B}(\mathcal{H})$ the map $T \to \lim_n \langle \Gamma_n(X), T\rangle$ is a bounded linear functional on the subspace of $\mathcal{T}(\mathcal{H})$ consisting of finite linear combinations of the operators $\xi_i \otimes \xi_j$. Since the dual of $\mathcal{T}$ is $\mathcal{B}(\mathcal{H})$ there is an operator $Z \in \mathcal{B}(\mathcal{H})$ such that $\langle Z, T\rangle = \lim_n \langle \Gamma_n(X), T\rangle$ for any T of the form $\xi_i \otimes \xi_j$. It follows that $(\alpha_{ij}(X\xi_j, \xi_i))$ is the matrix of Z and thus that (α_{ij}) is an H-multiplier.

(4) Consider first the case in which (α_{ij}) is a doubly infinite matrix $(i, j \in \mathbf{Z})$ and let $\xi_j(e^{i\theta}) = e^{ij\theta}$ be the usual basis of $L^2(0, 2\pi)$. If M_φ denotes the operator $f \to \varphi \cdot f$ on L^2 for a given $\varphi \in L^\infty$, then the Hadamard product of (α_{ij}) and the matrix of M_φ is the matrix associated with $M_{\tilde{\varphi}}$ where $\tilde{\varphi}$ is the function whose Fourier coefficients $(\tilde{\varphi}, \xi_n)$ vanish for $n < 0$ and agree with those of φ for $n \geqq 0$. Since there are $\varphi \in L^\infty$ for which $\tilde{\varphi} \notin L^\infty$ it follows that (α_{ij}) cannot be an H-multiplier of $L^2(0, 2\pi)$ and consequently cannot be an H-multiplier of $l^2(\mathbf{Z})$ either.

Consider now the matrix $\beta_{ij} = 1$ or 0 depending on whether or not $i \leqq j$ for $i, j \in \mathbf{Z}^+$. Then (β_{ij}) cannot be an H-multiplier of $l^2(\mathbf{Z}^+)$ because the doubly infinite matrix (α_{ij}) just mentioned is the sum of the direct sum $(\beta_{ij}) \oplus (\beta_{-j,-i})$ and a matrix which is an obvious H-multiplier of $l^2(\mathbf{Z})$.

There is another, perhaps more natural, way to see that the doubly infinite matrix (α_{ij}) of (4) is not a Hadamard multiplier of $l^2(\mathbf{Z})$ that we now sketch. It is enough to show that (α_{ij}) does not induce an operator α on the trace class matrices $\mathcal{T}$ on $l^2(\mathbf{Z})$. Now $\mathcal{T}$ is isometric with $l^2(\mathbf{Z}) \hat{\otimes} l^2(\mathbf{Z})$ so the convolution product gives rise to the map $(\rho S)_k = \Sigma_{i-j=k}\, s_{ij}$ of $\mathcal{T}$ into $A(\mathbf{Z})$ and in fact $A(\mathbf{Z})$ is isometric with $\mathcal{T}/\mathrm{Ker}(\rho)$. Clearly $\alpha(\mathrm{Ker}(\rho)) \subset \mathrm{Ker}(\rho)$ so α lifts to $\alpha' \in \mathcal{B}(A(\mathbf{Z}))$. Here α' is the operation of multiplication by the characteristic function of $\mathbf{Z}^+$ and it is well known that $A(\mathbf{Z})$ is *not* closed under this operation. ($L^1(\mathbf{T})$ is not closed under the Hilbert transform.) The harmonic analysis used here appears in [**13**; pp. 80–81] and [**9**; p. 64].

Hadamard multipliers are important for studying the range of a derivation mainly because of the following simple fact: (Recall that a *diagonal operator* is an operator for which there is an orthonormal basis of eigenvectors.)

LEMMA (3.3). *Suppose $A \in \mathcal{B}(\mathcal{H})$ is a diagonal operator with matrix* $\operatorname{diag}(\lambda_1, \lambda_2, \cdots)$ *with respect to the orthonormal basis $\{\xi_i\}$. If $B \in \mathcal{B}(\mathcal{H})$ then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ if and only if $B = \operatorname{diag}(\mu_1, \mu_2, \cdots)$ with respect to $\{\xi_i\}$ and (γ_{ij}) is an H-multiplier of $\mathcal{B}(\mathcal{H})$ where*

$$\gamma_{ij} = \begin{cases} (\mu_i - \mu_j)(\lambda_i - \lambda_j)^{-1} & \text{if} \quad \lambda_i \neq \lambda_j \\ 0 & \text{if} \quad \lambda_i = \lambda_j \end{cases}.$$

Proof. Suppose $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$. Then $B \in \{A\}''$ by (2.5) and hence B has a diagonal matrix with respect to the given basis. If $X \in \mathcal{B}(\mathcal{H})$ there is $Z \in \mathcal{B}(\mathcal{H})$ with $\delta_B(X) = \delta_A(Z)$. Computing (i,j) entries yields $(\mu_i - \mu_j)(X\xi_j, \xi_i) = (\lambda_i - \lambda_j)(Z\xi_j, \xi_i)$. Now we can choose Z so that $P(Z) = 0$ where P is a norm one projection from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ For such a choice Lemma (2.1) shows that $(Z\xi_j, \xi_i) = 0$ whenever $\lambda_i = \lambda_j$ and therefore $(Z\xi_j, \xi_i) = \gamma_{ij}(X\xi_j, \xi_i)$ with γ_{ij} defined as in the statement of the Lemma. This proves that the multiplier condition is necessary for $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ and it is clear that it is also sufficient.

LEMMA (3.4). *Let μ be a finite measure on $\mathbf{C}$ and let A be the operator $h(z) \to zh(z)$ on $L^2(\mu)$. Suppose that $f \in C(\sigma(A))$ and that $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$. If B is a diagonal operator with distinct eigenvalues $\lambda_1, \lambda_2, \cdots$ in $\sigma(A)$ then $\mathcal{R}(\delta_{f(B)}) \subset \mathcal{R}(\delta_B)$.*

Proof. Let $\{\xi_i\}$ be an orthonormal basis of $\mathcal{H}$ such that $B\xi_i = \lambda_i\xi_i$ for $i = 1, 2, \cdots$. Fix an integer $n > 0$ and let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be the normalized characteristic functions in $L^2(\mu)$ of disjoint neighborhoods $N_1, N_2, \cdots, N_n$ of the λ_i, each having diameter at most n^{-1}. Finally let $U = U_n$ be a partial isometry from $\mathcal{H}$ into $L^2(\mu)$ with

$$U\xi_i = \begin{cases} \varphi_i & 1 \leqq i \leqq n \\ 0 & i > n \end{cases}.$$

By (2.6) there is a map α from $\mathcal{T} = \mathcal{T}(L^2(\mu))$ into itself such that $(\delta_{f(A)} | \mathcal{T}) = \alpha(\delta_A | \mathcal{T}) = \delta_A\alpha$, and this induces a map β from $\mathcal{T}(\mathcal{H})$ into itself, namely $\beta(T) = U^*\alpha(UTU^*)U$. An easy calculation gives $(\beta(\xi_i \otimes \xi_j)\xi_l, \xi_k) = \beta_{ij}^{(n)}((\xi_i \otimes \xi_j)\xi_l, \xi_k)$

$$\beta_{ij}^{(n)} = \begin{cases} \mu(N_i)^{-1}\mu(N_j)^{-1} \int_{N_i} \int_{N_j} \lambda(z,w)\,d\mu(z)\,d\mu(w) & 1 \leqq i, j \leqq n \\ 0 & i > n \text{ or } j > n \end{cases}$$

and $\lambda(z,w) = (f(z) - f(w))(z - w)^{-1}(1 - \delta_{z,w})$. Now if Γ_n is the Hadamard multiplication on $\mathcal{T}(\mathcal{H})$ associated with the matrix $(\beta_{ij}^{(n)})$ it follows that Γ_n and β agree on the operators $\xi_i \otimes \xi_j$ and since both operators are ultraweakly continuous we have $\|\Gamma_n\| = \|\beta\| \leqq \|\alpha\|$. Hence the matrix $(\gamma_{ij}^{(n)})$ defined by $\gamma_{ij}^{(n)} = \beta_{ij}^{(n)} - \delta_{ij}\beta_{ij}^{(n)}$ has multiplier norm at most $2\|\alpha\|$. Since $\gamma_{ij}^{(n)} \to \gamma_{ij}$ where

$$\gamma_{ij} = \begin{cases} (f(\lambda_j) - f(\lambda_i))(\lambda_j - \lambda_i)^{-1} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

it follows from (3.2) that (γ_{ij}) is an H-multiplier on $\mathcal{T}(\mathcal{H})$ with multiplier norm at most $2\|\alpha\|$. Thus $(\delta_{f(B)} | \mathcal{T}) = \alpha_B(\delta_B | \mathcal{T}) = \delta_B\alpha_B$ for an operator α_B on $\mathcal{T}(\mathcal{H})$ with norm at most $2\|\alpha\|$ so that $\mathcal{R}(\delta_{f(B)}) \subset \mathcal{R}(\delta_B)$.

Our next result shows that the question whether $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ depends only on the values of f on $\sigma(A)$ and not on the normal operator A itself:

COROLLARY (3.5). *Let A and f be as in the statement of the Lemma. If B is any normal operator with $\sigma(B) \subset \sigma(A)$ then $\mathcal{R}(\delta_{f(B)}) \subset \mathcal{R}(\delta_B)$.*

Proof. Any normal operator B on $\mathcal{H}$ is a countable direct sum of operators $h(z) \to zh(z)$ on L^2 spaces and each of these is uniformly approximable by diagonal operators (approximate the identity function by a simple function.) Hence B itself is the uniform limit of a sequence $\{B_n\}$ of diagonal operators on $\mathcal{H}$ and clearly we may also choose the B_n to have distinct diagonal entries for each n. Now for each n there is an operator α_n on $\mathcal{T}(\mathcal{H})$ with $(\delta_{f(B_n)} | \mathcal{T}) = \alpha_n(\delta_{B_n} | \mathcal{T}) = \delta_{B_n}\alpha_n$ and $\sup_n \|\alpha_n\| < \infty$ by the proof of the Lemma. Fix $X \in \mathcal{B}(\mathcal{H})$ and let $Z_n = \alpha_n^*(X)$. Since the Z_n are bounded we can pass to a subsequence if necessary to insure that the sequence $\{Z_n\}$ converges ultraweakly to some $Z \in \mathcal{B}(\mathcal{H})$. Then

$$\delta_{f(B)}(X) = (f(B) - f(B_n))X + (B_n - B)Z_n + (BZ_n - Z_nB) + Z_n(B - B_n) + X(f(B_n) - f(B))$$

so that, taking ultraweak limits, $\delta_{f(B)}(X) = \delta_B(Z) \in \mathcal{R}(\delta_B)$ as required.

THEOREM (3.6). *Let A be a normal operator on $\mathcal{H}$ with dominating scalar spectral measure μ* (see [5; Theorem 10, p. 916].). *If $B \in \mathcal{B}(\mathcal{H})$*

then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ *if and only if* $B = f(A)$ *where* $f \in C(\sigma(A))$ *and* $(\alpha t)(z, w) = (f(z) - f(w))(z - w)^{-1} t(z, w)$ (taken as 0 when $z = w$) *is a trace class kernel with respect to* μ *whenever* $t(z, w)$ *is such a kernel.*

Proof. Suppose first that A is the operator $h(z) \to zh(z)$ on $L^2(\mu)$. If $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then $B \in \{A\}''$ by Theorem (2.5) so that $B = f(A)$ for some $f \in L^\infty(\mu)$. Also, by (2.5) there is an operator α from the trace class operators $\mathcal{T}$ on $L^2(\mu)$ into itself such that $(\delta_B \mid \mathcal{T}) = \alpha(\delta_A \mid \mathcal{T}) = (\delta_A \mid \mathcal{T})\alpha$. If $t(z, w)$ is the kernel of a trace class operator on $L^2(\mu)$ then $(z - w)(\alpha t)(z, w) = (f(z) - f(w))t(z, w)$ almost everywhere $\mu \times \mu$. Now if μ has mass at z_0 and ξ_0 is the corresponding normalized characteristic function, then $(\alpha t)(z_0, z_0) = \langle \xi_0 \otimes \xi_0, \alpha t \rangle = \langle \alpha^*(\xi_0 \otimes \xi_0), t \rangle = 0$ because the range of α^* is contained in the range of $1 - P$, where P is the projection of $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ used to construct α, and therefore $0 = P\alpha^*(\xi_0 \otimes \xi_0) = \alpha^*(P(\xi_0 \otimes \xi_0)) = \alpha^*(\xi_0 \otimes \xi_0)$. Thus $(\alpha t)(z, w) = (f(z) - f(w))(z - w)^{-1} t(z, w)$ (taken as 0 when $z = w$) and this holds $\mu \times \mu$ almost everywhere. To complete the proof we appeal to Theorem (4.1) below which implies that the multiplier condition on f just established forces f to be equal μ, a.e. to a continuous function on $\sigma(A)$.

Conversely, if αt is a trace class kernel for each trace class kernel t then $t \to \alpha t$ defines a bounded operator on the trace class by the closed graph theorem and $\alpha(\delta_A)t(z, w) = (f(z) - f(w))t(z, w) = \delta_{f(A)} t(z, w)$ for $z \neq w$ by definition of α and for $z = w$ since both sides of the equation are 0. Thus $\delta_B = \delta_A \alpha^*$ and $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$.

Consider now the general case. If $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then there is a reducing subspace $\mathcal{H}_0$ of $\mathcal{H}$ on which A is unitarily equivalent to the operator $h(z) \to zh(z)$ on $L^2(\mu)$. The subspace $\mathcal{H}_0$ reduces B and since $\delta_B(\mathcal{H}_0) \subset \delta_A(\mathcal{H}_0)$ the first part of the argument shows that $B = f(A)$ where f has the asserted properties.

Conversely if $B = f(A)$ with f of the given form then $\mathcal{R}(\delta_{B_0}) \subset \mathcal{R}(\delta_{A_0})$ where A_0, B_0 are the restrictions of A, B to $\mathcal{H}_0$. Corollary (3.5) then implies that $\mathcal{R}(\delta_{\tilde{B}}) \subset \mathcal{R}(\delta_{\tilde{A}})$ where $\tilde{A}, \tilde{B}$ are the direct sum of countably many copies of A_0 and B_0 respectively. Since A is the restriction of $\tilde{A}$ to a reducing subspace it follows that $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$.

COROLLARY (3.7). *Let* A *be a normal operator on* $\mathcal{H}$ *and let* $B \in \mathcal{B}(\mathcal{H})$. *Then* $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ *if and only if there is a constant* c *such that* $\|\delta_B(X)\| \leqq c \|\delta_A(X)\|$ *for all* $X \in \mathcal{B}(\mathcal{H})$.

Proof. Suppose $\|\delta_B(X)\| \leqq c \|\delta_A(X)\|$ for all X. Then $B \in \{A\}''$ so that $B = f(A)$ for some bounded Borel function f. Also if $T \in \mathcal{T}$ then $\delta_B(T) = \delta_A(S)$ for some $S \in \mathcal{T}$ by (1.4) and, in the notation of (3.6), αt is the kernel of the trace class operator S_1 satisfying

$$\langle (1-P)(K), S\rangle = \langle K, S_1\rangle$$

for all $K \in \mathcal{K}$, i.e., $S_1 = S - (P^*(S)\,|\,\mathcal{K})$. Thus $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$. The converse is clear from (2.6).

4. We now consider the problem of determining the elements of $C(\sigma(A))$ that satisfy the multiplier condition of Theorem (3.6). It seems that, just as there is no satisfactory way of recognizing which periodic functions have absolutely convergent Fourier series, so no easy description of the class of functions in Theorem (3.6) exists.

THEOREM (4.1). *Let μ be a finite measure on* **C** *of compact support and let $f \in L^\infty(\mu)$ satisfy the criterion of Theorem* (3.6). *Then f is equal* a.e. *μ to a continuous function which satisfies a Lipschitz condition and is differentiable relative to $\sigma(A)$ at each nonisolated point of $\sigma(A)$.*

Proof. Let $\Lambda(z, w) = (f(z) - f(w))(z-w)^{-1}$ for $z \neq w$ and $\Lambda(z, z) = 0$. The function Λ is μ-measurable and if ξ, η are unit vectors in $L^2(\mu)$ then the trace class norm of the operator with kernel $\Lambda(z, w)\xi(z)\eta(w)$ is no less than its Hilbert-Schmidt norm

$$\{\textstyle\int\int |\Lambda(z, w)|^2\,|\xi(z)|^2\,|\eta(w)|^2\, d\mu(z)d\mu(w)\}^{\frac{1}{2}}$$

for which the upper bound as ξ, η vary is $K = \text{ess sup}\,|\Lambda(z, w)|$. Thus $\Lambda \in L^\infty(\mu \times \mu)$ and $|f(z) - f(w)| \leqq K\,|z-w|$ for almost all (z, w). Put

$$E = \{z \in \text{Supp}(\mu) \colon |f(z) - f(w)| \leqq K\,|z - w| \text{ for almost all } w \in \text{Supp}(\mu)\}.$$

The complement of E is of measure 0. If $z_1, z_2 \in E$ we have

$$\begin{aligned} |f(z_1) - f(z_2)| &\leqq |f(z_1) - f(w)| + |f(w) - f(z_2)| \\ &\leqq K(|z_1 - w| + |z_2 - w|) \end{aligned}$$

except for values of w in a set of measure 0. If $\mu(\{z_1\}) = 0$ then we can find a sequence of values of w outside this exceptional set converging to z_1 so $|f(z_1) - f(z_2)| \leqq K\,|z_1 - z_2|$. If $\mu(\{z_1\}) \neq 0$ we get the same inequality from the fact that z_2 belongs to E. As E is dense in Supp(μ) and f is uniformly continuous on E there is a continuous function g on Supp(μ) equal to f on E, that is, equal to f a.e. By continuity $|g(z) - g(w)| \leqq K\,|z - w|$ for z, w in $\sigma(A)$.

Suppose now that λ is a nonisolated point of $\sigma(A)$ and that g is not differentiable at λ. Then, as g is a Lipschitz function, we can find a sequence $\{\lambda_i\}$ $(-\infty < i < \infty)$ of distinct points in $\sigma(A) - \{\lambda\}$ with $\lambda_i \to \lambda$ as $i \to \pm\infty$ and

$$\lim_{i\to-\infty} (g(\lambda_i) - g(\lambda))(\lambda_i - \lambda)^{-1} = p$$

$$\lim_{i\to+\infty} (g(\lambda_i) - g(\lambda))(\lambda_i - \lambda)^{-1} = q \neq p.$$

Replacing g by $h(z) = (g(z) - pz)(q - p)^{-1}$ if necessary we can assume that $p = 0$, $q = 1$ since g satisfies the criterion of Theorem (3.6) if and only if h does.

If

$$\gamma_{ij} = \begin{cases} (g(\lambda_i) - g(\lambda_j))(\lambda_i - \lambda_j)^{-1} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

then (γ_{ij}) is an Hadamard multiplier of $\mathscr{B}(l^2(\mathbf{Z}))$ by (3.3) and (3.4). This implies that the matrix (β_{ij}) where $\beta_{ij} = \gamma_{-i,j}$ for $i, j \geqq 1$ is an H-multiplier of $\mathscr{B}(l^2(\mathbf{Z}^+))$. Indeed if $\{\varphi_i : -\infty < i < \infty\}$ is an orthonormal basis of $l^2(\mathbf{Z})$ and $\{\xi_i : i \geqq 1\}$ is an orthonormal basis of $l^2(\mathbf{Z}^+)$ then the operator Γ' on $\mathscr{B}(l^2(\mathbf{Z}^+))$ corresponding to H-multiplication by (β_{ij}) is given by $\Gamma'(X) = V^*\Gamma(VXU^*)U$ where Γ is the operator on $\mathscr{B}(l^2(\mathbf{Z}^+))$ associated with (γ_{ij}) and U, V are the isometries from $l^2(\mathbf{Z}^+)$ into $l^2(\mathbf{Z})$ defined by $U\xi_i = \varphi_i$, $V\xi_i = \varphi_{-i}$ for $i \geqq 1$.

Fix a positive integer n. Since

$$\lim_{i\to\infty}\lim_{j\to\infty} \beta_{ij} = 0, \qquad \lim_{j\to\infty}\lim_{i\to\infty} \beta_{ij} = 1$$

we can find one-to-one maps π, σ from $\mathbf{Z}^+$ into itself such that

$$|\beta_{\pi(i),\sigma(j)} - 1| < n^{-1} \qquad \text{for} \qquad i > j \geqq 1$$

$$|\beta_{\pi(i),\sigma(j)}| < n^{-1} \qquad \text{for} \qquad 1 \leqq i \leqq j.$$

(See [**8**; p. 694].) Since $(\beta_{\pi(i),\sigma(j)})$ is clearly an H-multiplier with multiplier norm at most equal to the multiplier norm of the matrix (β_{ij}), and since n can be taken arbitrarily large here, it follows from (3) of Lemma (3.2) that (α_{ij}) is a Hadamard multiplier where

$$\alpha_{ij} = \begin{cases} 1 & \text{for } i > j \\ 0 & \text{for } i \leqq j \end{cases}.$$

This contradicts (4) of Lemma (3.2) and completes the proof.

COROLLARY (4.2). *Suppose that f is continuous on the closed unit disk D and that* $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ *for some normal operator A with* $\sigma(A) = D$. *Then f is analytic at each point of D. In fact, f and its derivative f′ belong to* H^∞, *f satisfies the Lipschitz condition* $|f(e^{i\alpha}) - f(e^{i\beta})| \leqq K|\alpha - \beta|$ *on the unit circle, and the Taylor series of f is absolutely convergent on D.*

Proof. That f and f' belong to H^∞ is clear from the theorem. The other assertions about f are consequence of these, a theorem of Hardy and Littlewood, and Hardy's inequality. See [6; pp. 48, 78].

There is a natural anti-involution $\tau \to \tau^*$ on $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ defined by $\tau^*(X) = \tau(X^*)^*$ for $X \in \mathcal{B}(\mathcal{H})$. With respect to this involution it is easy to check that δ_A and $(\delta_A)^*$ commute if and only if A is normal so that the term "normal derivation" is unambiguous. It is known [2] that normal derivations on $\mathcal{B}(\mathcal{H})$ exhibit some of the properties of normal operators on $\mathcal{H}$, for example the orthogonality of range and kernel mentioned earlier. But Theorem (4.1) indicates that whereas a normal operator on $\mathcal{H}$ and its adjoint always have the same range, this property fails in general for normal derivations, even those induced by diagonal operators, because $z \to \bar{z}$ is not analytic. (However, the ranges of δ_A and δ_{A^*} have the same norm closure. In fact, $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)^-$ for any B in the C^*-algebra generated by the normal operator A.) This fact can be expressed in a slightly different way to provide a negative answer to a question raised in a conversation with the authors of [3].

COROLLARY (4.3). *There exist a diagonal operator A* (with distinct eigenvalues) *and a sequence* $\{X_n\}$ *in* $\mathcal{B}(\mathcal{H})$ *such that* $\|AX_n - X_nA\| \to 0$ *but* $\|AX_n^* - X_n^*A\| > 1$ *for each* $n = 1, 2, \cdots$.

Proof. There exists a diagonal operator A with $\mathcal{R}(\delta_A *)$ not contained in $\mathcal{R}(\delta_A)$ by the preceding remarks. For such a choice of A Corollary (3.7) implies that for each n there is an operator Z_n with $\|\delta_A * (Z_n)\| > n\|\delta_A(Z_n)\|$. Then $\delta_A(Z_n) \neq 0$ by the Fuglede theorem so the choice $X_n = Z_n / n\|\delta_A(Z_n)\|$ satisfies the required conditions.

REMARK. The sequence $\{X_n\}$ cannot be chosen to be uniformly bounded however [R. L. Moore, private communication.]

In [3] a counterexample is constructed to show that if A is a normal operator on $\mathcal{H}$ and P is a projection in $\{A\}''$, then in general one cannot find a positive number δ corresponding to each $\epsilon > 0$ so that the conditions $X \in \mathcal{B}(\mathcal{H})$, $\|AX - XA\| < \delta$ imply $\|PX - XP\| < \epsilon$. Or e-quivalently, by (3.7), the condition $P \in \{A\}''$ is not sufficient for $\mathcal{R}(\delta_P) \subset \mathcal{R}(\delta_A)$. Theorem (4.1) helps to explain this situation more fully:

COROLLARY (4.4). *Let A be a normal operator on $\mathcal{H}$ with spectral measure $E(\cdot)$ and let P be a projection on $\mathcal{H}$. Then $\mathcal{R}(\delta_P) \subset \mathcal{R}(\delta_A)$ if and only if there are disjoint closed sets Δ_0, Δ_1 with $\Delta_0 \cup \Delta_1 = \sigma(A)$ and $P = E(\Delta_1)$.*

Proof. If $\mathcal{R}(\delta_P) \subset \mathcal{R}(\delta_A)$ then by (4.1) there is a continuous function f on $\sigma(A)$ with $P = f(A)$. The spectral mapping theorem implies that $\sigma(A)$ is the union of the disjoint closed sets $\Delta_0 = f^{-1}(0)$ and $\Delta_1 = f^{-1}(1)$. Also $P = f(A) = \int f(\lambda)dE_\lambda = E(\Delta_1)$.

Conversely if $P = E(\Delta_1)$ where Δ_0, Δ_1 are disjoint closed sets whose union is the spectrum of A then by the Riesz-Dunford functional calculus $P = f(A)$ for some function f that is analytic in a neighborhood of $\sigma(A)$. Hence $\mathcal{R}(\delta_P) = \mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ by (3.6) or by the result of Weber **[20]** mentioned in the Introduction.

The sufficiency of the condition on P may also be established directly by considering the decomposition $\mathcal{H} = E(\Delta_0)\mathcal{H} \oplus E(\Delta_1)\mathcal{H}$ and appealing to the theorem of Lumer and Rosenblum **[11]** on the solvability of the operator equations $A_1Z - ZA_0 = X$, $A_0Y - YA_1 = W$ for operators A_0, A_1 with disjoint spectra.

A theorem of Anderson **[1]** shows that if N is a normal operator then $\mathcal{R}(\delta_N) \subset \bigcup \mathcal{R}(\delta_A)$ where the union is taken over the set of all self adjoint operators A in $\mathcal{B}(\mathcal{H})$. That is, any commutator of the form $NX - XN$ can also be written $AY - YA$ for some $A = A^*$ and $Y \in \mathcal{B}(\mathcal{H})$. Theorem (4.1) implies that one cannot improve this to: $\mathcal{R}(\delta_N) \subset \mathcal{R}(\delta_A)$ for some $A = A^*$. Indeed if $\sigma(N)$ has infinite one-dimensional Hausdorff measure then we cannot have $\sigma(N) = f(\sigma(A))$, and hence cannot have $N = f(A)$, for any self-adjoint A and Lipschitz f.

Theorem (4.1) also permits a somewhat simpler proof of the theorem of Anderson and Foias **[1]** which determines when the range of a normal derivation is closed.

COROLLARY (4.5). *Let A be a normal operator on $\mathcal{H}$. Then $\mathcal{R}(\delta_A)$ is norm closed in $\mathcal{B}(\mathcal{H})$ if and only if the spectrum of A is finite.*

Proof. Suppose that $\mathcal{R}(\delta_A)$ is norm closed in $\mathcal{B}(\mathcal{H})$. If P is a norm 1 projection of $\mathcal{B}(\mathcal{H})$ onto the commutant of A then there is a constant $c > 0$ such that $\|\delta_A(X)\| \geqq c\,\|X\|$ for all $X \in \mathcal{R}(1-P)$. For $B \in \{A\}''$ and $X \in \mathcal{B}(\mathcal{H})$ we have $P(BX - XB) = BP(X) - P(X)B = 0$ and so $\mathcal{R}(\delta_B) \subset \mathcal{R}(1-P)$. Hence

$$c\,\|\delta_B(X)\| \leqq \|\delta_A(\delta_B(X))\| = \|\delta_B(\delta_A(X))\| \leqq \|\delta_B\|\,\|\delta_A(X)\|.$$

Therefore $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ by (3.7) so that $B = f(A)$ for some continuous function f by (3.6). Thus $\{A\}''$, the von Neumann algebra generated by A, coincides with the C^*-algebra generated by A. It follows that $\sigma(A)$ is extremally disconnected and therefore is a finite subset of $\mathbf{C}$.

Conversely if $\sigma(A)$ is finite then A is a linear combination of orthogonal projections and this easily implies that δ_A has closed range.

REMARK. With a different argument one can show that if A is normal with infinite spectrum then $\overline{\mathcal{R}(\delta_A) + \{A\}'}$ is not norm dense in $\mathcal{B}(\mathcal{H})$. (See **[2]**.) Hence $\mathcal{R}(\delta_A) \subsetneq \mathcal{R}(1-P)$ for any projection P of norm 1 from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$.

We conclude this section with an example that confirms a remark of J. Taylor [**18**; p. 29].

EXAMPLE (4.6). Let A be the operator of multiplication by the sequence $\lambda_i = i^{-1} (i \neq 0)$, $\lambda_0 = 0$ in $l^2(\mathbf{Z})$, and let f be the Lipschitz function $f(x) = x^+ = \frac{1}{2}(x + |x|)$. Then $\mathcal{R}(\delta_{f(A)}) \not\subset \mathcal{R}(\delta_A)$. That is, there does not exist a constant c such that $\|f(A)X - Xf(A)\| \leqq c\,\|AX - XA\|$ for all operators X on $l^2(\mathbf{Z})$.

This follows at once from (4.1) since f is not differentiable at $x = 0$. The result can also be proved directly by observing that

$$\gamma_{ij} = \begin{cases} (f(\lambda_i) - f(\lambda_j))(\lambda_i - \lambda_j)^{-1} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j \end{cases}$$

cannot be an H-multiplier of $\mathcal{B}(l^2(\mathbf{Z}))$ because this would imply that the matrix $(j(i+j)^{-1})$ is an H-multiplier of $\mathcal{B}(l^2(\mathbf{Z}^+))$ which is impossible because of the relations

$$\lim_{j \to \infty} j(i+j)^{-1} = 1, \qquad \lim_{i \to \infty} j(i+j)^{-1} = 0.$$

(See the proof of (4.1).)

5. We conclude by giving a condition which is sufficient for f to satisfy the criterion of Theorem (3.6) for self-adjoint operators.

THEOREM (5.1). *Let f be a complex valued function on* $\mathbf{R}$ *with continuous third derivative. If A is a self-adjoint operator on $\mathcal{H}$ then* $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$.

Proof. By (3.5) it is sufficient to prove the theorem for the case in which A is the operator $f(x) \to x \cdot f(x)$ on $L^2(I)$ where I is a compact subinterval of $\mathbf{R}$. Define

$$\Lambda(x, y) = \begin{cases} (f(x) - f(y))(x - y)^{-1} & \text{for } x \neq y \\ f'(x) & \text{for } x = y. \end{cases}$$

Then Λ is a $C^{(2)}$ function on $\mathbf{R} \times \mathbf{R}$ and, without altering Λ in a neighborhood of $\sigma(A) \times \sigma(A)$ we can assume that Λ is doubly periodic with periods p, q say. The Fourier coefficients $\lambda_{kl} = p^{-1}q^{-1} \int_0^q \int_0^p \Lambda(x, y) \exp 2\pi i(kxp^{-1} + lyq^{-1}) dx\, dy$ satisfy $\{k^2\lambda_{kl}\} \in l^2(\mathbf{Z}^2)$, $\{l^2\lambda_{kl}\} \in l^2(\mathbf{Z}^2)$ because $\partial^2\Lambda/\partial x^2$ and $\partial^2\Lambda/\partial y^2$ belong to L^2. Since $\{(k^2 + l^2)^{-1}\} \in l^2(\mathbf{Z}^2)$ it follows that $\{\lambda_{kl}\} \in l^1(\mathbf{Z}^2)$.

Now if $t(x, y)$ is the kernel of a trace class operator T on $L^2(I)$ then $(\alpha_{kl} t)(x, y) = t(x, y) \exp(-2\pi i(kxp^{-1} + lyq^{-1}))$ is the kernel of the operator UTV where U and V are unitary so α_{kl} is an operator of norm 1 in $\mathcal{B}(\mathcal{T})$. Hence $\alpha = \Sigma_{k,l}\lambda_{kl}\alpha_{kl}$ is an operator in $\mathcal{B}(\mathcal{T})$ and this is $t \to \Lambda t$ because $\Lambda = \Sigma \lambda_{kl} \exp(-2\pi i(kxp^{-1} + lyq^{-1}))$. Thus Λt is a trace class kernel whenever t is and it follows as in the proof of Theorem (3.6) that $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$.

Note that now that we know $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ Theorem (3.6) shows that f also satisfies the multiplier condition of that theorem. That is, the function $\Lambda(z, w)$, taken to be 0 when $z = w$, rather than $f'(z)$, also multiplies trace class kernels.

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