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PRODUCT INTEGRALS AND THE SOLUTION OF INTEGRAL EQUATIONS

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Functions are from R to N or $R \times R$ to N, where R denotes the set of real numbers and N denotes a normed complete ring. If $\beta > 0$, H and G are functions from $R \times R$ to N, f and h are functions from R to N, each of H, G and dh has bounded variation on $[a, b]$ and $|H| < 1 - \beta$ on $[a, b]$, then the following statements are equivalent:

(1) f is bounded on $[a, b]$, each of $\int_a^b H$, $\int_a^b G$ and $(LR)\int_a^b (fG + fH)$ exists and

$$f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

for $a \leqq x \leqq b$, and

(2) each of ${}_x\Pi^y(1 + \sum_{j=1}^{\infty} H^j)$, ${}_x\Pi^y(1 + G)$ and

$$(R)\int_x^y dh\left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^y(1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right)$$

exists for $a \leqq x < y \leqq b$ and

$$f(x) = h(a)\,{}_a\Pi^x(1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right)$$
$$+ (R)\int_a^x dh\left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^x(1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right)$$

for $a \leqq x \leqq b$.

This result is obtained without requiring the existence of integrals of the form

$$\int_a^b \left| G - \int G \right| = 0 \quad \text{and} \quad \int_a^b |1 + G - \Pi(1 + G)| = 0.$$

This article is part of a sequence of results on the solution of integral equations initiated by two papers by H. S. Wall **[28][29]** on continuous continued fractions and harmonic matrices. He studied certain techniques for solving integral equations which are associated with product integration and his results have been extended in various directions by J. S. MacNerney **[18][19][20][21][22]**, J. W. Neuberger

[24] [25] [26], T. H. Hildebrandt **[13]**, J. R. Dorroh **[4]**, B. W. Helton **[5] [6] [7]**, D. B. Hinton **[14]**, J. V. Herod **[11]**, C. W. Bitzer **[2] [3]**, D. L. Lovelady **[16] [17]** and J. A. Reneke **[27]**. The results here connect closely with those of B. W. Helton [**5**, §5, pp. 307–315].

B. W. Helton [**5**, Theorem 5.1, p. 310] solved the integral equation

$$\text{(a)} \qquad f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

by using product integral techniques. In his development, the existence of integrals of the form

$$\text{(b)} \qquad \int_a^b \left| G - \int G \right| = 0 \quad \text{and} \quad \int_a^b |1 + G - \Pi(1+G)| = 0$$

plays an important part. For real valued functions, A. Kolmogoroff [**15**, p. 669] has shown that if $\int_a^b G$ exists, then $\int_a^b \left| G - \int G \right|$ exists and is zero. Further, W. D. L. Appling [**1**, Theorem 2, p. 155] and B. W. Helton [**5**, Theorem 4.1, p. 304] have shown that there exist other classes of functions such that the existence of $\int_a^b G$ is sufficient to assure that $\int_a^b \left| G - \int G \right|$ exists and is zero. Also, B. W. Helton [**5**, Theorem 4.2, p. 305] has shown that for some settings the existence of ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ is sufficient to assure that $\int_a^b |1 + G - \Pi(1+G)|$ exists and is zero. However, it has been shown by W. D. L. Appling [**1**, Theorem 2, p. 155] and the author [**8**, pp. 153–154] that the existence of $\int_a^b G$ and ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ is not sufficient to imply the existence of the integrals in (b). In the following, we solve the integral equation in (a) without requiring the existence of the integrals in (b).

All integrals and definitions are of the subdivision-refinement type, and functions are from either R to N or $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. Lower case letters are used to denote functions from R to N, and capital letters are used to denote functions from $R \times R$ to N. Unless noted otherwise, functions on $R \times R$ are assumed to be defined only for elements $\{a, b\}$ of $R \times R$ such that $a < b$. If $D = \{x_q\}_{q=0}^n$ is a subdivision of $[a, b]$, then $D(I) = \{[x_{q-1}, x_q]\}_{q=1}^n$, $f_q = f(x_q)$ and $G_q = G(x_{q-1}, x_q)$. Further, $\{x_{qr}\}_{r=0}^{n(q)}$ represents a subdivision of $[x_{q-i}, x_q]$ and $G_{qr} = G(x_{q,r-l}, x_{qr})$.

The statement that $\int_a^b G$ exists means there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that if J is a refinement of D, then

$$\left| L - \sum_{J(I)} G \right| < \epsilon.$$

The statement that ${}_a\Pi^b(1 + G)$ exists means there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that if J is a refinement of D, then

$$\left| L - \prod_{J(I)} (1 + G) \right| < \epsilon.$$

The statement $(LR) \int_a^b (fG + fH)$ exists means $\int_a^b C$ exists, where

$$C(r, s) = f(r)G(r, s) + f(s)H(r, s).$$

We adopt the conventions that

$$\int_a^a G = 0 \quad \text{and} \quad {}_a\Pi^a(1 + G) = 1.$$

Further,

$$\sum_{q=i}^{j} G_q = 0 \quad \text{and} \quad \prod_{q=i}^{j} (1 + G_q) = 1,$$

where $i > j$.

The statements that G is bounded on $[a, b]$, $G \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $|G_q| < B$ for $q = 1, 2, \cdots, n$,

(2) $|\Pi_{q=i}^j (1 + G_q)| < B$ for $1 \leqq i \leqq j \leqq n$, and

(3) $\Sigma_{q=1}^n |G_q| < B$,

respectively. Similarly, statements of the form $|G| < \beta$ are to be interpreted in terms of subdivisions and refinements. Observe that every function in OB° is also in OP°.

The statement that $G \in OM^*$ on $[a, b]$ means ${}_x\Pi^y(1 + G)$ exists for $a \leqq x < y \leqq b$ and if $\epsilon > 0$ then there exists a subdivision D of $[a, b]$ such that if $\{x_q\}_{q=0}^n$ is a refinement of D and $0 \leqq p < q \leqq n$, then

$$|{}_{x_p}\Pi^{x_q}(1 + G) - \Pi_{i=p+1}^q (1 + G_i)| < \epsilon.$$

Also, $G \in OL^\circ$ on $[a,b]$ only if $\lim_{x\to p^+} G(p,x)$, $\lim_{x\to p^-} G(x,p)$, $\lim_{x,y\to p^+} G(x,y)$ and $\lim_{x,y\to p^-} G(x,y)$ exist for $a \leqq p \leqq b$, and $G \in OA^\circ$ on $[a,b]$ only if $\int_a^b G$ exists and $\int_a^b \left| G - \int G \right|$ exists and is zero. For additional background with respect to this paper, see work by B. W. Helton **[5]** **[6]** and J. S. MacNerney **[20]**. Further, additional background on product integration is given by P. R. Masani **[23]**.

LEMMA 1. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a,b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$* [**10,** Theorem 4].

LEMMA 2. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a,b]$, $G \in OB^\circ$ on $[a,b]$ and either $\int_a^b G$ exists or ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and ${}_x\Pi^y(1+HG)$ and ${}_x\Pi^y(1+GH)$ exist for $a \leqq x < y \leqq b$* [**10,** Theorem 5].

LEMMA 3. *If G is a function from $R \times R$ to N, $G \in OB^\circ$ on $[a,b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $G \in OM^*$ on $[a,b]$* [**10,** Theorem 1].

LEMMA 4. *If $\epsilon > 0$, H is a function from $R \times R$ to N and $H \in OL^\circ$ on $[a,b]$, then there exist a subdivision $\{t_i\}_{i=0}^p$ of $[a,b]$ and a sequence $\{k_j\}_{j=1}^p$ such that if $1 \leqq j \leqq p$ and $t_{j-1} < x < y < t_j$, then*

$$|H(x,y) - k_j| < \epsilon$$

[**6,** Lemma, p. 498].

LEMMA 5. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a,b]$ and $G \in OA^\circ$ and OB° on $[a,b]$, then $HG \in OA^\circ$ on $[a,b]$* [**6,** Theorem 2, p. 494].

LEMMA 6. *If F and U are functions from $R \times R$ to N, F and U are in OB° on $[a,b]$, $F \in OA^\circ$ on $[a,b]$, ${}_x\Pi^y(1+U)$ exists for $a \leqq x < y \leqq b$ and*

$$(R)\int_x^y F_s\Pi^y(1+U)$$

exists for $a \leqq x < y \leqq b$, *then*

$$\int_a^b \left| (R) \int_x^y F_s \Pi^y (1+U) - F(x,y) \right|$$

exists and is zero [**5**, Lemma, p. 307].

The main result now follows.

THEOREM. *If* $\beta > 0$, H *and* G *are functions from* $R \times R$ *to* N, f *and* h *are functions from* R *to* N, *each of* H, G *and* dh *is in* $OB°$ *on* $[a,b]$ *and* $|H| < 1-\beta$ *on* $[a,b]$, *then the following statements are equivalent*:

(1) f *is bounded on* $[a,b]$, *each of* $\int_a^b H$, $\int_a^b G$ *and*

$$(LR) \int_a^b (fG + fH)$$

exists and

$$f(x) = h(x) + (LR) \int_a^x (fG + fH)$$

for $a \leqq x \leqq b$, *and*

(2) *each of* ${}_x\Pi^y(1 + \Sigma_{j=1}^{\infty} H^j)$, ${}_x\Pi^y(1+G)$ *and*

$$(R) \int_x^y dh \left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^y (1+G) \left(1 + \sum_{j=1}^{\infty} H^j\right)$$

exists for $a \leqq x < y \leqq b$ *and*

$$f(x) = h(a)\,{}_a\Pi^x (1+G) \left(1 + \sum_{j=1}^{\infty} H^j\right)$$
$$+ (R) \int_a^x dh \left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^x (1+G) \left(1 + \sum_{j=1}^{\infty} H^j\right)$$

for $a \leqq x \leqq b$.

Before proving the theorem, we point out the results of considering left and right integrals, respectively. If $H \equiv 0$, then we have the integral equation

(a) $$f(x) = h(x) + (L) \int_a^x fG.$$

This equation involves only a left integral, and its solution is

$$\text{(b)} \qquad f(x) = h(a)_a\Pi^x(1+G) + (R)\int_a^x dh_s\Pi^x(1+G).$$

On the other hand, if $G \equiv 0$, then we have the integral equation

$$\text{(c)} \qquad f(x) = h(x) + (R)\int_a^x fG.$$

This equation involves only a right integral, and its solution is

$$\text{(d)} \quad f(x) = h(a)_a\Pi^x\left(1+\sum_{j=1}^{\infty} H^j\right) + (L)\int_a^x dh_r\Pi^x\left(1+\sum_{j=1}^{\infty} H^j\right).$$

If z is in N and $|z| < 1$, then $1 + \Sigma_{j=1}^{\infty} z^j$ exists and is $(1-z)^{-1}$. Thus, in (d) and in the theorem itself, it is possible to substitute $(1-H)^{-1}$ for $1 + \Sigma_{j=1}^{\infty} H^j$. To obtain some feeling for why invertibility-related conditions are placed on H but not on G, consider the first approximations to equations (a) and (c). For (a), we have that

$$f(x) \doteq h(x) + f(a)G(a,x);$$

while for (c), we have that

$$f(x) \doteq h(x) + f(x)H(a,x),$$

and hence that

$$f(x) \doteq h(x)[1 - H(a,x)]^{-1}.$$

For additional discussion of product integrals, inverses and integral equations, the reader is referred to papers by J. V. Herod **[12]** and the author **[9]**.

The main result is now established.

Proof. To simplify notation in the following work, we use the interval functions T, U and V to denote

$$(1+G)\left(1+\sum_{j=1}^{\infty} H^j\right),$$

$$G + \sum_{j=1}^{\infty} H^j + G\sum_{j=1}^{\infty} H^j$$

and

$$1+\sum_{j=1}^{\infty} H^j,$$

respectively. Further, we use C to denote the interval function

$$C(r,s)=f(r)G(r,s)+f(s)H(r,s).$$

Proof (1)→(2). Since $\int_a^b H$ exists and $H \in OB^\circ$ on $[a,b]$, it follows that $H \in OL^\circ$ on $[a,b]$, and hence, $1+\Sigma_{j=1}^\infty H^j \in OL^\circ$ on $[a,b]$. Thus, the existence of

$$\int_a^b H\left(1+\sum_{j=1}^{\infty} H^j\right)=\int_a^b \sum_{j=1}^{\infty} H^j$$

follows from Lemma 2. Therefore, the existence of ${}_x\Pi^y V$ for $a \leqq x < y \leqq b$ follows from Lemma 1. Also, Lemma 1 implies the existence of ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ from the existence of $\int_a^b G$. Lemma 2 can be used to establish the existence of $\int_a^b G\,\Sigma_{j=1}^\infty H^j$. Therefore, since each of

$$\int_a^b G,\ \int_a^b \sum_{j=1}^{\infty} H^j \quad\text{and}\quad \int_a^b G\sum_{j=1}^{\infty} H^j$$

exists, we have that $\int_a^b U$ exists, and thus, the existence of ${}_x\Pi^y T$ for $a \leqq x < y \leqq b$ can be established by applying Lemma 1. Finally, since $V(r,s)\,{}_s\Pi^y T$ is in OL° on $[a,b]$, the existence of

$$(R)\int_x^y dh\, V\,{}_s\Pi^y T$$

for $a \leqq x < y \leqq b$ can be obtained from the existence of $\int_a^b dh$ through the use of Lemma 2.

Suppose $a \leqq x \leqq b$. We now show that

$$f(x)=h(a)\,{}_a\Pi^x T+(R)\int_a^x dh\, V\,{}_s\Pi^x T.$$

If $a=x$, the result follows immediately. Therefore, suppose $a<x$.

Let $\epsilon > 0$. Since $|H| < 1-\beta$ on $[a,x]$, G, H and dh are in OB° on $[a,x]$ and f and V are bounded on $[a,x]$, there exist a subdivision D_1 of $[a,x]$ and a number B such that if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

(1) $|H_i| < 1-\beta$ for $i = 1,2,\cdots,n$,

(2) $\Sigma_{i=1}^n |dh_i V_i| < B$,

(3) $\Sigma_{i=1}^n |C_i V_i| < B$,

(4) $\Sigma_{i=1}^n \left| \left[\int_{x_{i-1}}^{x_i} C \right] V_i \right| < B$, and

(5) $|V_i \Pi_{k=i+1}^n T_k| < B$ for $i = 1,2,\cdots,n$.

Since ${}_r\Pi^s T$ exists for $a \leqq r \leqq s \leqq x$ and $U \in OB^\circ$ on $[a,x]$, it follows from Lemma 3 that there exists a subdivision D_2 of $[a,x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 \leqq p < q \leqq n$, then

(1) $|{}_{x_p}\Pi^{x_q} T - \Pi_{i=p+1}^q T_i| < \epsilon(16B)^{-1}$,

(2) $|\Pi_{i=p+1}^q T_i - {}_{x_p}\Pi^{x_q} T| < \epsilon(16B)^{-1}$, and

(3) $|h(a)_a\Pi^x T - h(a)\Pi_{i=1}^n T_i| < \epsilon/4$.

Since $(R)\int_a^x dh\, V_s\Pi^x T$ exists, there exists a subdivision D_3 of $[a,x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then

$$\left| (R)\int_a^x dh\, V_s\Pi^x T - \sum_{i=1}^n dh_i\, V_{i\, x_i}\Pi^x T \right| < \epsilon/8.$$

Since $V(r,s)_s\Pi^x T$ is in OL° on $[a,x]$, it follows from Lemma 4 that there exist a subdivision $D_4 = \{t_i\}_{i=0}^p$ of $[a,x]$ and a sequence $\{k_j\}_{j=1}^p$ such that if $1 \leqq j \leqq p$ and $t_{j-1} < r < s < t_j$, then

$$|V(r,s)_s\Pi^x T - k_j| < \epsilon(16B)^{-1}.$$

Since $C \in OB^\circ$ on $[a,x]$ and $\int_a^x C$ exists, there exist subdivisions $\{r_j\}_{j=0}^{p+1}$ and $\{s_j\}_{j=0}^{p+1}$ of $[a,x]$ such that

(1) $t_{j-1} < r_j < s_j < t_j$ for $j = 1,2,\cdots,p$, and

(2) $\Sigma_{k=1}^{n(j)} \left| C_{jk} - \int_{x_{j,k-1}}^{x_{jk}} C \right| < \epsilon[8B(p+1)]^{-1}$ for $j = 1,2,\cdots,p+1$ and each refinement $\{x_{jk}\}_{k=0}^{n(j)}$ of $\{s_{j-1}, t_{j-1}, r_j\}$.

Further, for $j = 1,2,\cdots,p$, there exist subdivisions E_j of $[r_j, s_j]$ such that if F_j is a refinement of E_j, then

$$\sum_{j=1}^p \left| \sum_{F_j(I)} C - \int_{r_j}^{s_j} C \right| |k_j| < \epsilon/8.$$

Let D denote the subdivision

$$\bigcup_{i=1}^4 D_i \cup \{r_j\}_{j=0}^{p+1} \cup \{s_j\}_{j=0}^{p+1} \bigcup_{j=1}^p E_j$$

of $[a,x]$, and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. For $j = 1, 2, \cdots, p$, let K_j be the set such that $i \in K_j$ only if $r_j < x_i \leqq s_j$. Let K and L denote the sets

$$\bigcup_{j=1}^{p} K_j \quad \text{and} \quad \{i\}_{i=1}^{n} - \bigcup_{j=1}^{p} K_j,$$

respectively.

We now establish two inequalities that are necessary to complete the proof. First,

$$\begin{aligned}
&|(R)\int_a^x dh\, V_s \Pi^x T - \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k| \\
&\leqq \left| \sum_{i=1}^{n} dh_i\, V_{i\,x_i}\Pi^x T - \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k \right| \\
&\quad + \left| (R)\int_a^x dh\, V_s \Pi^x T - \sum_{i=1}^{n} dh_i\, V_{i\,x_i}\Pi^x T \right| \\
&< \sum_{i=1}^{n} |dh_i\, V_i| \left| {}_{x_i}\Pi^x T - \prod_{k=i+1}^{n} T_k \right| + \epsilon/8 \\
&< B[\epsilon(16B)^{-1}] + \epsilon/8 < \epsilon/4.
\end{aligned}$$

Second,

$$\begin{aligned}
&\left| \sum_{i=1}^{n} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_i \prod_{k=i+1}^{n} T_k \right| \\
&\leqq \left| \sum_{i\in K} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_i \prod_{k=i+1}^{n} T_k \right| \\
&\quad + \sum_{i\in L} \left| C_i - \int_{x_{i-1}}^{x_i} C \right| \left| V_i \prod_{k=i+1}^{n} T_k \right| \\
&< \left| \sum_{i\in K} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_i \prod_{k=i+1}^{n} T_k \right| \\
&\quad + (p+1)\{\epsilon[8B(p+1)]^{-1}\}B \\
&\leqq \left| \sum_{i\in K} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_{i\,x_i}\Pi^x T \right| \\
&\quad + \sum_{i\in K} \left| \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_i \right| \left| \prod_{k=i+1}^{n} T_k - {}_{x_i}\Pi^x T \right| \\
&\quad + \epsilon/8
\end{aligned}$$

$$\begin{aligned}
&< \left| \sum_{i \in K} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_{i\,x_i} \Pi^x T \right| \\
&\quad + 2B[\epsilon(16B)^{-1}] + \epsilon/8 \\
&\leqq \sum_{j=1}^{p} \left| \sum_{i \in K_j} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_{i\,x_i} \Pi^x T \right| + \epsilon/4 \\
&\leqq \sum_{j=1}^{p} \left| \sum_{i \in K_j} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] k_j \right| \\
&\quad + \sum_{j=1}^{p} \sum_{i \in K_j} \left| C_i - \int_{x_{i-1}}^{x_i} C \right| \left| V_{i\,x_i} \Pi^x T - k_j \right| + \epsilon/4 \\
&< \sum_{j=1}^{p} \left| \sum_{i \in K_j} C_i - \int_{r_j}^{s_j} C \right| |k_j| \\
&\quad + 2B[\epsilon(16B)^{-1}] + \epsilon/4 \\
&< \epsilon/8 + 3\epsilon/8 = \epsilon/2.
\end{aligned}$$

If we employ the iterative technique used by B. W. Helton [**5**, p. 311], we have that

$$\begin{aligned}
f(x) = \sum_{i=1}^{n} &\left[\int_{x_{i-1}}^{x_i} C - C_i \right] V_i \prod_{k=i+1}^{n} T_k \\
&+ h(a) \prod_{i=1}^{n} T_i + \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left| h(a)\,{}_a\Pi^x T + (R) \int_a^x dh\, V_s \Pi^x T - f(x) \right| \\
&\quad \leqq \left| h(a)\,{}_a\Pi^x T - h(a) \prod_{i=1}^{n} T_i \right| \\
&\qquad + \left| (R) \int_a^x dh\, V_s \Pi^x T - \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k \right| \\
&\qquad + \left| \sum_{i=1}^{n} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_i \prod_{k=i+1}^{n} T_k \right| \\
&\quad < \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon.
\end{aligned}$$

Therefore, (1) implies (2).

Proof (2)$\rightarrow$(1). It follows from the bounded variation of the various functions involved that f is bounded on $[a, b]$. Since $\Sigma_{j=1}^{\infty} H^j \in OB^\circ$ on $[a, b]$, it follows from Lemma 1 that

$$\int_a^b \sum_{j=1}^{\infty} H^j = \int_a^b H\left(1 + \sum_{j=1}^{\infty} H^j\right)$$

exists. Recall that $(1 + \Sigma_{j=1}^{\infty} H^j)^{-1}$ exists and is $1 - H$. Thus, since $1 - H \in OL^\circ$ on $[a, b]$, it follows from Lemma 2 that $\int_a^b H$ exists. Further, it follows from Lemma 1 that $\int_a^b G$ exists. The existence of $\int_a^b C$ now follows from the existence of $\int_a^b G$ and $\int_a^b H$ by applying Lemma 2.

Suppose $a \leqq x \leqq b$. We now show that

$$f(x) = h(x) + (LR) \int_a^x (fG + fH).$$

If $a = x$, the result follows immediately. Therefore, suppose $a < x$.

Let $\epsilon > 0$. There exist a subdivision D_1 of $[a, x]$ and a number B such that if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

(1) $|H_i| < 1 - \beta$ for $i = 1, 2, \cdots, n$,
(2) $\Sigma_{i=1}^n |G_i| < B$,
(3) $\Sigma_{i=1}^n |H_i| < B$,
(4) $\Sigma_{i=1}^n |dh_i V_i| < B$, and
(5) $|{}_{x_p}\Pi^{x_q} T| < B$ for $0 \leqq p < q \leqq n$.

Since $\int_a^x G$ exists and $\Sigma_{j=1}^{\infty} H^j \in OL^\circ$ on $[a, b]$, it follows from Lemma 2 that $\int_a^x G\, \Sigma_{j=1}^{\infty} H^j$ exists. Thus, the existence of $\int_a^x U$ follows from the existence of

$$\int_a^x G, \int_a^x \sum_{j=1}^{\infty} H^j \quad \text{and} \quad \int_a^x G \sum_{j=1}^{\infty} H^j.$$

Therefore,

$${}_r\Pi^t (1 + U) = {}_r\Pi^t T$$

exists for $a \leqq r \leqq t \leqq x$ by Lemma 1. Now, it follows from Lemma 3 that $U \in OM^*$ on $[a, x]$. Hence, there exists a subdivision D_2 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 \leqq p < q \leqq n$, then

$$\left| f(a)\,{}_{x_p}\Pi^{x_q} T - f(a) \prod_{i=p+1}^{q} T_i \right| < \epsilon(6B)^{-1}.$$

Since dh is in OA° and OB° on $[a, x]$ and $V \in OL^\circ$ on $[a, x]$, it follows from Lemma 5 that $dh\, V \in OA^\circ$ on $[a, x]$. Thus, since $U \in OB^\circ$ on $[a, x]$ and ${}_s\Pi^t T$ exists for $a \leqq s < t \leqq x$, it follows from Lemma 6 that

$$\int_a^x \left| (R)\int_u^v dh\, V\,{}_s\Pi^v T - dh(v)V(u, v) \right| = 0.$$

From the existence of this integral and the fact that $U \in OM^*$ on $[a, x]$, it follows that there exists a subdivision D_3 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_3 and $0 \leqq p < q \leqq n$, then

(1) $\Sigma_{i=1}^n \left| (R)\int_{x_{i-1}}^{x_i} dh\, V\,{}_s\Pi^{x_i} T - dh_i\, V_i \right| < \epsilon(12B^2)^{-1}$, and

(2) $\left| {}_{x_p}\Pi^{x_q} T - \Pi_{k=p+1}^q T_k \right| < \epsilon(12B^2)^{-1}$.

Thus, if $\{x_i\}_{i=0}^n$ is a refinement of $D_1 \cup D_3$ and $0 < p \leqq n$, then

$$\begin{aligned}
&\left| (R)\int_a^{x_p} dh\, V\,{}_s\Pi^{x_p} T - \sum_{i=1}^{p} dh_i\, V_i \prod_{k=i+1}^{p} T_k \right| \\
&\leqq \left| (R)\int_a^{x_p} dh\, V\,{}_s\Pi^{x_p} T - \sum_{i=1}^{p} dh_i\, V_i\,{}_{x_i}\Pi^{x_p} T \right| \\
&\quad + \sum_{i=1}^{p} |dh_i\, V_i| \left| {}_{x_i}\Pi^{x_p} T - \prod_{k=i+1}^{p} T_k \right| \\
&< \left| \sum_{i=1}^{p} \left[(R)\int_{x_{i-1}}^{x_i} dh\, V\,{}_s\Pi^{x_i} T - dh_i\, V_i \right] {}_{x_i}\Pi^{x_p} T \right| \\
&\quad + B[\epsilon(12B^2)^{-1}] \\
&\leqq \sum_{i=1}^{p} \left| (R)\int_{x_{i-1}}^{x_i} dh\, V\,{}_s\Pi^{x_i} T - dh_i\, V_i \right| \left| {}_{x_i}\Pi^{x_p} T \right| + \epsilon(12B)^{-1} \\
&< B[\epsilon(12B^2)^{-1}] + \epsilon(12B)^{-1} = \epsilon(6B)^{-1}.
\end{aligned}$$

It follows from the existence of the integrals involved that there exists a subdivision D_4 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_4, then

$$\left| f(a) \prod_{i=1}^{n} T_i - f(a)\,{}_a\Pi^x T \right| < \epsilon/6$$

and

$$\left| \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k - (R)\int_a^x dh\, V_s\, {}_s\Pi^x T \right| < \epsilon/6.$$

Let D denote the subdivision $\cup_{i=1}^4 D_i$ of $[a, x]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Observe that

$$\begin{aligned} \prod_{i=m}^{n} T_i &= 1 + \sum_{i=m}^{n} \left(\prod_{k=m}^{i-1} T_k \right) U_i \\ &= 1 + \sum_{i=m}^{n} \left(\prod_{k=m}^{i-1} T_k \right) G_i + \sum_{i=m}^{n} \left(\prod_{k=m}^{i} T_k \right) H_i. \end{aligned}$$

Further,

$$\begin{aligned} &\sum_{i=1}^{n} dh_i\, V_i \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k-1} T_j \right) G_k \\ &\qquad = \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{n} dh_i\, V_i \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k} T_j \right) H_k \\ &\qquad = \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i} T_k \right] H_i. \end{aligned}$$

These identities can be established by induction and are used in subsequent manipulations.

We now work out a further identity to aid in establishing the existence of the desired integral. By employing the previously stated identities, we have that

$$
\begin{aligned}
& f(a) \prod_{i=1}^{n} T_i + \sum_{i=1}^{n} dh_i V_i \prod_{k=i+1}^{n} T_k \\
&= f(a) \left[1 + \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \right] \\
&\quad + \sum_{i=1}^{n} dh_i V_i \left[1 + \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k-1} T_j \right) G_k + \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k} T_j \right) H_k \right] \\
&= f(a) \left[1 + \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \right] \\
&\quad + \sum_{i=1}^{n} dh_i V_i + \sum_{i=1}^{n} dh_i V_i \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k-1} T_j \right) G_k \\
&\quad + \sum_{i=1}^{n} dh_i V_i \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k} T_j \right) H_k \\
&= f(a) \left[1 + \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \right] \\
&\quad + \sum_{i=1}^{n} dh_i + \sum_{i=1}^{n} dh_i V_i H_i \\
&\quad + \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \\
&\quad + \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j V_j \prod_{k=j+1}^{i} T_k \right] H_i \\
&= f(a) \left[1 + \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \right] \\
&\quad + h(x) - h(a) + \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \\
&\quad + \sum_{i=1}^{n} \left[\sum_{j=1}^{i} dh_j V_j \prod_{k=j+1}^{i} T_k \right] H_i \\
&= f(a) \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + f(a) \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \\
&\quad + h(x) + \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \\
&\quad + \sum_{i=1}^{n} \left[\sum_{j=1}^{i} dh_j V_j \prod_{k=j+1}^{i} T_k \right] H_i.
\end{aligned}
$$

Now, by employing the identity developed in the preceding paragraph, we have that

$$
\begin{aligned}
&\left| h(x) + \sum_{i=1}^{n} f(x_{i-1})G_i + \sum_{i=1}^{n} f(x_i)H_i - f(x) \right| \\
&\quad < \left| h(x) + \sum_{i=1}^{n} f(x_{i-1})G_i + \sum_{i=1}^{n} f(x_i)H_i \right. \\
&\qquad \left. - \left[f(a) \prod_{i=1}^{n} T_i + \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k \right] \right| + \epsilon/6 + \epsilon/6 \\
&\quad = \left| h(x) + \sum_{i=1}^{n} f(x_{i-1})G_i + \sum_{i=1}^{n} f(x_i)H_i \right. \\
&\qquad - \left\{ f(a) \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + f(a) \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \right. \\
&\qquad + h(x) + \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \\
&\qquad \left. \left. + \sum_{i=1}^{n} \left[\sum_{j=1}^{i} dh_j\, V_j \prod_{k=j+1}^{i} T_k \right] H_i \right\} \right| + \epsilon/3 \\
&\quad \leqq \sum_{i=2}^{n} \left| f(a)\, {}_a\Pi^{x_{i-1}} T - f(a) \prod_{k=1}^{i-1} T_k \right| |G_i| \\
&\qquad + \sum_{i=1}^{n} \left| f(a)\, {}_a\Pi^{x_i} T - f(a) \prod_{k=1}^{i} T_k \right| |H_i| \\
&\qquad + \sum_{i=2}^{n} \left| (R) \int_a^{x_{i-1}} dh\, V\, {}_v\Pi^{x_{i-1}} T - \sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k \right| |G_i| \\
&\qquad + \sum_{i=1}^{n} \left| (R) \int_a^{x_i} dh\, V\, {}_v\Pi^{x_i} T - \sum_{j=1}^{i} dh_j\, V_j \prod_{k=j+1}^{i} T_k \right| |H_i| \\
&\qquad + \epsilon/3 \\
&\quad < B[\epsilon(6B)^{-1}] + B[\epsilon(6B)^{-1}] + B[\epsilon(6B)^{-1}] + B[\epsilon(6B)^{-1}] \\
&\qquad + \epsilon/3 \\
&\quad = \epsilon.
\end{aligned}
$$

Therefore, $(LR)\int_a^x (fG + fH)$ exists and is $f(x) - h(x)$. Hence, (2) implies (1).

B. W. Helton states three additional theorems on the solution of integral equations by product integration [**5**, Theorems 5.2, 5.3, 5.4, pp.

313–314]. The techniques used in the present paper to avoid requiring the existence of the integrals

$$\int_a^b \left| G - \int G \right| = 0 \quad \text{and} \quad \int_a^b |1 + G - \Pi(1 + G)| = 0$$

can also be applied to these results.

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