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ON ONE-SIDED PRIME IDEALS

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This paper contains some results on prime right ideals in weakly regular rings, especially V-rings, and in rings with restricted minimum condition. Theorem 1 gives information about the structure of V-rings: A V-ring with maximum condition for annihilating left ideals is a finite direct sum of simple V-rings. A characterization of rings with restricted minimum condition is given in Theorem 2: A nonprimitive right Noetherian ring satisfies the restricted minimum condition iff every critical prime right ideal $\neq(0)$ is maximal. The proof depends on the simple observation that in a nonprimitive ring with restricted minimum condition all prime right ideals $\neq(0)$ contain a (two-sided) prime ideal $\neq(0)$. An example shows that Theorem 2 is not valid for right Noetherian primitive rings. The same observation on nonprimitive rings leads to a sufficient condition for rings with restricted minimum condition to be right Noetherian.

It remains an open problem whether there exist nonnoetherian rings with restricted minimum condition (clearly in the commutative case they are Noetherian).

Theorem 1 is a generalization of the well known: A right Goldie V-ring is a finite direct sum of simple V-rings (e.g., **[2]**, p. 357). Theorem 2 is a noncommutative version of a result due to Cohen [**1**, p. 29]. Ornstein has established a weak form in the noncommutative case [**11**, p. 1145].

In §§1, 2, 3 the unity in rings is assumed (except in Proposition 2.1), but most of the results are valid for rings without unity, as shown in §4.

Of basic importance for the following is Lambek's and Michler's work **[8]** from which also most notions have been taken.

1. Basic concepts. A right ideal L of the ring R is called *prime* (*semiprime*), if $xRy\ (xRx) \subseteq L$ implies $x \in L$ or $y \in L\ (x \in L)$ for all $x, y \in R$. As in the two-sided case an equivalent definition is: $AB \subseteq L\ (A^2 \subseteq L)$ implies A or $B \subseteq L\ (A \subseteq L)$ for all right ideals $A, B \subseteq R$.

A right ideal M of R is called *critical*, if R/M is a "supporting module" [**3**, p. 35] for a hereditary torsion theory, i.e., if there exists a hereditary torsion theory (T, F) such that $R/M \in F$, but $R/N \in T$ for all $N \supsetneqq M$. An important among under the critical right ideals are the 1-*critical* ones. A right ideal L is 1-critical, if R/L is not Artinian, but R/N for every $N \supsetneqq L$. It is easy to see that then R/L is a supporting module for the hereditary torsion theory, generated by the class of simple R-modules.

A ring R satisfies the *proper restricted minimum condition*, if (0) is a 1-critical right ideal of R. If additionally R is allowed to be right Artinian, R satisfies the *restricted minimum condition.*

A right ideal L of a ring R is *irreducible* (*indecomposable*), if every submodule $\neq (0)$ of R/L is essential (if R/L is indecomposable). Every critical right ideal is irreducible. The following facts are needed:

(a) Every prime (semiprime) right ideal L contains a two-sided prime (semiprime) ideal, namely $R^{-1}L$. The proof is not hard.

(b) If L is an irreducible right ideal of the right Noetherian ring R, there is a critical prime right ideal of the form $s^{-1}L$, $s \in R \setminus L$ [**8**, p. 370].

(c) Every irreducible semiprime right ideal is prime [**8**, p. 370].

(d) Any ring R with proper restricted minimum condition is a right Öre domain [**11**, p. 1149].

For a subset T of a ring R and a right or left ideal $L \subseteq R$ one defines $T^{-1}L := \{x \in R;\ Tx \subseteq L\}$, $LT^{-1} := \{x \in R;\ xT \subseteq L\}$.

2. Weakly regular rings and V-rings.

LEMMA 2.1. *The following conditions for a ring R are equivalent*:

(a) $L = L^2$ *for every right ideal L of R.*

(b) *Every right ideal is semiprime.*

(c) *Every right ideal is the intersection of prime right ideals.*

A ring satisfying these equivalent conditions is called weakly regular.

Proof. (a) $\Rightarrow$ (b). Let N be any right ideal. $xRx \subseteq N$ implies $xRxR \subseteq N$, so $xR \subseteq N$, i.e. $x \in N$.

(b) $\Rightarrow$ (c). Every right ideal is the intersection of irreducible semiprime right ideals. In view of §1, (c) these are prime.

(c) $\Rightarrow$ (a). The intersection of prime right ideals is semiprime. L^2 semiprime implies $L^2 = L$.

Besides the regular rings the best known weakly regular ones are the simple rings and the *V-rings.* A ring R is called V-ring, if every homomorphic image of R_R has zero radical. Equivalently all simple R-modules are injective (see (10)).

PROPOSITION 2.1. *A ring R with maximum condition for annihilating left ideals and $x \in xR$ for all $x \in R$ has a right unity.*

Proof. Let be $L_e = \{x - ex;\ x \in R\}$, $e \in R$; $0L_e^{-1}$ is maximal for some e. Assume $0L_e^{-1} \subsetneq R$, i.e. $yL_e \neq (0)$ for some $y \in R$. Then $(y - ye)R \neq (0)$ and so $y - ye \neq 0$. There exists an $e' \in R$, such that $y - ye = (y - ye)e'$; $y = yf$ with $f = e + e' - ee'$, so $y \in 0L_f^{-1}$; $z \in 0L_e^{-1}$ implies $z = ze$, thus $z = zf$, i.e. $z \in 0L_f^{-1}$. Altogether one has $0L_e^{-1} \subsetneq 0L_f^{-1}$, but this is impossible. Because of $0L_e^{-1} = R$, $(r - re)R = (0)$ for all $r \in R$, that means e is a right unity.

The proof is essentially the same as the one of A. Kertész [**5**, p. 237] for: A left Noetherian ring has a right unity, iff $x \in xR$ for all $x \in R$.

PROPOSITION 2.2. *A ring R with maximum condition for annihilating left ideals is weakly regular, iff it is a finite direct sum of simple rings.*

Proof. Let I be an ideal of R and L a right ideal of the ring I. Then $LR = LRLR \subseteq LI \subseteq L$, i.e., every right ideal of I is a right ideal of R. Let be $0 \neq x \in I, Z$ be the ring of integers. $Zx + xI$ is a right ideal of I, therefore of R. Now $x \in Zx + xI = (Zx + xI)^2 = Zx^2 + x^2I + xIx + xIxI = xI$. Because of Proposition 2.1 there is an idempotent $e \in I$ with $Ie = Re = I$; $ex = xe$ for all $x \in R$, otherwise there were an $y \in R$ with $I \ni z = ey - ye \neq 0$, thus $eze = ez = 0$. Therefore $(0) = RezR = IzR \supseteq (zR)^2 = zR$. This is a contradiction. It follows that the second summand in $R = I \oplus R(1 - e)$ is an ideal, and the assertion is true because of a basic argument.

COROLLARY 2.1. *A weakly regular ring with maximum condition for annihilating left ideals is a simple ring, iff it is prime.*

According to (7) the simple rings are just these ones, in which every right ideal is prime.

A consequence of Proposition 2.2 is

THEOREM 1. *A V-ring R with maximum condition for annihilating left ideals is a finite direct sum of simple V-rings.*

COROLLARY 2.2 [**2**, p. 357]. *A right Goldie V-ring R is a finite direct sum of simple rings.*

Proof. R satisfies the maximum condition for annihilating left ideals [e.g. **4**, p. 173].

The same argument holds for left Goldie (right) V-rings.

3. Rings with restricted minimum condition.

LEMMA 3.1. *If R is a ring with restricted minimum condition, R is primitive or every indecomposable prime right ideal $\neq (0)$ is maximal.*

Proof. Let L be any prime right ideal $\neq (0)$. As R/L is Artinian, it contains a minimal submodule E/L. If R is not primitive, $Er \subseteq L$ holds for some $0 \neq r \in R$, i.e. $E^{-1}L \neq (0)$; $xR + L = E$ for some $x \in E \setminus L$; $E^{-1}L \subseteq L$, for $y \in E^{-1}L$ implies $(xR + L)y = xRy + Ly \subseteq L$, thus $y \in L$; $L \supseteq R^{-1}L = E^{-1}L \neq (0)$, because every ideal contained in L is contained in $R^{-1}L$; the other inclusion is obvious. By §1, (a) $E^{-1}L$ is a prime ideal, so maximal, as $R/E^{-1}L$ is an Artinian ring. As L is indecomposable, R/L is an indecomposable $R/E^{-1}L$-module, thus simple. This means, L is a maximal right ideal of R.

LEMMA 3.2. *If R is a right Noetherian ring and every critical prime right ideal $\neq (0)$ maximal, R satisfies the restricted minimum condition.*

Proof. Suppose the assertion is false. Then there is a right ideal $L \neq (0)$, maximal with respect to R/L is nonartinian. Therefore L is 1-critical. Because of §1, (b) a critical prime right ideal N of the form $N = s^{-1}L$, $s \notin L$, exists. $R/s^{-1}L \cong (sR + L)/L$ is nonartinian. This is a contradiction, as in a right Noetherian ring every right ideal of the form $s^{-1}L$, $L \neq (0)$, $s \notin L$, is different from zero.

THEOREM 2. *In a nonprimitive right Noetherian ring R the following conditions are equivalent:*

(a) *Every critical prime right ideal $\neq (0)$ is maximal.*
(b) *Every irreducible prime right ideal $\neq (0)$ is maximal.*
(c) *Every indecomposable prime right ideal $\neq (0)$ is maximal.*
(d) *R satisfies the restricted minimum condition.*

REMARK. Compare it with Theorem 3.6 (a noncommutative version of the Krull-Akizuki Theorem) in [**8**, p. 373].

The following example shows that there are primitive right Noetherian rings which satisfy the restricted minimum, but contain a critical nonmaximal prime right ideal $\neq (0)$: Let $K(z)$ be the ring of rational functions over a field K with $\mathrm{Char}(K) = 0$ and R be the ring of differential polynomials in one indeterminate x over $K(z)$, i.e. $xf = fx + f', f \in K(z)$; R is known to be a simple principal left and right ideal domain. Hence it is a ring with proper restricted minimum condition [see **11**, p. 1149]. Since every right ideal is prime, it is enough to find a critical nonmaximal right ideal. $L = (z + x)xR$ is not maximal; it is properly contained only in R and $(z + x)R$; for $(g_0 + g_1x)(h_0 + h_1x) = (z)x + x^2, g_0, g_1 \neq 0, h_0, h_1 \in K(z)$, leads to $hk + k' = 0, h + k = z$ with $h = (g_0 - g_1')g_1^{-1}, k = g_1h_0$. The only solution in $K(z)$ is $h = z, k = 0$. It follows $(g_0 + g_1x)R = (z + x)R$. To show that R/L is a supporting module for the torsion theory generated by the simple module $E = R/(z + x)R, E \not\cong F = R/xR$ must be proved: There is an element $e \neq 0$ in F, e.g., $e = 1 \bmod (xR)$, with $ex = 0$, but no such an element in E; it can be checked by similar methods as above.

In (7) and (9) it was stated: Any ring is right Noetherian iff every prime right ideal is finitely generated. An easy consequence of this is:

Proposition 3.1. *A nonprimitive ring R with restricted minimum condition is right Noetherian iff the square of every principal right ideal is finitely generated.*

Proof. If R is Artinian, it is always Noetherian. Alternatively R satisfies the restricted minimum condition. It was already proven (Lemma 3.1) that every prime right ideal $L \neq (0)$ contains an ideal $RxR \neq (0)$. L modulo RxR is finitely generated, as R/xR is right Noetherian. RxR itself is a finitely generated right ideal, because $RxR \cong (xR)^2$ as R-modules; thus L is finitely generated, too. The other direction is trivial.

4. The results for rings without unity. Clearly the definition of a prime (semiprime, irreducible, indecomposable) right ideal is the same as in rings with unity. Torsion theories on Mod-R can be regarded as torsion theories on Mod-R_1, where R_1 denotes the usual unitary overring of R. So the other concepts are defined, too. Setting $R^{-1}L \cap L$ instead of $R^{-1}L$, §1, (a) remains true. (b) only holds for rings with unity, (c) only for rings R with $x \in xR$, $x \in R$. (d) can be generalized in the following way:

Proposition 4.1. *Any ring R with $R^2 \neq (0)$ and proper restricted minimum condition is a right Öre domain and can be imbedded as an ideal into a unitary ring with proper restricted minimum condition.*

Proof. Let be $0 \neq x \in R$ and $(0) \neq \{x\}^{-1}0$. Then $xR \cong R/\{x\}^{-1}0$ is an Artinian R-module, so must be zero (otherwise R/xR Artinian implies R Artinian), i.e. $0R^{-1} \neq (0)$. The intersection of any two nonzero right ideals M, L cannot be zero by a similar argument. It remains to show that $R^2 = (0)$, if $0R^{-1} \neq (0)$: $0R^{-1}$ is a trivial R-module, hence as a group nonartinian with Artinian proper homomorphic images. It is easy to see that $0R^{-1}$ must be some proper subgroup of Q, the additive group of the rational numbers. Every $0 \neq r \in 0R^{-1}$ defines a group homomorphism $R/0R^{-1} \to 0R^{-1}$ by $\bar{x} \to xr$. It obviously is well defined and monic, so the additive group of $0R^{-1}$ is a subgroup of Q. No proper subgroup $\neq (0)$ of Q is the additive group of an Artinian ring [e.g., **5**, p. 225], hence $R/0R^{-1}$ must be zero, i.e., $R^2 = (0)$.

Let K be the (right) quotient field of R and E be the subring generated by 1; $E \cong Z$ or $E \cong Z(p)$; R is a right essential ideal in $S = E + R$, and every right ideal of R is a right ideal of S. If $\mathrm{Char}(K) = 0$, then $xR \cap xE \neq (0)$ for all $0 \neq x \in R$, otherwise the trivial right R-module $(xE + xR)/xR \cong xE/(xR \cap xE)$ were not Artinian. Thus there is an $r \in R$ and an $0 \neq n \in E$ such that $xr = xn$; hence $0 \neq r = n \in R \cap E$, and $S/R \cong E/(R \cap E)$ is finite, especially an Artinian S-module. The same follows immediately, if $\mathrm{Char}(K) = p$. $(R + L)/L \cong R/(L \cap R) \neq R$ is Artinian for all right ideals L of S, likewise $S/(R + L)$. So S/L is an Artinian S-module, and the assertion is proved.

Lemma 2.1, Proposition 2.2 and Corollary remain true for rings without unity, as easily can be checked. Because of Proposition 2.1 in Proposition 2.2 the existence of the unity follows necessarily.

There is a great difference between V-rings with and without unity. The latter are not weakly regular in general (and the simple modules need not be injective), as is shown by the ring R with four elements and exactly two right unities; it also is a counterexample that the next Theorem remains valid for $0R^{-1} = (0)$.

THEOREM 1′. *A V-ring R with $R^{-1}0 = (0)$ and maximum condition for annihilating left ideals is a finite direct sum of simple V-rings with unity.*

Proof. $R^2 \subseteq M$ for any maximal right ideal $M \supseteq R^3$, otherwise $R^2 + M = R$ implies $M \supseteq R^3 + MR = R^2$. So $R^3 = R^2 =: S$. The ring S has a unity: $xS = \bigcap N$, where N denotes the set of maximal R-submodules of S containing xS; $S \supseteq S \cap NS^{-1} \supseteq N$; $S \cap NS^{-1} \neq S$, as $S = S^2 \not\subseteq N$, therefore $S \cap NS^{-1} = N$. Now $x \in S \cap (xS)S^{-1} = \bigcap (S \cap NS^{-1}) = \bigcap N = xS$; this holds for all $x \in S$. Because of Proposition 2.1 S contains a right unity e. If $y = er - re \neq Q$, $0 = Seye = Sey = Sy$. This is impossible, as $S \cap S^{-1}0 \subseteq R^{-1}(R^{-1}0) = 0$. So e is the unity of S. Now $R = S \oplus R(1-e)$, and the ideal $R(1-e)$ is isomorphic to R/R^2 as a ring. $R(1-e) = (0)$, as $R(1-e)R(1-e) = R(R(1-e)) = (0)$ and $R^{-1}0 = (0)$. Theorem 1 completes the proof.

Proposition 3.1 remains true for rings R with $R^2 \neq (0)$. This follows from Proposition 4.1 and from the fact that the unitary overring S of R is nonprimitive, right Noetherian with restricted minimum condition and the square of every principal right ideal finitely generated iff R is.

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