PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No 1, 1975

MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS II

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If A is a ring satisfying a polynomial identity, what identity is satisfied by the matrix ring A_n? Theorem: If A satisfies the standard identity of degree k, then A_n satisfies the standard identity of degree $2kn^2 - n^2 + 1$.

Definition: Suppose that $\{r_1, \cdots, r_q\}$ is a sequence of elements of a ring. *To parenthesize the sequence into j clumps* is to insert j pairs of adjacent, nonoverlapping parentheses. The subsequence within one pair of parentheses constitutes a *clump*. It is odd or even, depending on the number of entries. The *value* of the clump is the product of the entries. If the value is zero, the clump *vanishes*.

In the following let Z represent the integers.

LEMMA 1. *Let k, m, and n be positive integers. Let $\{u_1, \cdots, u_m\}$ be a nonvanishing sequence of matrix units e_{ij} in Z_n.*

(i) *If $m = kn$, there exists i such that the sequnce can be parenthesized into k clumps, each of value e_{ii}.*

(ii) *If $m = (kn - n + 1)n$, there exist i and j such that the sequence can be parenthesized into k clumps, each of value e_{ii}, and each beginning with e_{ij}.*

Proof of (i). Case 1. Suppose there exists i such that at least $k + 1$ of the entries in the sequence have i as initial subscript. Call the first $k + 1$ such entries $y_1, y_2, \cdots, y_{k+1}$. Then parenthesize the sequence as follows: start with y_1. Enclose it in parentheses, together with all entries to the right, if any, up to y_2. Next parenthesize y_2 with all entries up to y_3, etc. We form k clumps, each beginning with a y. Since each clump has to the right an entry with i as initial subscript, and the sequence is nonvanishing, each clump has value e_{ii}.

Case 2. Suppose that for all i, at most k of the entries have i as initial subscript. Since the sequence has kn entries, every i from l through n occurs exactly k times as an initial subscript.

Case 2a. The last entry is an idempotent e_{ii}. There are previous entries $y_1, \cdots, y_{k-1}$, each with i as initial subscript. Start with y_1 and enclose it in parentheses with all entries to the right, up to y_2. Continue, forming $k-1$ clumps, each of value e_{ii}. Then form a final clump consisting of the single e_{ii} at the end.

Case 2b. The last entry is e_{ij}, with $i \neq j$. Then there are k previous entries $y_1, \cdots, y_k$ with j as initial subscript. Parenthesize, forming $k-1$ clumps, beginning with $y_1, y_2, \cdots, y_{k-1}$, respectively. Then form a final clump, beginning with y_k and ending with the last e_{ij}. The result is k clumps, wach of value e_{ij}.

Proof of (ii). Let $m = (kn - n + 1)n$. Let $\{u_1, \cdots, u_m\}$ be a nonvanishing sequence of matrix units. Let $t = kn - n + 1$. By (i) there exists i such that the sequence can be parenthesized into t clumps, each of value e_{ii}. Let $y_1, \cdots, y_t$ be the first entries in these clumps. Each y has i as initial subscript. The second subscript can be any integer from 1 through n. Now

$$t = kn - n + 1 = (k-1)n + 1.$$

Thus for some j, at least k of the y's have j as second subscript. Suppose that $y_{f(1)}, \cdots, y_{f(k)}$ are all e_{ij}. Make new clumps as follows: start with $y_{f(1)}$ and enclose it in parentheses together with all entries to the right, up to $y_{f(2)}$. Continue, forming $k-1$ clumps. In the old parenthesizing $y_{f(k)}$ was the initial entry in a clump of value e_{ii}. Let this old clump be the kth clump in the new parenthesizing. The result is k clumps, each of value e_{ii}, and each beginning with e_{ij}.

Theorem 3.2 of **[2]** established that if A is an algebra satisfying a standard identity, so is A_n. The following theorem improves this result in three ways: (1) the degree of the identity satisfied by A_n is much lower. (2) The theorem holds for rings, not just algebras over fields. (3) The proof is simpler.

THEOREM 1. *If A is a ring satisfying the standard identity of degree k, then A_n satisfies thestandard identity of degree $2kn^2 - n^2 + 1$.*

Proof. Let

$$t = 2kn^2 - n^2 + 1 = (2k-1)n^2 + 1.$$

Choose t simple tensors in $A \otimes Z_n$ of form $a \otimes e_{ij}$, where $a \in A$, and e_{ij} is a matrix unit. Evaluate on these simple tensors the standard polynomial of degree t. Consider only nonvanishing terms.

Case 1. Suppose that for some i, at least k simple tensors have form

$$a_1 \otimes e_{ii}, \cdots, a_k \otimes e_{ii}.$$

Let $y = e_{ii}$. Call the remaining elements

$$b_1 \otimes z_1, b_2 \otimes z_2, \cdots.$$

Insert parentheses on the right side of each term: start with the first y and enclose it with all z's to the right, if any. Similarly parenthesize the next y with its z's, etc. The last y forms a singleton clump. Thus k clumps are created, each beginning with e_{ii}, and each of value e_{ii}. If there are any z's in the clump, call them the z *sub-clump.* It also has value e_{ii}.

Let V be the number of even clumps, and let D be the number of odd clumps. Then $V + D = k$. Each even clump yields two new odd clumps: the initial y and the z sub-clump. The result is $2V + D$ adjacent odd clumps, each of value e_{ii}. Note that $2V + D \geqq V + D = k$.

In each term find the first set of k adjacent odd clumps of value e_{ii}. Create a corresponding set of clumps on the left side. Call two terms equivalent if the following conditions hold on their left sides:

1. The elements to the left of the clumps are the same elements in the same order.

2. The k clumps are the same, but in any order.

3. The elements to the right of the clumps are the same elements in the same order.

Consider a fixed equivalence class. The sum of the terms in the class is a simple tensor whose right side has the common value for the class. The left side is the product of the following:

1. The product of all elements left of the clumps.

2. The standard polynomial of degree k, evaluated on the values of the k clumps, in some order.

3. The product of all the elements right of the clumps. (Because all these clumps are odd, Corollary to Lemma 4 of **[4]** ensures correctness of signs of terms.) Since the second factor vanishes, the conclusion follows.

Case 2. Suppose that Case 1 does not hold. Since there are $(2k-1)n^2+1$ simple tensors, by Lemma 1 (ii) there exist i and j such that at least $2k$ simple tensors have form $a \otimes e_{ij}$. Evidently, $i \neq j$. Let $w_{ii} = e_{ii} + e_{ij}$. Then w_{ii} is idempotent, and

$$e_{ij} = e_{ii} + e_{ij} - e_{ii} = w_{ii} - e_{ii}.$$

In each term replace e_{ij} by $w_{ii} - e_{ii}$. Let N be the original number of e_{ij}'s. Each old term, upon expansion, yields 2^N new terms. Every new term has on the right a monomial in w's and e's. If there are at least k of the e_{ii}'s in the term, it is suitable for Case 1. Otherwise there are at least k of the w_{ii}'s. In this case, define new elements as follows:

$$w_{jj} = -e_{ij} + e_{jj}$$

$$w_{ji} = -e_{ii} - e_{ij} + e_{ji} + e_{jj}.$$

If $i \neq p \neq j$, let

$$w_{pi} = e_{pi} + e_{pj}$$

$$w_{jp} = -e_{ip} + e_{jp}.$$

For the remaining integers from 1 through n, let $w_{pq} = e_{pq}$.

The w's constitute another set of matrix units in Z_n. Each old matrix unit e_{pq} is a linear combination of the w's with integral coefficients. Replace all the e's by w's. The conclusion follows by the linearity of the standard polynomial and by Case 1.

DEFINITION. The *unitary identity of degree k is*

$$\sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)} = 0,$$

where the sum is over all permutations π of the integers 1 through k.

THEOREM 2. *If A is a ring satisfying the unitary identity of degree k, then A_n satisfies the unitary identity of degree kn.*

Proof. The proof uses Lemma 1 (i) and is similar to Theorem 1 of **[4]**.

THEOREM 3. *If A is an algebra over a field with at least k elements, and A satisfies $x^k = 0$, then A_n satisfies $x^{kn} = 0$.*

Proof. The proof uses Lemma 1(i) and is similar to Theorem 1.2 of **[3]**. Note: That paper uses without definition the term "homogeneous component" of a polynomial. If $f(x_1, \cdots, x_j)$ is a polynomial, the homogeneous component of degree n_1 in x_1, degree n_2 in x_2, etc., is the sum of all terms with degree n_1 *in* x_1, etc.

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Received July 1, 1974. Presented to the Society, January 17, 1974. The writer is grateful to Professor Uri Leron for advice on this paper.

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