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STRONGLY SUPERFICIAL ELEMENTS

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The concepts of a strongly superficial element and a very strongly superficial element are introduced. A number of their properties are established and three applications are given.

1. Introduction. Superficial elements have proved to be a useful and important concept in a number of problems in commutative algebra, for example, the study of characteristic functions and multiplicities. This paper is concerned with two special kinds of such elements: a very strongly superficial (v.s.s.) element of degree k for an ideal A in a ring R; and, a strongly superficial (s.s.) element for A^k. After listing a number of properties of s.s. and v.s.s. elements, we present in Theorem (2.5) and (2.6) a number of characterizations of such elements. In §3 we give three applications of the theorems. Namely, we first show that a known result about s.s. elements for an ideal generated by an R-sequence in a locally Macaulay ring holds in every Noetherian ring (3.2). Next we show that if A is an ideal in a Noetherian ring R, then the zero ideal in the A-form ring of R has no irrelevant prime divisor if and only if there exists a v.s.s. element of some positive degree for A (3.5). The final application is concerned with certain ideals in Rees rings of R ((3.8) and (3.9)).

2. s.s. and v.s.s. elements. All rings in this paper are assumed to be commutative with a unit element.

DEFINITION. 2.1. Let A be an ideal in a ring R, and let k be a positive integer. A *superficial element of degree k for A* is an element $x \in A^k$ for which there exists a nonnegative integer c such that $(A^{n+k}: xR) \cap A^c = A^n$, for all integers $n \geqq c$. If $c = 0$ (where $A^0 = R$), then x is said to be a *very strongly superficial* (v.s.s.) *element of degree k for a*. If $A^{nk}: xR = A^{nk-k}$, for all integers $n \geqq 1$, then x is said to be a *strongly superficial* (s.s) *element for A^k*.

It is easily seen that, if $A^n \neq A^{n+1}$ (for each integer $n \geqq 0$) and x is a superficial element of degree k for A, then $x \notin A^{k+1}$. (In particular, a v.s.s. element of degree k for A is not in A^{k+1}.) It is also clear that a v.s.s. element of degree k for A is a s.s. element for A^k. Some further properties of such elements are given in the following remark.

REMARK 2.2. Let A be an ideal in a Noetherian ring R, let k be a positive integer, and assume x is a s.s. element for A^k.

(2.2.1) If $k = 1$, then x is a v.s.s. element of degree 1 for A, so the concepts of a s.s. element for A and of a v.s.s. element of degree 1 for A are the same.

(2.2.2) If y is a v.s.s. element of degree n for A, where $n = mk$, for some positive integer m, then it is readily seen that xy is a s.s. element for A^{n+k}. Also, x^n is a s.s. element for A^{nk}, for each integer $n \geqq 1$.

(2.2.3) If $xy = 0$, for some $y \in R$, then $y \in \bigcap_{n \geqq 1} (A^{nk} : xR) = \bigcap_{n \geqq 1} A^{nk-k}$. Therefore, if $\bigcap_{n \geqq 1} A^n = (0)$, then a s.s. element is not a zero-divisor.

(2.2.4) If x is a v.s.s. element of degree k for A, then statements analogous to (2.2.2)–(2.2.3) hold.

Theorems (2.5) and (2.6) below give several necessary and sufficient conditions for x to be a v.s.s. element of degree k for A (respectively, a s.s. element for A^k). To prove these results, the following lemma and definitions are needed.

LEMMA 2.3. *Let A be an ideal in a ring R, let $x \in A^k$, and assume x is a nonzero-divisor in R. Then $A^{nk} : xR = A^{nk-k}$, for all $n \geqq 1$, if and only if $x^n R[A^k/x] \cap R = A^{nk}$, for all integers $n \geqq 1$.*

Proof. Let $R' = R[A^k/x]$. To show the "only if" part, fix an integer $n \geqq 1$. Since $x^n R' \cap R \supseteq A^{kn}$, let $r \in x^n R' \cap R$. Then $r = x^n r'$, for some $r' = a/x^j$, where $a \in A^{kj}$, so $rx^j = x^n a \in A^{(n+j)k}$. Therefore, $r \in A^{(n+j)k} : x^j R = A^{nk}$. To see the "if" part, let $r \in A^{nk} : xR$. Then $rx = a \in A^{nk}$. Hence $r = x^{n-1}(a/x^n) \in x^{n-1}R' \cap R = A^{(n-1)k}$. The opposite containment is always true, because $x \in A^k$.

DEFINITION. 2.4. Let $A = (a_1, \cdots, a_n)$ be an ideal in a Noetherian ring $\mathcal{R}$, let u be an indeterminate, and let $t = u^{-1}$.

(2.4.1) The graded Noetherian ring $\mathcal{R} = \mathcal{R}(R, A) = R[ta_1, \cdots, ta_n, u]$ is the *Rees ring of R with respect to A.* The elements of R in $t^r A^r$ are said to be *homogeneous elements of degree r* ($-\infty < r < \infty$, where $A^r = R$ if $r \leq 0$) and a *homogeneous ideal* is an ideal which can be generated by homogeneous elements. A homogeneous ideal H in $\mathcal{R}$ is said to be *irrelevant* in case it contains every homogeneous element of sufficiently large degree. Otherwise, H is said to be *relevant.*

(2.4.2) The graded subring $\mathcal{S} = \mathcal{S}(R, A) = R[ta_1, \cdots, ta_n]$ of $\mathcal{R}$ is the *restricted Rees ring of R with respect to A.*

(2.4.3) The *form ring of R with respect to A* (or, *A-form ring of R*), $\mathcal{F}(R, A)$, is the graded ring $\bigoplus_{i=0}^{\infty} A^i/A^{i+1}$. It is known [3, Theorem 2.1] that $\mathcal{F} = \mathcal{F}(R, A) \cong \mathcal{R}/u\mathcal{R}$, where $\mathcal{R} = \mathcal{R}(R, A)$, and in this isomorphism the A-form of an element x in R corresponds to the coset $xt^k + u\mathcal{R}$ in $\mathcal{R}/u\mathcal{R}$. (The assumption in [3] that R be local is not essential.)

THEOREM. 2.5. *Let A be an ideal in a Noetherian ring R, let $\mathcal{R} = \mathcal{R}(R, A)$, and let $\mathcal{S} = \mathcal{S}(R, A)$. Fix a positive integer k, fix $x \in A^k$, and consider the following statements*:

(i) *x is a* v.s.s. *element of degree k for A.*
(ii) *x is a* v.s.s. *element of degree k for $A\mathcal{S}$ in $\mathcal{S}$.*
(iii) *x is a* v.s.s. *element of degree k for $u\mathcal{R}$ in $\mathcal{R}$.*
(iv) *xt^k is not in any prime divisor of $A^i\mathcal{S}$, for all $i \geqq 1$.*
(v) *u, xt^k is an $\mathcal{R}$-sequence.*
(vi) *$A^{n+k} \cap xR = xA^n$, for every integer $n \geqq 1$.*
(vii) *$A^{n+k}\mathcal{S} \cap x\mathcal{S} = xA^n\mathcal{S}$, for every integer $n \geqq 1$.*
(viii) *$u^{n+k}\mathcal{R} \cap x\mathcal{R} = (xu^n)\mathcal{R}$, for every integer $n \geqq 1$.*
(ix) *x is a nonzero-divisor and $x^nR' \cap R = A^{nk}$, for every integer $n \geqq 1$, where $R' = R[A^k/x]$.*

Then the following hold:

(2.5.1) (i)-(v) *are equivalent and each implies* (vi)-(viii).

(2.5.2) (vi)-(viii) *are equivalent and, if x is a nonzero-divisor, then each implies* (i)-(v) *and* (ix).

(2.5.3) If $k = 1$, *then* (ix) *implies* (i)-(viii).

Proof. (i)$\rightarrow$(iii). $x = u^k(xt^k) \in u^k\mathcal{R}$, and $u^{n+k}\mathcal{R} : x\mathcal{R} = u^n\mathcal{R} : xt^k\mathcal{R} \supseteq u^n\mathcal{R}$. For the opposite inclusion, let $yt^r \in u^{n+k}\mathcal{R} : x\mathcal{R}$. Then, with $m = n + k + r$, there exists $a \in A^m$ such that $xyt^r = u^{n+k}at^m$. Therefore, $xy = a \in A^m$; hence $y \in A^{n+r+k} : xR = A^{n+r}$, by (i). Therefore, $yt^{n+r} \in \mathcal{R}$, so $yt^r \in u^n\mathcal{R}$. Hence, since $u^n\mathcal{R} : x\mathcal{R}$ is homogeneous, (iii) holds.

(iii) implies $u\mathcal{R} = u^{k+1}\mathcal{R} : x\mathcal{R} = u^{k+1}\mathcal{R} : u^k(xt^k)\mathcal{R} = (u^{k+1}\mathcal{R} : u^k\mathcal{R}) : xt^k\mathcal{R} = u\mathcal{R} : xt^k\mathcal{R}$. Hence (iii) implies (v), since u is not a zero-divisor in $\mathcal{R}$.

(v)$\rightarrow$(iv). Let $i \geqq 1$ and let $at^n \in A^i\mathcal{S} : xt^k\mathcal{S}$. Then $at^nxt^k \in A^i\mathcal{S} = u^i\mathcal{R} \cap \mathcal{S}$ (this can be seen much as in the remainder of this paragraph). Hence $at^n \in u^i\mathcal{R} : xt^k\mathcal{R} =$ (by (v)) $u^i\mathcal{R}$, and so $at^{n+i} \in \mathcal{R}$, thus $a \in A^{n+i}$. Therefore $a = \Sigma\, b_g c_g$, where $b_g \in A^i$ and $c_g \in A^n$, hence $at^n = \Sigma\, b_g(c_g t^n) \in A^i\mathcal{S}$, and so (iv) holds.

(iv)$\rightarrow$(ii). Since $A^i\mathcal{S} : x\mathcal{S} \supseteq A^{i-k}\mathcal{S}$, for all $i \geqq k$, and both ideals are homogeneous, let yt^r be an arbitrary homogeneous element in $A^i\mathcal{S} : x\mathcal{S}$. Then $xyt^r \in A^{i-k}A^k\mathcal{S}$, say $xyt^r = \Sigma_{g,j} a_{g,j} b_{g,j}(c_{g,j}t^r)$, where each $a_{g,j} \in A^{i-k}$, $b_{g,j} \in A^k$, and $c_{g,j}t^r \in \mathcal{S}$, so $xt^k yt^r = \Sigma_{g,j} a_{g,j}(b_{g,j}t^k)(c_{g,j}t^r)$, where each $a_{g,j} \in A^{i-k}$ and $b_{g,j}c_{g,j}t^{r+k} \in \mathcal{S}$. Hence $xt^k yt^r \in A^{i-k}\mathcal{S}$, so that $yt^r \in A^{i-k}\mathcal{S}$, by (iv), and so (ii) holds.

Since $\mathcal{S} \subseteq R[t]$, (ii) implies, for all $i \geqq k$, $A^{i-k} = A^{i-k}\mathcal{S} \cap R = (A^i\mathcal{S}: x\mathcal{S}) \cap R = (A^i\mathcal{S} \cap R): (x\mathcal{S} \cap R) = A^iR: xR$, hence (ii) implies (i). Therefore, (i)-(v) are equivalent, and if (i) holds, then

(*) $A^{n+k} \cap xR = x(A^{n+k}: xR) = xA^n$, for all $n \geqq 1$, hence (i) implies (vi). Similarly (ii) implies (vii) and (iii) implies (viii). Therefore, (2.5.1) holds.

(2.5.2) (viii)$\rightarrow$(vii) much as in the proof that (v)$\rightarrow$(iv); and (vii)$\rightarrow$(vi), since $\mathcal{S} \subseteq R[t]$. For (vi)$\rightarrow$(viii), let $at^i \in u^{n+k}\mathcal{R} \cap x\mathcal{R} = x(u^{n+k}\mathcal{R}: x\mathcal{R})$, so $at^i = xbt^i \in u^{n+k}\mathcal{R}$, for some $bt^i \in u^{n+k}\mathcal{R}: x\mathcal{R}$. Therefore, with $g = n+i+k$, $a = bx \in A^g$, so $b \in A^g: xR$, hence $xb \in x(A^g: xR) = A^g \cap xR =$ (by (vi)) xA^{n+i}. Therefore $xb = \Sigma xc_f d_f$, where $xc_f \in xA^n$ and $d_f \in A^i$, hence $at^i = xbt^i \in xA^n\mathcal{R} \subseteq xu^n\mathcal{R}$, and so $u^{n+k}\mathcal{R} \cap x\mathcal{R} = x(u^{n+k}\mathcal{R}: x\mathcal{R}) \subseteq xu^n\mathcal{R}$, since $u^{n+k}\mathcal{R} \cap x\mathcal{R}$ is homogeneous. Hence (viii) holds, since $xu^n\mathcal{R} \subseteq u^n\mathcal{R}$ and since $x \in A^k$ implies $u^{n+k}\mathcal{R}: x\mathcal{R} \supseteq u^n\mathcal{R}$. Further, if x is a nonzero-divisor, then (vi) implies (i), by (*), and so (ix) holds, by (2.3).

Finally, for (2.5.3), if $k = 1$, then (ix) implies (i), by (2.3).

THEOREM 2.6. *Let R, A, x and k be as in* (2.5), *let* $\mathcal{R} = \mathcal{R}(R, A^k)$, *let* $\mathcal{S} = \mathcal{S}(R, A^k)$, *and consider the following statements*:

(i) *x is a* s.s. *element for A^k.*
(ii) *x is a* s.s. *element for $A^k\mathcal{S}$.*
(iii) *x is a* s.s. *element for $u\mathcal{R}$.*
(iv) *xt is not in any prime divisor of $A^{ki}\mathcal{S}$, for each $i \geqq 1$.*
(v) *u, xt is an $\mathcal{R}$-sequence.*
(vi) *$A^{nk} \cap xR = xA^{nk-k}$, for all integers $n \geqq 1$.*
(vii) *$A^{nk}\mathcal{S} \cap x\mathcal{S} = xA^{nk-k}\mathcal{S}$, for all integers $n \geqq 1$.*
(viii) *$u^n\mathcal{R} \cap x\mathcal{R} = xu^{n-1}\mathcal{R}$, for all integers $n \geqq 1$.*
(ix) *x is a nonzero divisor and $x^nR' \cap R = A^{nk}$, for all integers $n \geqq 1$, where $R' = R[A^k/x]$.*

Then the following statements hold:

(2.6.1) (i)-(v) *are equivalent and each implies* (vi)-(viii).

(2.6.2) (vi)-(viii) *are equivalent and, if x is not a zero divisor, then each implies* (i)-(v) *and* (ix).

(2.6.3) (ix) *implies* (i)-(viii).

Proof. This follows from (2.2.1) and (2.5).

3. Applications.

In this section we give three applications of Theorems (2.5) and (2.6).

REMARK. 3.1. Let $a_1, \cdots, a_m$ be an R-sequence in a Noetherian ring R, and let $A = (a_1, \cdots, a_m)R$.

(3.1.1) [2, Corollary 3.7]. If R is locally Macaulay, then a_1 is a s.s. element for A.

(3.1.2) If R is a Macaulay local ring, then each of the following statements hold:

(i) ([5, Lemma 6, p. 402] and [1, Theorem 119].) Each a_i is a s.s. element for A.

(ii) [5, Lemma 5, p. 401]. The prime divisors of $A^n (n \geqq 1)$ are the prime divisors of A and each has height m.

We note that it follows from (3.1.2) that parts (i) and (ii) of (3.1.2) also hold for an ideal generated by an R-sequence in a locally Macaulay ring. However, it follows from (3.2) below that (3.1.2) (i) holds even if R is not locally Macaulay.

PROPOSITION[1] 3.2. *Let R be a Noetherian ring, let $a_1, \cdots, a_m$ be an R-sequence, and let $A = (a_1, \cdots, a_m)R$. Then $A^n : a_iR = A^{n-1}$, for every integer $n \geqq 1$ and for every $i = 1, \cdots, m$.*

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[1] I am grateful to the referee for mentioning that this result was proved in D. Taylor, "Ideals generated by monomials in an R-sequence," Thesis, University of Chicago, 1966. Since her thesis isn't readily available, the referee kindly provided the following proof of a generalization of (3.2): Let R be a commutative ring with identity, M an R-module, $a_1, \cdots, a_m$ an M-sequence in R, and $A = (a_1, \cdots, a_m)R$. Then, for all positive integers n and for $i = 1, \cdots, m$, $A^nM : a_iR = A^{n-1}M$.

Proof. Let $S = Z[x_1, \cdots, x_m]$, $I = (x_1, \cdots, x_m)S$, $\phi : S \to R$ by $\phi(x_i) = a_i$ (so that R, M become S-modules) and consider the commutative diagram:

$$\begin{tikzcd} & S \arrow[r, "x_i"] \arrow[d] & S \arrow[d] & S/(I^n + x_iS) \arrow[d, phantom, "\cong"] & \\ 0 \arrow[r] & S/I^{n-1} \arrow[r, "\alpha"] & S/I^n \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

In order that $A^nM : a_iR = A^{n-1}M$ it suffices that the bottom row remain exact upon applying $\otimes_R M$.

Hence, it suffices that $\mathrm{Tor}_1^S(E, M) = 0$. But E is easily seen to have a filtration all of whose factors are $\cong F = S/(x_1, \cdots, x_m)S$ ($F \cong Z$, of course). Thus, a sufficient condition for $A^nM : a_iR = A^{n-1}M$, all n, is that $\mathrm{Tor}_1^S(F, M) = 0$, which is immediate if $a_1, \cdots, a_m$ is a regaular sequence on M.

Proof. Let $\mathcal{R} = \mathcal{R}(R, A)$. By [**4**, Theorem 3.5.1], $a_i t$ is not in any prime divisor of $u\mathcal{R}$, for every $i = 1, \cdots, m$. Hence, we are done by (2.5) (i) and (v).

Clearly, (3.2) and (2.5) show that, with R, $a_1, \cdots, a_m$ and A as in (3.2), each a_i is a s.s. element for $A\mathcal{S}$ in $\mathcal{S} = \mathcal{S}(R, A)$ and for $u\mathcal{R}$ in $\mathcal{R} = \mathcal{R}(R, A)$.

DEFINITION 3.3. Let A be an ideal in a Noetherian ring R. For all integers s, the *s-component* H_s of a homogeneous ideal H in $\mathcal{R} = \mathcal{R}(R, A)$ is the ideal in R, $H_s = \{b \in R \mid t^s b \in H\}$.

It is easy to see that a homogeneous ideal H in $\mathcal{R}$ is irrelevant if and only if $H_s = A^s$, for all (or, for some) sufficiently large s. Equivalently, H is irrelevant if and only if $H \supseteq (A^*)^s = (A^s)^*$, for all (or, for some) sufficiently large s, where $B^* = BR[u, t] \cap \mathcal{R}$, for each ideal B in R. (2.5) (v) shows that a sufficient condition for $u\mathcal{R}$ to have no irrelevant prime divisors is the existence of a v.s.s. element x of some degree k for A. That is, if xt^k is not in any prime divisor P of $u\mathcal{R}$, then clearly no power of xt^k can belong to P. (3.4) below shows that the converse also holds.

THEOREM 3.4. *Let A be an ideal in a Noetherian ring R, and let $\mathcal{R} = \mathcal{R}(R, A)$. A necessary and sufficient condition for $u\mathcal{R}$ to have no irrelevant prime divisor is that there exists a* v.s.s. *element of some positive degree for A.*

Proof. By the preceding discussion, it suffices to prove the "necessary part." Let $A^* = AR[u, t] \cap \mathcal{R}$, let $P_1, \cdots, P_h$ be the prime divisors of $u\mathcal{R}$, and let $N_g = \{c_r t^r;\ c_r t^r \in P_g \text{ and } r \geqq 1\}$ be the set of all homogeneous elements of positive degree contained in P_g, for each $g = 1, \cdots, h$. If we can find a homogeneous element of positive degree in $\mathcal{R}$ and not in any of the N_g, then we are done by (2.5) (i) and (v).

Since P_h is relevant by hypothesis, $P_h \not\supseteq A^*$; therefore, there exists some $a \in A$ such that $at \notin N_h$. If $at \notin G = \bigcup_{g=1}^{h} N_g$, we are done. If $at \in G$, then, say, $at \in I = \bigcap_{i=1}^{m} N_i$ and $at \notin J = \bigcup_{j=m+1}^{h} N_j$. We can assume there are no containment relations among the N_g; thus $J' \not\subseteq I'$, where $I' = \bigcup_{i=1}^{m} N_i$ and $J' = \bigcap_{j=m+1}^{h} N_j$. To see this, note that each homogeneous element in $N_{m+1} \cdots N_h$ is in J', because the N_j are subsets of ideals. Therefore, if $J' \subseteq I'$, then $(N_{m+1} \cdots N_h)\mathcal{R} \subseteq \bigcup_{i=1}^{m} P_i$; hence there exists an i $(1 \leqq i \leqq m)$ and a j $(m+1 \leqq j \leqq h)$ such that $N_j \subseteq N_i$ which contradicts the assumption. Therefore, let bt^e be a homogeneous element of positive degree such that $bt^e \in J'$ and $bt^e \notin I'$. Then $xt^e = (at)^e + bt^e$ satisfies (2.5) (v).

COROLLARY 3.5. *Let A be an ideal in a Noetherian ring R, and let $\mathcal{F}$ be the form ring of R with respect to A. Then a necessary and sufficient condition for there to exist a* v.s.s. *element of some positive ddegree for A is that the zero ideal in $\mathcal{F}$ has no irrelevant prime divisor.*

(An irrelevant (homogeneous) ideal in $\mathcal{F}$ is defined in an analogous manner to (2.4.1).)

Proof. This follows immediately from (3.4) and the fact that $\mathcal{F} \cong \mathcal{R}/u\mathcal{R}$ [3, Theorem 2.1], where $\mathcal{R} = \mathcal{R}(R, A)$.

COROLLARY 3.6. *Let A be an ideal in a Noetherian ring R, and assume there exists an element x in A such that $A^m : xR = A^{m-1}$, for all integers $m \geqq r$, where r is some fixed positive integer. Then the following statements hold, for each integer $i \geqq r$:*

(3.6.1) *x^i is a* v.s.s. *element of degree one for A^i.*

(3.6.2) *If M is a maximal ideal in R such that $A \subseteq M$, then $u\mathcal{R}^{(i)} : \mathcal{M}_i = u\mathcal{R}^{(i)}$, where $\mathcal{R}^{(i)} = \mathcal{R}(R, A^i)$, and $\mathcal{M}_i = (MR[u,t] \cap \mathcal{R}^{(i)}, u)\mathcal{R}^{(i)}$.*

Proof. (3.6.1) is clear, because $(A^i)^{n+1} : x^iR = (A^{in+i} : xR) : x^{i-1}R = (A^i)^n$, for all integers $n \geqq 1$. (3.6.2) follows from (3.6.1) and (3.4).

We conclude this paper with the following three observations.

LEMMA 3.7. *Let A be an ideal in a ring R, and assume x is an element in R such that $A^n : xR = A^n$, for every integer $n \geqq 1$. If $(A,x)R \neq R$, then x is a* v.s.s. *element of degree one for $(A,x)R$.*

Proof.

$$(A,x)^nR : xR = (A^n, x(A,x)^{n-1})R : xR$$
$$= A^n : xR + (A,x)^{n-1}R = A^n + (A,x)^{n-1} = (A,x)^{n-1},$$

for every integer $n \geqq 1$.

COROLLARY 3.8. *Assume A is an ideal in a Noetherian ring R containing a* v.s.s. *element x of degree k. Then xt^k is a* v.s.s. *element of degree one for both $(u, xt^k)\mathcal{R}$ in $\mathcal{R} = \mathcal{R}(R, A)$ and $(A, xt^k)\mathcal{S}$ in $\mathcal{S} = \mathcal{S}(R, A)$.*

Proof. Clear by (3.7) and the equivalence of (2.5) (i), (iv), and (v).

Let A be an ideal in a Noetherian ring R, and let $\mathcal{R} = \mathcal{R}(R, A)$. It is easily shown (cf. [3]) that for every ideal B in R, $B^* = BR[u, t] \cap \mathcal{R}$ is such that $B^*: u\mathcal{R} = B^*$, and $B^{*n}R[u, t] = B^nR[u, t]$, but it is not in general true that $B^{*n} = (B^n)^*$. However, it follows from considering homogeneous elements that $A^{*n} = (A^n)^*$, for each $n \geqq 1$.

COROLLARY 3.9. *Let A and B be ideals in a Noetherian ring R such that $A + B \neq R$, and let $\mathcal{R} = \mathcal{R}(R, A)$. If $(B^*)^n = (B^n)^*$, for each $n \geqq 1$, then u is a* v.s.s. *element of degree one for $(B^*, u)\mathcal{R}$.*

Proof. $(B^*, u)\mathcal{R} \neq \mathcal{R}$, since $A + B \neq R$. Therefore (3.9) follows from (3.7), because $(B^*)^n: u\mathcal{R} = (B^n)^*: u\mathcal{R} = (B^n)^* = (B^*)^n$, for each $n \geqq 1$.

It is also clear, by the preceding discussion, that (3.9) holds, in particular, whenever $B = A$.

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