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LOGARITHMIC CONVEXITY RESULTS FOR HOLOMORPHIC SEMIGROUPS

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The classical logarithmic convexity inequality, for solutions of $u' = -Au$ with A a self adjoint operator on Hilbert space, yield that u is small at intermediate times, $0 < t \leqq T$, provided that u is small at T and bounded at 0. Use of the Carleman inequality for analytic functions allows one to easily generalize this result to the case of operators A which are generators of holomorphic semigroups on Banach space.

The basic logarithmic convexity result states that $\log\|u(t)\|$ is a convex function of t for solutions of the ordinary differential equation on Hilbert space, $u' = -Au$, where A is a self adjoint operator. The simplest and earliest proof known to the author appears in [1]; it involves merely differentiating $\log\|u(t)\|$ twice and use of symmetry and the Cauchy-Schwartz inequality.

Log convexity is equivalent to the following inequality: if $0 \leqq t \leqq T$,

$$\text{(1)} \qquad \|u(T)\| \leqq \epsilon, \quad \|u(0)\| \leqq E, \quad \text{then} \quad \|u(t)\| \leqq \epsilon^{t/T} E^{1-t/T}.$$

It thus provides a stability estimate for the problem of backward solution of $u' = -Au$ with a prescribed bound, for if u and v are two solutions to this equation, both closely fitting measured data g at time T and satisfying prescribed bounds at time 0; i.e.,

$$\text{(2)} \qquad \begin{aligned} &\|u(T) - g\| \leqq \epsilon, \qquad \|v(T) - g\| \leqq \epsilon \\ &\|u(0)\| \leqq E, \qquad \|v(o)\| \leqq E, \end{aligned}$$

then at intermediate times we have

$$\text{(3)} \qquad \|u(t) - v(t)\| \leqq 2\epsilon^{t/T} E^{1-t/T}, \quad 0 \leqq t \leqq T.$$

We wish to show that the basic log convexity inequality (1) generalizes quite easily, by use of the Carleman inequality, to the class of operators *A on Banach space* which are *generators of holomorphic semigroups*. Such operators are usually defined in terms of existence and certain bounds for the resolvent operator $(A - zI)^{-1}$ in certain sectors of the complex plane, see Kato [4], or see Friedman [2] for a more concise and introductory treatment. From these bounds it follows that there exist constants $M \geqq 1$, k real, and $0 < \psi \leqq \pi/2\, k$ such that:

(4) (i) A generates a semigroup $e^{-\tau A}$ which is strongly continuous at $\tau = 0$ and satisfies the semigroup property, not only for real τ, but also for all complex $\tau = t + is$ in the closure of the sector $\Gamma_\psi = \{\tau \neq 0: \arg \tau < \psi\}$,

(4) (ii) $e^{-\tau A}$ is analytic with respect to τ in Γ_ψ,

(4) (iii) $\|e^{-\tau A}\| \leqq Me^{kt}$ on $\bar{\Gamma}_\psi$.

This is a particularly large and interesting class of operators. It includes, for example (see [2]), essentially all elliptic operators on $L^2(\Omega)$ corresponding to zero Dirichlet data on $\partial\Omega$ for which the Gärding inequality applies, and all elliptic operators on $L^p(\Omega)$ corresponding to regular elliptic boundary value problems.

In the Hilbert space case it suffices that A be a "sectorial operator," i.e.,

(5) (i) the numerical range of A lies in the sector $\{z: \arg(z + k) \leqq \pi/2 - \psi)$,

(5) (ii) A is closed,

(5) (iii) the resolvent $(A - zI)^{-1}$ exists at at least one point z outside this sector. Under these hypotheses (4) holds with $M = 1$.

THEOREM. *Let A be the generator of a holomorphic semigroup on Banach space, with corresponding* $M \geqq 1$, $0 < \psi \leqq \pi/2$, $0 < \psi \leqq \pi/2$, *and real k, in* (4). *Let $u(t)$ be a solution of the ordinary differential equation $u' = -Au$ (that is, $u(t) = e^{tA}u(0)$, $t \geqq 0$) satisfying*

$$\|u(T)\| \leqq \epsilon, \ \|u(0)\| \leqq E. \tag{6}$$

Then

$$\|u(t)\| \leqq Me^{k(t-Tw(t))}\epsilon^{w(t)}E^{1-w(t)}, \quad 0 \leqq t \leqq T, \tag{7}$$

where $w(\tau)$ is the harmonic function on the "bent strip"

$$S = \{\tau = t + is: |\arg \tau| < \psi, |\arg(\tau - T)| > \psi\}$$

which is bounded and continuous on $\bar{S}$, and which assumes the values 0 *and* 1 *respectively on the left and right hand boundary arcs of S.*

Proof. It suffices to assume that $k = 0$, for the general case then follows by considering $e^{-k\tau}u(\tau)$ instead of $u(\tau)$ itself.

The vector valued function $u(\tau) = e^{-\tau A}u(0)$ is analytic on S, continuous and bounded on $\bar{S}$, and bounded in norm by ME and $M\epsilon$ respectively on the left and right hand boundary arcs of S. The same conditions then hold for the complex valued function $f(\tau) = v^*(u(\tau))$, where v^* is any element of unit norm in the dual Banach space. The Carleman inequality (whose proof after all merely involves dominating the subharmonic function $\log|f(z)|$ by the harmonic function $(\log \epsilon)w(z) + (\log E)\,(1 - w(z))$, see **[3]**) then yields that

$$|f(\tau) \leqq \epsilon^{w(\tau)}E^{1-w(\tau)} \quad \text{on} \quad \bar{S}. \tag{8}$$

Since the norm of a vector u is the supremum of its values $|v^*(u)|$ over all v^* of unit norm, we obtain (7) as desired.

REMARK. Notice that when A is self-adjoint and semi-bounded from below, then the numerical range of A lies on the segment $[-k, \infty)$ of the real axis, $\psi = \pi/2$, S is the vertical strip $\{\tau = t + is: 0 < t < T\}$, $w(\tau) \equiv t/T$, and we hence obtain (1) as a special case of (7).

REFERENCES

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