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SOME THREE-POINT SUBSET PROPERTIES CONNECTED WITH MENGER'S CHARACTERIZATION OF BOUNDARIES OF PLANE CONVEX SETS

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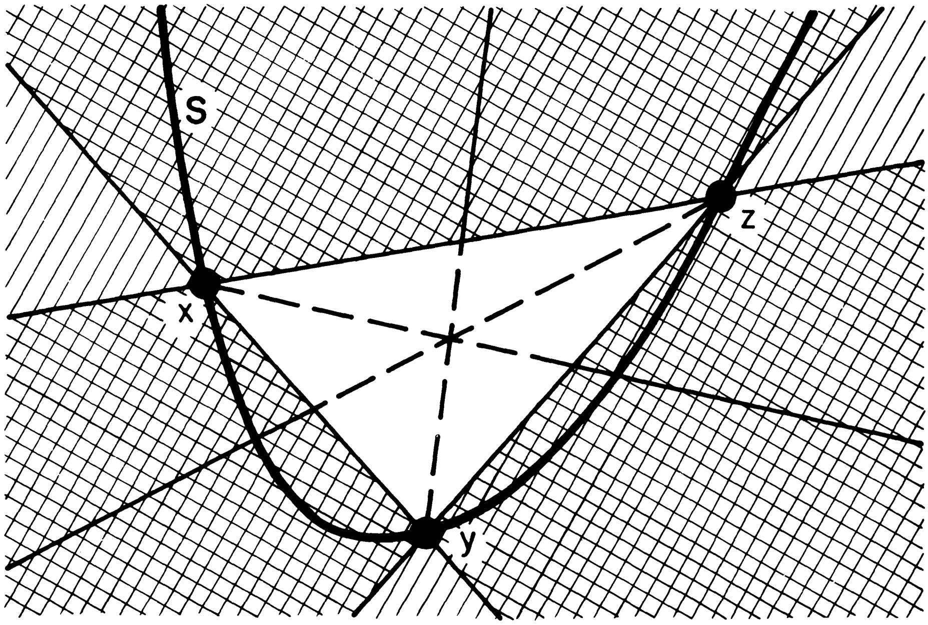
An elementary characterization of those plane sets which are boundaries of convex sets is given together with other results of the same character.

1. Introduction. A theorem of K. Menger [1], see also F. A. Valentine [3], states that a plane compact set S is the boundary of some (bounded) convex set if and only if S satisfies certain simple conditions expressible in terms of the three-point subsets of S. The main result of the present note is an extension of Menger's theorem to possibly unbounded closed sets (Theorem 3). We shall start by studying subsets of boundaries of convex sets, the main tool being one of Menger's conditions. Our final result (Theorem 4) is an extension of a theorem of Valentine [3], giving a description of those sets satisfying another of Menger's conditions. The results of this note may be related to an unpublished work of W. M. Swan [2].

The assertion of Theorem 3 is illustrated in the figure: A closed set S containing three noncollinear points is the boundary of a convex set if and only if, for any noncollinear points $x, y, z \in S$, firstly, each of the six closed areas with strong hatching (the dotted lines are medians in triangle xyz) contains points from S other than x, y, z, and secondly, the interior of triangle xyz contains no points from S. Instead of the second condition we might have required that the interior of each of the three weakly hatched angles contained no points from S.

2. Terminology. Everything takes place in the plane. The interior, boundary, and convex hull of a set S are denoted by int S, bd S, and conv S, respectively. The closed and open segments with end-points x and y are denoted by $[x, y]$ and $]x, y[$, respectively. If x, y, z are noncollinear points, $L(x, y)$ and $H(x, y; z)$ denote the line through x and y, and the closed half-plane H with $x, y \in \operatorname{bd} H$, $z \in H$, respectively. A *convex curve* is a connected subset of the boundary of a convex set.



3. Results and proofs.

PROPOSITION. *A plane set S fulfils*

$$(\S) \qquad \forall\, x, y, z \in S: S \cap \operatorname{int} \operatorname{conv}\{x, y, z\} = \varnothing$$

if and only if S is either a subset of the boundary of a convex set, or an X-set, that is a set $\{x_1, x_2, x_3, x_4, x_5\}$ with $]x_1, x_2[\cap]x_3, x_4[= \{x_5\}$.

Proof. The "if" statement is obvious. To prove the "only if" statement, assume (§) and $S \not\subseteq \operatorname{bd} \operatorname{conv} S$. There are noncollinear $p, w, x \in S$ with $p \in \operatorname{int} \operatorname{conv} S$. Let V denote the complement of $H(p, x; w) \cup H(p, w; x)$. Then $S \cap V = \varnothing$ by (§). Since $p \in \operatorname{int} \operatorname{conv} S$, there is an open half-plane H with $p \in \operatorname{bd} H$, $V \subseteq H$ such that there are $y, z \in S \cap H$ with $y \in H(p, w; x)$, $z \in H(p, x; w)$. By (§) we see that $p \in]x, z[$, $p \in]w, y[$ and that S cannot contain more points than p, w, x, y, z. Thus S is an X-set.

THEOREM 1. *A connected set S is a convex curve if and only if* (§).

Proof. Since an X-set is not connected, the assertion follows by the proposition.

THEOREM 2. *A connected set S is a convex curve if*

$$(\S') \qquad \forall\, x, y \in S\colon S \cap\,]x, y[\, = \emptyset.$$

Proof. Suppose S is a connected set fulfilling (§′), but not being a convex curve. By Theorem 1 there are $w, x, y, z \in S$ with $w \in \operatorname{int} \operatorname{conv}\{x, y, z\}$. Let the points r and t be determined by $r \in [x, y]$, $w \in [r, z]$ and $t \in [y, z]$, $w \in [x, t]$, respectively, and the point u by $u \in [y, w] \cap [r, t]$. By (§′), $S \cap L(w, z) = \{w, z\}$, $S \cap L(x, w) = \{x, w\}$, and $u \notin S$. Since S is connected, either $]u, r[$ or $]u, t[$, say $]u, r[$, contains a point $p \in S$. By (§′), $S \cap L(y, p) = \{y, p\}$, hence x and w are not in the same component of S, a contradiction.

THEOREM 3. *A closed set S containing three noncollinear points is the boundary of a convex set if and only if* (§) *and*

$$(*) \qquad \forall\, x, y, z \in S\colon S \cap \surd(x, y, z) \neq \emptyset,$$

where we define $\surd(x, y, z)$ to be the whole plane if x, y, z are collinear, and otherwise, letting V denote the closed convex cone with vertex y whose boundary contains z and the midpoint $x \oplus z$ of $[x, z]$, to be V deprived of z and the part contained in the open half-plane I with $x, z \in \operatorname{bd} I$ and $y \in I$.

Proof. First we shall prove the "only if" statement. Suppose $S = \operatorname{bd} C$ where C is a closed convex set. Then (§) is obvious. To prove $(*)$, let $x, y, z \in S$ be noncollinear. Letting V denote the complement of $H(x, z; y) \cup H(y, z; x)$, we have $C \cap V = \emptyset$. Let $r \in V$. Now $x \oplus z \in C$ and $r \notin C$ so $S \cap [x \oplus z, r] \neq \emptyset$, whence $S \cap \surd(x, y, z) \neq \emptyset$.

Next we shall prove the "if" statement. Observing that $(*)$ implies that S is not an X-set and using (§) we get $S \subseteq \operatorname{bd} \operatorname{conv} S$ by Proposition 2. Assume $a \in (\operatorname{bd} \operatorname{conv} S) \setminus S$.

Let H be a closed half-plane with $a \in \operatorname{bd} H$, $S \subseteq H$. There is a point in $S \cap \operatorname{bd} H$, and hence a point $x \in \operatorname{bd} H$ fulfilling $S \cap [x, a] = \{x\}$, for assume $S \cap \operatorname{bd} H = \emptyset$, and let (x_n) be a sequence in S such that the distance from x_{n+1} to H is less than half of the distance from x_n to H. We may assume that all the x_n, $n \geqq 2$ are in the same open half-plane K with $x_1, a \in \operatorname{bd} K$. In addition, consider a half-plane N with $x_2 \in \operatorname{bd} N$, $S \subseteq N$. Then it is seen that (x_n) has an accumulation-point lying in $\operatorname{bd} H$, a contradiction.

There are $y, z \in S$ such that $y \in \operatorname{int} H$, $z \notin H(x, y; a)$, for the assumption $v \in S$, $a \in\,]x, v[$ would contradict $(*)$. Choose $p \in S \cap \surd(z, x, y)$ and let M denote the closed half-plane with $x \oplus y \in \operatorname{bd} M$,

$M \subseteq H$. Now $p \in M \cap \operatorname{int} H(x, y; z)$ since $S \cap L(x, y) = \{x, y\}$. Choose $q \in S \cap \vee(y, p, x)$, then $q \in H \setminus \operatorname{int}(H(y, a; x) \cup M)$.

Consider a closed half-plane R with $q \in \operatorname{bd} R$, $S \subseteq R$, and consider all the half-planes T such that $a \in T$ and $\operatorname{bd} T$ cuts $]q, a[$ and $]x, a[$. Since all the T contain points from S, there is a point $u \in S$ with $a \in]x, u[$, a contradiction.

THEOREM 4. *For a connected set S such that*

$$(**) \qquad \forall\, x, y, z \in S\colon y \in]x, z[\Rightarrow [x, z] \subseteq S,$$

at least one of the following four conditions holds:

(1) $\operatorname{int} S$ *is convex and* $S \subseteq \operatorname{cl} \operatorname{int} S$.

(2) *S is a convex curve.*

(3) *S is the union of two non-disjoint linear elements, a linear element being a connected subset of a line.*

(4) *S is the union of three linear elements P, Q, R with a common endpoint contained in S and* $\operatorname{int}\operatorname{conv}(P \cup Q \cup R)$.

Proof. Suppose S is a connected set fulfilling $(**)$, but none of the four conditions.

We have $\operatorname{int} S = \varnothing$ since otherwise (1) were true.

By Theorem 1 and non(2) there are points $a, b, c, d \in S$ with $d \in \operatorname{int}\operatorname{conv}\{a, b, c\}$. An argument resembling the one in Theorem 2 shows that one of the segments $[a, d]$, $[b, d]$, and $[c, d]$, say $[a, d]$, is part of S.

Put $R = S \cap L(a, d)$ and let i be the point determined by $i \in L(a, d) \cap [b, c]$. Now $i \notin S$, for otherwise $S \cap (L(a, d) \cup L(b, c))$ fulfils (3) so $S \setminus (L(a, d) \cup L(b, c)) \neq \varnothing$, whence $\operatorname{int} S \neq \varnothing$, a contradiction. By a similar argument we see that for a point m, $R \cap]m, b[\neq \varnothing$ implies $m \notin S$. Hence, letting n denote the endpoint of R nearest to i, there is an open half-line T with endpoint n and $S \cap T = \varnothing$ such that T together with $L(a, d) \setminus R$ and $\{n\}$ separates c from R. Thus $n \in S$. By non(4), S cannot contain both $[b, n]$ and $[c, n]$, hence we may assume $S \cap L(b, n) = \{b, n\}$.

Let $j \in R \setminus \{n\}$ and let the points k and l be determined by $k \in [j, b] \cap L(n, c)$ and $l \in [k, i] \cap [n, b]$, respectively. By non(3) we get $S \cap L(c, n) = \{c, n\}$ and $S \cap]k, l[= \varnothing$. Moreover $S \cap]l, i[$ is empty, for if it contained a point p, then, since $S \cap]n, i[= \varnothing$, $(**)$ would imply that S had no point in $L(p, b) \cap H(n, i; c)$ contradicting that S is a connected set containing b and c. But then we have a new contradiction to the connectedness of S, and the proof is complete.

References

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