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A CHARACTERIZATION OF THE MAXIMAL MONOIDS AND MAXIMAL GROUPS IN β_X

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This paper gives a characterization of the maximal monoids and maximal groups in β_X, the semigroup of binary relations. The characterization is applied to show that the maximal group containing the partial order α is isomorphic to the group of order automorphisms of (X, α).

Introduction. In establishing some background research in which the setting of this current paper can be placed, one cites that much work has been done on the semigroup β_X, the binary relations on the set X. In particular, the works of Montague and Plemmons [4], Plemmons and Schein [5], Schwarz [6], as well as Clifford [3] appear dominant in the study of the maximal groups contained in β_X. The basic result of these papers shows that the maximal group of β_X containing the partial order α, is isomorphic to Auto (X, α), the group of order automorphisms of (X, α). Thus, as Birkhoff's Theorem provides that every group is isomorphic to a group of automorphisms on some partially ordered set (X, α), it follows that every group is isomorphic to a maximal group in β_X, for some X.

The work of this paper is concerned with maximal monoids as well as maximal groups in β_X. The work differs from previously published papers in that the concern here is to provide a unified characterization of these two algebraic systems of β_X. The characterization then provides a natural avenue to obtain the isomorphism theorems of Montague and Plemmons [4], Plemmons and Schein [5], and Clifford [3]. Further, by viewing the isomorphism theorems through our characterization, the researcher obtains some measure of intuition as to their validity.

As a convenient method of presentation, the paper is divided into two parts, a section of genral results, followed by a section of applications in which our general results are applied to β_X for X infinite and then X finite.

General results. Let $\alpha \in \beta_X$. We associate with α a Boolean matrix whose (x, y)th entry is 1 if and only if $(x, y) \in \alpha$. This correspondence provides an isomorphism between the semigroup β_X and the semigroup B_X of $X \times X$ Boolean matrices. Thus, without loss of generality, we characterize the maximal monoids and maximal group in B_X.

Let V_X be the collection of $X \times 1$ Boolean vectors. If $A \in B_X$, A^x denotes the xth column of A. By $z = \Sigma_{x \in \tau} A^x$ we mean the vector in V_X with the property that the yth entry in z is 1 if and only if there is an $x \in \tau$ such that A^x has yth entry 1. Set $R(A) = \{Ax \mid x \in V_X\}$. By End $R(A)$ we mean all cone endomorphisms from $R(A)$ into $R(A)$, i.e. $\{\psi \mid \psi(\Sigma_{x \in \tau} a_x z^x) = \Sigma_{x \in \tau} a_x \psi(z^x)$ where $z^x \in R(A)$ and $a_x \in B$, B the Boolean algebra, for all $x \in \tau$, and index set$\}$. Similarly, Aut $R(A) = \{\psi \in \text{End}\, R(A) \mid \psi$ is one-to-one and onto$\}$.

It should be noted here that $\psi(ax + by) = a\psi(x) + b\psi(y)$ for all $x, y \in R(A)$ and $a, b \in B$ does not guarantee $\psi \in \text{End}\, R(A)$. For example, let $X = \{1, 2, \cdots, n, \cdots\}$. Define

$\psi(x) = x$ if x has only finitely many nonzero entries, and

$\psi(x) = e = (1, 1, \cdots, 1, \cdots)^t$ if x has infinitely many nonzero entries.

Thus $\psi(ax + by) = a\psi(x) + b\psi(y)$ for all $x, y \in V_X$ and $a, b \in B$. Now let $e_i \in V_X$ be the vector with precisely one 1, which is in the ith position. Set $z = \Sigma_{i=1}^{\infty} e_{2i}$. Then $\psi(z) = e$, yet $\Sigma_{i=1}^{\infty} \psi(e_{2i}) = \Sigma_{i=1}^{\infty} e_{2i} \neq e$, and hence $\psi \notin \text{End}\, R(I)$ where I is the identity matrix.

Finally, if $I \in B_X$ is an idempotent,

$$M(I) = \{A \in B_X \mid AI = IA = A\}$$

and

$$G(I) = \{\text{units in } M(I)\}.$$

$M(I)$ is the maximal monoid with identity I and of course, $G(I)$ is the maximal group in B_X which contains I.

For further background in this area, the reader is referred to **[1]**.

Our initial results characterize $M(I)$.

LEMMA 1. *Set* $z = \Sigma_{x \in \tau} z^x$ *where* $z^x \in V_X$ *for* $x \in \tau$. *If* $A \in B_X$, $Az = \Sigma_{x \in \tau} Az^x$.

Proof. For $w \in V_X$, let w_y denote the yth entry in w. Now $z_y = 1$ if and only if $z_y^x = 1$ for some $x \in \tau$. By calculation,

$$Az = \sum_{z_y = 1} A^y = \sum_{\substack{z_y^x = 1 \\ x \in \tau}} A^y = \sum_{x \in \tau} Az^x.$$

Hence the lemma follows.

Our characterization of $M(I)$ is contained in the next theorem.

THEOREM 1. End $R(I)$ *is isomorphic to* $M(I)$ *for any idempotent* $I \in B_X$.

Proof. For each $\psi \in \operatorname{End} R(I)$, consider the matrix A_ψ whose xth column is $\psi(I^x)$. Pick $\sigma \in \operatorname{End} R(I)$. Then $A_\psi \cdot A_\sigma$ has xth column $\Sigma_y [\sigma(I^x)]_y \psi(I^y)$ where $[\sigma(I^x)]_y$ is the yth entry of $\sigma(I^x)$. Now

$$\sum_y [\sigma(I^x)]_y \psi(I^y) = \psi\Big[\sum_y [\sigma(I^x)]_y I^y\Big] = \psi[I\sigma(I^x)]$$
$$= \psi[\sigma(I^x)] = \psi \circ \sigma(I^x).$$

Thus $A_\psi A_\sigma = A_{\psi\circ\sigma}$.

Now, if 1 is the identity endomorphism, $A_1 = I$ and so $A_\psi I = IA_\psi = A_\psi$ from which it follows that $A_\psi \in M(I)$.

Consider the map $\pi(\psi) = A_\psi$ of $\operatorname{End} R(I)$ into $M(I)$. Of course π is a monomorphism. To show π is onto, pick $A \in M(I)$ and set $\psi(z) = Az$ for each $z \in R(I)$. Then by Lemma 1, $\psi \in \operatorname{End} R(I)$. Since $AI = A$, the xth column of A is $\psi(I^x)$ and hence $A = A_\psi$.

COROLLARY 1. Aut $R(I)$ *is isomorphic to* $G(I)$ *for any idempotent* $I \in B_X$.

Proof. The group of units of $M(I)$ is $G(I)$ while the group of units of $\operatorname{End} R(I)$ is Aut $R(I)$. Since a monoid isomorphism maps units to units the result follows.

We say that a set $\mathcal{S} = \{S^x \mid S^x \in V_X \text{ for } x \in T, \text{ some index set}\}$ is independent if and only if no $S^x = 0$ and

$$\sum_{x\in N\subseteq T} S^x = S^y \quad \text{implies} \quad y \in N.$$

Further, if $w, z \in V_X$ we say $w \leqq z$ if and only if the xth entry of z being 1 implies the xth entry of w is 1. Note that $\Sigma_{x\in N\subseteq T} S^x \geqq S^x$ for each $x \in N$ and thus in independence proofs one needs only to check that $S^y = \Sigma_{x\in N\subseteq T} S^x$ implies $S^y \leqq S^x$ for some $x \in N$. For $S, T \subseteq V_X$ we say that ψ is an order map from S to T if and only if for $w \leqq z$ in S, $\psi(w) \leqq \psi(z)$ in T.

With these definitions, our direction of research is now to refine the characterization of $M(I)$ and thereby to provide a link from Aut $R(I)$ to Auto (X, I). The instrument for this link is given in the next theorem.

THEOREM 2. *Let* $S \in B_X$ *and* $\mathcal{S} = \{S^x \mid x \in \tau, \tau \text{ some index set}\}$ *be independent such that*

(1) $R(S) = \{w \mid w = \Sigma_{x\in N\subseteq\tau} S^x\} \cup \{0\}$.

(2) $\Sigma_{x\in N\subseteq\tau} S^x \geqq S^y$ *implies some* $S^x \geqq S^y$, $x \in N$.

Then if ψ is an order map from $\mathcal{S}$ into $R(S)$ there is a unique $\bar{\psi} \in \operatorname{End} R(S)$ so that $\bar{\psi} = \psi$ on $\mathcal{S}$.

Proof. Define $\bar{\psi}(S^x) = \psi(S^x)$ for each $x \in \tau$.

$$\bar{\psi}(0) = 0.$$

$$\bar{\psi}\Big(\sum_{x \in N \subseteq \tau} S^x\Big) = \sum_{x \in N \subseteq \tau} \psi(S^x).$$

To see that $\bar{\psi}$ is well defined, note that if $\Sigma_{x \in N \subseteq \tau} S^x = \Sigma_{x \in M \subseteq \tau} S^x$ then for each $y \in M$ there is an $x \in N$ so that $S^x \geqq S^y$. Thus $\bar{\psi}(S^x) = \psi(S^x) \geqq \psi(S^y) = \bar{\psi}(S^y)$ and hence $\bar{\psi}(\Sigma_{x \in N} S^x) \geqq \bar{\psi}(\Sigma_{x \in M} S^x)$. Similarly, $\bar{\psi}(\Sigma_{x \in M} S^x) \geqq \bar{\psi}(\Sigma_{x \in N} S^x)$ and so $\bar{\psi}(\Sigma_{x \in N} S^x) = \bar{\psi}(\Sigma_{x \in M} S^x)$.

To see that $\bar{\psi} \in \operatorname{End} R(S)$ pick $z^x \in R(S)$, $x \in N$ where N is some index set. Suppose $z^x = \Sigma_{y \in N_x} S^y$, N_x an index set. Consider

$$\bar{\psi}\Big(\sum_{x \in N} z^x\Big) = \bar{\psi}\Big(\sum_{\substack{y \in N_x \\ x \in N}} S^y\Big) = \sum_{\substack{y \in N_x \\ x \in N}} \psi(S^y) = \sum_{x \in N} \sum_{y \in N_x} \psi(S^y)$$

$$= \sum_{x \in N} \bar{\psi}\Big(\sum_{x \in N_x} S^y\Big) = \sum_{x \in N} \bar{\psi}(z^x).$$

Thus $\bar{\psi} \in \operatorname{End} R(S)$.

Finally, the uniqueness of $\bar{\psi}$ is clear as any $\sigma \in \operatorname{End} R(S)$ must satisfy $\sigma(\Sigma_{x \in M \subseteq T} S^x) = \Sigma_{x \in M \subseteq T} \sigma(S^x)$ and hence is uniquely determined on the independent set.

We now apply the results of this section to illustrate the utility of our characterization and to obtain the isomorphism theorem of Montague, Plemmons, Schein, and Clifford, as a consequence.

Applications. The organization of this section is to separate the applications of the general results to B_X into two cases, i.e. X infinite and X finite.

Case 1. X infinite. Our first lemma shows that a partial order satisfies the hypotheses of Theorem 2.

LEMMA 2. *If I is a partial order, $\mathcal{S} = \{I^x \mid x \in X\}$ is independent with $\Sigma_{x \in N \subseteq X} I^x \geqq I^y$ implying some $I^x \geqq I^y$ for $x \in N$.*

Proof. Let $I^y \in \mathcal{S}$ such that $I^y = \Sigma_{x \in N \subseteq X} I^x$. As $(y, y) \in I$, $(y, x) \in I$ for some $x \in N$. Now $(z, y) \in I$ implies $(z, x) \in I$. So $I^x \geqq I^y$ and hence $\mathcal{S}$ is independent.

Now consider

$$\sum_{x \in N \subseteq X} I^x \geqq I^y \qquad \text{for some } y.$$

Then as in the above argument, there is some $x \in N$ so that $I^x \geqq I^y$. Hence the result follows.

From this lemma, we can argue our first isomorphism theorem.

THEOREM 3. *If I is a partial order,* $\operatorname{Aut} R(I)$ *is isomorphic to Auto* $(\mathcal{S}, \leqq)$ *where* $\mathcal{S} = \{I^x \mid X \in X\}$.

Proof. By Lemma 2, $\mathcal{S}$ satisfies the hypothesis of Theorem 2. Thus, any $\psi \in \operatorname{Auto}(\mathcal{S}, \leqq)$ extends to a unique $\bar{\psi} \in \operatorname{Aut} R(I)$. Conversely, any $\bar{\psi} \in \operatorname{Aut} R(I)$, when restricted to $\mathcal{S}$, determines a $\psi \in \operatorname{Auto}(\mathcal{S}, \leqq)$. Thus $\operatorname{Auto}(\mathcal{S}, \leqq)$ is isomorphic to $\operatorname{Aut} R(I)$.

The Plemmons, Schein, Clifford result is an immediate application.

COROLLARY 3. *If I is a partial order, $G(I)$ is isomorphic to* $\operatorname{Auto}(X, I)$.

Proof. From previous results, $G(I)$ is isomorphic to $\operatorname{Auto}(\mathcal{S}, \leqq)$, $\mathcal{S} = \{I^x \mid x \in X\}$. Finally, note that $(x, y) \in I$ if and only if $I^x \leqq I^y$. Thus $\operatorname{Auto}(\mathcal{S}, \leqq)$ is isomorphic to $\operatorname{Auto}(X, I)$.

To illustrate the utility of our characterization, we provide the following example.

EXAMPLE. Let $X = [0, 1]$ and let I be the natural partial order $\geqq$ on X. Clearly, $\{I^x \mid x \in X\}$ is independent. Further, $R(I) = \{I^x \mid x \in X\} = \mathcal{S}$ and hence $\{\psi \mid \psi \colon \mathcal{S} \to R(I),\ \psi \text{ an order map}\} = \{\psi \mid \psi \colon \mathcal{S} \to \mathcal{S},\ \psi \text{ an order map}\}$. Thus, it follows that $M(I)$ is isomorphic to the semigroup of nondecreasing functions $f\colon X \to X$ so that $f(0) = 0$. Hence $G(I)$ is the group of continuous strictly increasing functions $f\colon X \to X$, so that $f(0) = 0$ and $f(1) = 1$.

Case 2. X finite. For this case we use the following result of Schwarz [7].

Idempotent Theorem. If I is an idempotent then there is a permutation matrix P so that

$$P^tIP = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ A_{21} & A_2 & \cdots & 0 & 0 \\ \\ A_{s-1,1} & A_{s-1,2} & \cdots & A_{s-1} & 0 \\ A_{s,1} & A_{s,2} & \cdots & A_{s,s-1} & A_s \end{bmatrix} \text{ where}$$

(1) each A_k is composed entirely of 1's or $A_k = (0)$, the 0-matrix of order one,

(2) each A_{kj} is composed entirely of 1's or entirely of 0's,

(3) if $A_{jk} > 0$ and a_k a column in $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_k \\ \vdots \\ A_{s,k} \end{bmatrix}$, a_j a column in $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_j \\ \vdots \\ A_{s,j} \end{bmatrix}$, then $a_k \geqq a_j$,

(4) if $A_i = A_j = (0)$, $i > j$, and $A_{ij} \neq (0)$ then there is a k, $i > k > j$ so that A_k, A_{ik} and A_{kj} are composed entirely of 1's.

Let I be a nonzero finite idempotent. Without loss of generality we assume I has the form given in the Idempotent theorem. As a consequence of this theorem we note that I has a nonzero main diagonal element and, in fact the next theorem shows that this characterizes the independent columns of I.

To show that the situation is different for infinite idempotents consider the set $X = [0, 1]$ and define $I \in B_X$ by $(x, y) \in I$ if and only if $x > y$. Then I is an idempotent with no set of independent columns that generate $R(I)$ and I has no nonzero diagonal elements.

THEOREM 4. *Let I be a finite nonzero idempotent and $\mathcal{S}$ the set of distinct vectors I^x such that $(x, x) \in I$. Then $\mathcal{S}$ is an independent set satisfying the conditions of Theorem* 2.

Proof. From the remarks above, $\mathcal{S} \neq \varnothing$. Clearly $\mathcal{S}$ is independent and satisfies Condition 2. Thus, we need only prove $R(I) = \{w \mid w = \Sigma_{I^x \in N \subseteq \mathcal{S}} I^x\} \cup \{0\}$. Let I^y be nonzero and not in $\mathcal{S}$. Let $T = \{I^x \mid (x, x) \in I \text{ and } (x, y) \in I\}$. From (4) of the Idempotent Theorem, $T \neq \varnothing$. We show $\Sigma_T I^x = I^y$. It is seen from (3) of the Idempotent Theorem that $I^y \geqq I^x$ for all $I^x \in T$. Thus pick any $(z, y) \in I$. Consider I^z. If $(z, z) \in I$ then $I^z \in T$. If $(z, z) \notin I$, then by (4) of the Idempotent Theorem there is an I^x, $y < x < z$ with $(x, x) \in I$ and $(z, x) \in I$. Hence, $(z, x) \in \Sigma_T I^x$ and the result follows.

We now give the finite analogue of Theorem 3. Note that in this case, the partial order assumption is not required. Indeed the result holds for any idempotent.

THEOREM 5. *If I is a finite nonzero idempotent,* $\operatorname{Aut} R(I)$ *is isomorphic to* $\operatorname{Auto}(\mathcal{S}, \leqq)$ *where $\mathcal{S}$ is as given in Theorem* 4.

The Montague, Plemmons result follows as a consequence.

COROLLARY 4. *If I is a finite partial order, $G(I)$ is isomorphic to* $\operatorname{Auto}(X, I)$.

In conclusion, we provide an example indicating how our characterization theorem can be applied for the case X is finite.

EXAMPLE. Consider the idempotent

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then

$$R(I) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

with

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ independent and } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \geqq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, using Theorem 2,

$$M(I) = \left(I, [0], \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

and

$$G(I) = \{I\}.$$

Other such examples are easily constructed.

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