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RELATIVELY INVARIANT MEASURES

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A homomorphism of minimal flows $X \xrightarrow{\phi} Y$, has a relatively invariant measure if there exists a positive projection from $\mathscr{C}(X)$ onto $\mathscr{C}(Y)$ which commutes with translasion. Such a relatively invariant measure does not always exists. However, some elementary facts from the theory of compact convex sub-sets of a locally convex topological vector space are used to show that given a homomorphism of minimal flows $X \xrightarrow{\phi} Y$ there exists a commutative diagram

$$\begin{array}{ccc} X^{\sim} & \xrightarrow{\theta^{\sim}} & X \\ {\scriptstyle\phi^{\sim}}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y^{\sim} & \xrightarrow[\theta]{} & Y \end{array}$$

where θ and $\theta^{\sim}$ are strongly proximal homomorphisms and $\phi^{\sim}$ has a relatively invariant measure, (RIM). Homomorphisms which have invariant measures are studied and questions of existence and uniqueness are investigated.

Similar diagrams, where θ and $\theta^{\sim}$ are replaced by other types of proximal extensions, and $\phi^{\sim}$ is replaced by an open map with certain additional properties, are studied in [12] and [2].

In section one we introduce notions and definitions. Section two is devoted to the proof of the main theorem about affine flows and then some corollaries for homomorphisms of minimal flows are deduced. Another corollary is a generalization of the Ryll Nardzewskie fixed point theorem. This results are extensions of results in [6].

In section three the notation of a relatively invariant measure is discussed and it is shown that metric distal extension has a relatively invariant measure (see [8] and [1]). A homomorphism with a RIM which has at least one finite fiber is shown to be almost periodic. In section four we show the existence of the commutative diagram mentioned above. This is used to show the existence of a universal strongly proximal extension for any given minimal flow. We conclude with some questions about the uniqueness of a RIM, and the existence of almost periodic extensions.

1. Definitions. Let T be a topological group, X a compact Hausdorff space. We say that (T,X) is a *flow* if there exists a jointly continuous map from $T \times X$ onto X, denoted $(t,x) \to tx$, such that $s(tx) = (st)x$ and $ex = x$ for all $x \in X$ and $s,t \in T$; e is the identity element of T. Let Q be a compact convex sub-set of a locally convex topological vector space. We say that (T,Q) is an *affine flow* if (T,Q) is a flow and if in addition the map $x \to tx$ from Q onto Q is an affine map for each $t \in T$. In particular an affine flow is a flow and one can talk about minimal sets of Q, proximal points in Q, etc.

Usually when referring to a flow (T,X) (or an affine flow (T,Q)) we shall omit the group T and write just X (Q respectively).

A sub-set of a flow X is *minimal* set if it is nonempty, closed, invariant and contains no proper closed invariant sub-set. A sub-set of an affine flow is *irreducible* if it is nonempty, closed, convex and invariant and contains no proper sub-set with these properties.

A continuous equivariant map from the flow X into the flow Y is a *homomorphism.* If Q and P are affine flows and ϕ from Q into P, is a homomorphism which is also an affine map we say that ϕ is an *affine-homomorphism.*

Let $Q \xrightarrow{\phi} P$ be an affine homomorphism. We say that a sub-set Q_0 of Q is *P-irreducible* (with respect to ϕ) if Q_0 is closed, convex, invariant, $\phi(Q_0) = P$ and Q_0 contains no proper sub-set with these properties.

Let Q be an affine flow, $X = \overline{\text{ex}}(Q)$ the closure of the set of extreme points of Q. Clearly X is a closed invariant set and by the Krein Milman theorem $\overline{\text{co}}(X)$, the closed convex hull of X, is equal to Q. We say that Q is a *primitive affine flow* if X is a minimal set.

Given a flow X we denote by $\mathscr{C}(X)$ the algebra of real valued continuous functions on X. If $f \in \mathscr{C}(X)$ and $t \in T$ then $f^t \in \mathscr{C}(X)$ is the function defined by $f^t(x) = f(tx)$. Let $\mathscr{M}(X)$ be the set of regular Borel probability measures on X. We consider $\mathscr{M}(X)$ as a closed convex sub-set of $\mathscr{C}(X)^*$, the dual space of $\mathscr{C}(X)$, equipped with the weak * topology. The action of T on X induces an action of T on $\mathscr{M}(X)$ in the following way. Let $\mu \in \mathscr{M}(X)$, $t \in T$ and $f \in \mathscr{C}(X)$, then define $t\mu \in \mathscr{M}(X)$ by

$$\int f d(t\mu) = \int f^t d\mu.$$

For a point $x \in X$ we denote the point mass at x by δ_x. Clearly $t\delta_x = \delta_{tx}$ and thus the homeomorphism $x \to \delta_x$ of X into $\mathscr{M}(X)$ is also an isomorphism of flows. Sometimes we shall identify X with the sub-set $\{\delta_x \mid x \in X\}$ of $\mathscr{M}(X)$.

With the above action $\mathscr{M}(X)$ is an affine flow. Since $\text{ex}(\mathscr{M}(X)) = X$, $\mathscr{M}(X)$ is primitive iff X is a minimal flow.

Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal flows (ϕ is necessarily onto). This homomorphism induces an affine homomorphism $\mathcal{M}(X) \xrightarrow{\hat{\phi}} \mathcal{M}(Y)$, as follows. Let $\mu \in \mathcal{M}(X)$ and let $f \in \mathscr{C}(Y)$ be given, $\hat{\phi}(\mu) \in \mathcal{M}(Y)$ is defined by

$$\int fd(\hat{\phi}(\mu)) = \int (f \circ \phi)\, d\mu.$$

We say that ϕ is a *strongly proximal homomorphism* (or extension) if for every measure $\mu \in \mathcal{M}(X)$ with $\hat{\phi}(\mu) =$ point mass on Y, there exists a net $t_i \in T$ such that $\lim t_i\mu =$ point mass on X. In particular X is a *strongly proximal flow* if X is a strongly proximal extension of the trivial flow.

Let $X \xrightarrow{\phi} Y$ be a strongly proximal extension and $x_1, x_2 \in X$ with $\phi(x_1) = \phi(x_2)$. The measure $\mu = (\delta_{x_1} + \delta_{x_2})/2$ satisfies $\hat{\phi}(\mu) = \delta_{\phi(x_1)}$. Hence there exists a net $t_i \in T$ such that $\lim t_i\mu = \delta_x$ for some $x \in X$. But $\lim t_i\mu = (\lim t_i\delta_{x_1} + \lim t_i\delta_{x_2})/2$ and since δ_x is an extreme point of $\mathcal{M}(X)$ this implies $\lim t_ix_i = \lim t_ix_2$, i.e. the points x_1 and x_2 are *proximal* points in X.

We say that a homomorphism is *proximal* if every two points with the same image are proximal points. Thus we have shown that a strongly proximal homomorphism is proximal.

A homomorphism $X \xrightarrow{\phi} Y$ is *distal* if whenever $x_1 \neq x_2$ and $\phi(x_1) = \phi(x_2)$ then x_1 and x_2 are not proximal. A flow is *distal* if it is a distal extension of the trivial flow.

2. Affine flows. Let Q be a compact convex sub-set of a locally convex topological vector space, and let X be a closed sub-set of Q such that $\overline{\text{co}}(X) = Q$. We shall use the following theorems from the general theory of convex sets (see for example **[11]**).

I (Krein Milman) $\overline{\text{co}}(\text{ex}(Q)) = Q$.

II (Milman) $\text{ex}(Q) \subseteq X$.

III For every measure $\mu \in \mathcal{M}(X)$ there exists a unique point $z \in Q$ such that for all affine functions f on Q, $f(z) = \int_X f(x)d\mu$. The map $\mu \xrightarrow{\beta} z$ sends $\mathcal{M}(X)$ onto Q, and is a weak * continuous affine map. The point z is called the barycenter of μ.

IV (Bauer) A point $x \in X$ is an extreme point of Q iff δ_X is the only measure in $\mathcal{M}(X)$ whose barycenter is x.

THEOREM 2.1. *Let $Q \xrightarrow{\phi} P$ be an affine homomorphism of an affine flow Q onto a primitive affine flow P. Then*

(1) *There exists a P-irreducible sub-set of Q.*

(2) *Every P-irreducible sub-set of Q is primitive.*

(3) *If $Q_0 \subseteq Q$ is P-irreducible, $X = \overline{\text{ex}}(Q_0)$ and $Y = \overline{\text{ex}}(P)$. Then $\phi(X) = Y$ and $(\phi | X): X \to Y$ is a strongly proximal homomorphism.*

(4) *If $Q_0 \subseteq Q$ is P-irreducible, $Y = \overline{\mathrm{ex}}(P)$ and $x_1, x_2 \in Q_0$ are such that $\phi(x_1) = \phi(x_2) \in Y$ then x_1 and x_2 are proximal points* (see **[6]** Theorem 5.3).

Proof. (1) Use Zorn's lemma.

(2) Let Q_0 be a P-irreducible sub-set of Q, $X = \overline{\mathrm{ex}}(Q_0)$ and $Y = \overline{\mathrm{ex}}(P)$. Let $x \in \phi^{-1}(Y) \cap Q_0$ then by the minimality of Y $\phi(\mathrm{cls}\{tx \mid t \in T\}) \supseteq Y$. Hence $\phi(\overline{\mathrm{co}}\{tx \mid t \in T\}) = \overline{\mathrm{co}}(Y) = P$. But Q_0 is P-irreducible and thus $\overline{\mathrm{co}}\{tx \mid t \in T\} = Q_0$. By II we conclude that

$$\mathrm{ex}(Q_0) \subseteq X \subseteq \mathrm{cls}\{tx \mid t \in T\} \subseteq \phi^{-1}(Y) \cap Q_0$$

Thus if $x \in X$ we have

$$X \subseteq \mathrm{cls}\{tx \mid t \in T\} \subseteq X \text{ and } X = \mathrm{cls}\{tx \mid t \in T\}.$$

This proves that X is a minimal set.

(3) Consider the map $\mathcal{M}(X) \xrightarrow{\beta} Q_0$ which sends a measure on X to its barycenter on Q_0. Let $y \in Y$, and let $\mu \in \mathcal{M}(X)$ be a measure with $\mathrm{Supp}(\mu) \subseteq \phi^{-1}(y) \cap X$. (i.e. $(\phi \mid X)^\wedge(\mu) = \delta_y$.)

If f is an affine function on P then $f \circ \phi$ is an affine function on Q. Hence $(f \circ \phi)(\beta\mu) = \int_X (f \circ \phi) d\mu = f(y)$ and since the affine functions on P separate points we can conclude that $\phi(\beta(\mu)) = y$. Hence $\phi(\beta(\mathrm{cls}\{t\mu \mid t \in T\})) = Y$ and $\phi(\overline{\mathrm{co}}(\beta(\mathrm{cls}\{t\mu \mid t \in T\}))) = P$.

Since Q_0 is P-irreducible we have

$$\overline{\mathrm{co}}(\beta(\mathrm{cls}\{t\mu \mid t \in T\})) = Q_0.$$

Now by II, this implies $\mathrm{ex}(Q_0) \subseteq X \subseteq \beta(\mathrm{cls}\{t\mu \mid t \in T\})$, and if $x_0 \in \mathrm{ex}(Q_o)$, then there exists $\nu \in \mathrm{cls}\{t\mu \mid t \in T\}$ such that $\beta(\nu) = x_0$. But by IV $\nu = \delta_{x_0}$ and $(\phi \mid X): X \to Y$ is a strongly proximal homomorphism.

(4) Let $X = \overline{\mathrm{ex}}(Q_0)$ and $y = \phi(x_1) = \phi(x_2)$. Choose $y_0 \in \mathrm{ex}(P)$, then there exists a net s_i in T such that $\lim s_i y = y_0$. We can assume that $\lim s_i x_1 = z_1$ and $\lim s_i x_2 = z_2$ exist. Let $\mu_i \in \mathcal{M}(X)$ satisfy $\beta(\mu_i) = z_i (i = 1,2)$, If we use β to denote also the barycenter map from $\mathcal{M}(Y)$ onto P, then $\beta((\phi \mid X)^\wedge(\mu_i)) = \phi(\beta(\mu_i) = \phi(z_i) = y_0$. Hence by IV $(\phi \mid X)^\wedge(\mu_i) = \delta_{y_0}$.

By (3) there exists a net t_j in T such that $\lim t_j\mu_1 = \lim t_j\mu_2 =$ point mass on X. By III we have

$$\lim t_j z_1 = \beta(\lim t_j\mu_1) = \beta(\lim t_j\mu_2) = \lim t_j z_2$$

and x_1 and x_2 are proximal.

PROPOSITION 2.2. *Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal flows.*

(1) *There exists an $\mathcal{M}(Y)$-irreducible affine sub-flow Q_0 of $\mathcal{M}(X)$. If $Y^{\sim} = \overline{\mathrm{ex}}(Q_0)$, then $Y^{\sim}$ is a minimal sub-flow of $\mathcal{M}(X)$, $\hat{\phi}(Y^{\sim}) = Y$ and the homomorphism $\theta = \hat{\phi} \mid Y^{\sim}$, $Y^{\sim} \xrightarrow{\theta} Y$ is strongly proximal.*

(2) *Conversely if $Y^{\sim} \subseteq \mathcal{M}(X)$ is a minimal set such that $\hat{\phi}(Y^{\sim}) = Y$ and such that $\theta = \hat{\phi} \mid Y^{\sim}$ is a strongly proximal homomorphism, then $Q = \overline{\mathrm{co}}(Y^{\sim})$ is an $\mathcal{M}(Y)$-irreducible affine sub-flow of $\mathcal{M}(X)$.*

Proof. (1) In Theorem 2.1. take $P = \mathcal{M}(Y)$, $Q = \mathcal{M}(X)$ and Q_0 a P-irreducible sub-set of Q with respect to $\hat{\phi}$. Since $\mathcal{M}(Y)$ is primitive $Y^{\sim} = \overline{\mathrm{ex}}(Q_0)$ is minimal, $\phi(Y^{\sim}) = Y$ and $Y^{\sim} \xrightarrow{\theta} Y$ is a strongly proximal homomorphism.

(2) Let $\beta : \mathcal{M}(Y^{\sim}) \to Q$ denote the barycenter map. Consider the diagram

$$\begin{array}{ccc} \mathcal{M}(Y^{\sim}) & \xrightarrow{\beta} & Q \subseteq \mathcal{M}(X) \\ \hat{\theta} \downarrow & \swarrow \hat{\phi} & \\ \mathcal{M}(Y) & & \end{array}$$

If $f \in \mathscr{C}(Y)$ then f can be considered as an affine function on $\mathcal{M}(Y)$ as follows. For $\nu \in \mathcal{M}(Y)$ $f(\nu) = \int f d\nu$. Thus $f \circ \hat{\phi}$ is an affine function on $\mathcal{M}(X)$ and hence also on Q. It follows that for every $f \in \mathscr{C}(Y)$ and $\xi \in \mathcal{M}(Y^{\sim})$

$$\int_Y f d\hat{\theta}(\xi) = \int_{Y^{\sim}} (f \circ \theta) d\xi = \int_{Y^{\sim}} (f \circ \hat{\phi}) d\xi$$
$$= (f \circ \hat{\phi})(\beta\xi) = f(\hat{\phi} \circ \beta(\xi)) = \int_Y f d(\hat{\phi} \circ \beta(\xi)).$$

Thus $\hat{\theta} = \hat{\phi} \circ \beta$ and the above diagram is commutative.

Let now $Q_0 \subseteq Q$ be an $\mathcal{M}(Y)$-irreducible sub-set of Q, $\nu \in \overline{\mathrm{ex}}(Q_0)$ and $\mu \in \mathcal{M}(Y^{\sim})$ such that $\beta(\mu) = \nu$. Then $\hat{\phi}(\nu)$ is a point mass on Y, say δ_y, and $\hat{\theta}(\mu) = (\hat{\phi} \circ \beta)(\mu) = \hat{\phi}(\nu) = \delta_y$. Since θ is strongly proximal homomorphism this implies that there exists a net t_i in T such that $\lim t_i\mu$ is a point mass on $Y^{\sim}$, say δ_η. Now $\lim t_i\nu = \lim t_i\beta\mu = \beta \lim t_i\mu = \beta(\delta_\eta) = \eta \in Y^{\sim}$ and since $\overline{\mathrm{ex}}(Q_0)$ is a closed invariant set this implies $Y^{\sim} \subseteq \overline{\mathrm{ex}}(Q_0)$. Therefore $Q = Q_0$, and Q is $\mathcal{M}(Y)$-irreducible.

COROLLARY 2.3. *Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal sets, then ϕ is strongly proximal iff $\mathcal{M}(X)$ is $\mathcal{M}(Y)$-irreducible.*

Proof. This follows immediately from Proposition 2.2 if we observe that $\overline{\text{ex}}(\mathcal{M}(X)) = X$, and $\overline{\text{co}}(X) = \mathcal{M}(X)$.

Theorem 2.1 can be applied to prove the following generalization of the Ryll Nardzewski fixed point theorem, in the same way as Theorem 5.3 in **[6]** was used in proving this fixed point theorem. (Theorem 7.3 in **[6]**.)

THEOREM 2.4. *Let E be a separable Banach space, Q a weakly compact convex sub-set of E. Suppose (T,Q) is an affine flow such that the action of T on Q is distal in the norm topology. Let $\phi: (T,Q) \to (T,P)$ be an affine homomorphism of Q onto a primitive affine flow P. Then there exists a minimal sub-flow X of Q such that $(\phi \mid X): X \to \overline{\text{ex}}(B)$ is an isomorphism of minimal flows.*

3. Relatively invariant measures.

DEFINITION. Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal flows; a linear map $P: \mathcal{C}(X) \to \mathcal{C}(Y)$ is called a *relatively invariant measure (RIM)* for ϕ if P satisfies the following properties

(1) $P(f) \geqq 0$ whenever $f \in \mathcal{C}(X)$ and $f \geqq 0$.

(2) $P(1) = 1$.

(3) $P(h \circ \phi) = h$ for $h \in \mathcal{C}(Y)$.

(4) $P(f^t) = (Pf)^t$ for $f \in \mathcal{C}(X)$ and $t \in T$.

DEFINITION. Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal flows. A homomorphism $\lambda: Y \to \mathcal{M}(X)$, $y \to \lambda_y$ is called a *section* for ϕ if for each $y \in Y, \hat{\phi}(\lambda_y) = \delta_y$.

PROPOSITION 3.1. *Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal flows. Then the following conditions are equivalent.*

(a) *There exists a section for ϕ.*

(b) *ϕ has a RIM.*

(c) *There exists a convex closed invariant sub-set Q of $\mathcal{M}(X)$ such that $(\hat{\phi} \mid Q): Q \to \mathcal{M}(Y)$ is an affine isomorphism onto.*

(d) *There exists an $\mathcal{M}(Y)$-irreducible affine sub-flow Q of $\mathcal{M}(X)$ such that $\hat{\phi} \mid \overline{\text{ex}}(Q): \overline{\text{ex}}(Q) \to Y$ is a flows isomorphism.*

Proof. (a)$\Rightarrow$(b) Let $\lambda: Y \to \mathcal{M}(X)$ be a section. Given a function $f \in \mathcal{C}(X)$, define a function Pf on Y as follows

$$(Pf)(y) = \int_X f d\lambda_y \qquad (y \in Y).$$

Clearly $Pf \in \mathscr{C}(Y)$. Since λ_y is a probability measure, properties (1) and (2) in the definition of a RIM are clearly satisfied by P. If $h \in \mathscr{C}(Y)$, then for each $y \in Y$

$$(P(h \circ \phi))(y) = \int_X (h \circ \phi) d\lambda_y = \int_Y h d\hat{\phi}(\lambda_y) = h(y).$$

Thus $P(h \circ \phi) = h$ and property (3) is satisfied. Finally if $f \in \mathscr{C}(X)$ and $t \in T$, then for each $y \in Y$

$$(P(f^t))(y) = \int_X f^t d\lambda_y = \int_X f dt\lambda_y = \int_X f d\lambda_{ty} = (Pf)(ty) = (Pf)^t(y).$$

Thus $P(f^t) = (Pt)^t$, property 4 is satisfied and P is a RIM for ϕ.

(b) $\Rightarrow$ (c) Let $P: \mathscr{C}(X) \to \mathscr{C}(Y)$ be a RIM. Define $\gamma: \mathscr{M}(Y) \to \mathscr{M}(X)$ by

$$\int_X f d(\gamma\mu) = \int_Y (Pf) d\mu \qquad (\mu \in \mathscr{M}(Y), f \in \mathscr{C}(X)).$$

Clearly γ is an affine weak * continuous map. Moreover if $t \in T$, then

$$\int_X f d(t\gamma\mu) = \int_X f^t d\gamma\mu = \int_Y P(f^t) d\mu = \int_Y (Pf)^t d\mu \quad --$$
$$= \int_Y (Pf)\, dt\mu = \int_X f d(\gamma t\mu).$$

Thus $\gamma t\mu = t\gamma\mu$ and γ is an affine homomorphism.
We now show that $\mu = \hat{\phi}\gamma\mu$ for $\mu \in \mathscr{M}(Y)$.
Indeed

$$\int_Y h\, d(\hat{\phi}\gamma\mu) = \int_X (h \circ \phi) d\gamma\mu = \int_Y P(h \circ \phi) d\mu = \int_Y h d\mu$$

for all $h \in \mathscr{C}(Y)$. If we denote $Q = \gamma(\mathscr{M}(Y))$ then Q is a convex closed invariant sub-set of $\mathscr{M}(X)$ and $\hat{\phi} \mid Q$ is $1-1$, hence an affine isomorphism.

(c) $\Rightarrow$ (d) Let Q be as in (c) then since $\hat{\phi} \mid Q$ is one to one it is clear that Q is $\mathscr{M}(Y)$-irreducible. Moreover $\hat{\phi} \mid \overline{\text{ex}}(Q)$: $\overline{\text{ex}}(Q) \to Y$ is a flow isomorphism.

(d) $\Rightarrow$ (a) Assuming Q as in (d) exists, define $\lambda: Y \to \mathscr{M}(X)$ by $\lambda = (\hat{\phi} | \overline{\text{ex}}(Q))^{-1}$ clearly λ is a section.

Definition. If $X \xrightarrow{\phi} Y$ is a homomorphism of minimal flows which has the properties (a)-(d) of Proposition 3.1 we say that ϕ is a *RIM extension* (or homomorphism).

REMARK. Let $X \xrightarrow{\phi} Y$ be a RIM extension then for a particular choice of a section $\lambda: Y \to \mathcal{M}(X)$, the following relations between the objects discussed in Proposition 4 holds:

A RIM for ϕ is given by

$$(Pf)(y) = \int_X f d\lambda_y \qquad (f \in \mathscr{C}(x), y \in Y)$$

The map $\gamma: \mathcal{M}(Y) \to \mathcal{M}(X)$, defined by

$$\int_X \mathrm{f} d\gamma\mu = \int_Y (\mathrm{Pf}) d\mu = \int_Y \int_X \mathrm{f} d\lambda_y d\mu$$

($\mathrm{f} \in \mathscr{C}(\mathrm{X}), \mu \in \mathcal{M}(\mathrm{Y})$), is an affine isomorphism into, and $Q = \gamma(\mathcal{M}(Y))$ is an $\mathcal{M}(Y)$-irreducible affine sub-flow of $\mathcal{M}(X)$. The minimal flow $Y^{\sim} = \overline{\mathrm{ex}}(Q)$ is isomorphic to Y via $\hat{\phi} \mid Y^{\sim}: Y^{\sim} \to Y$. Finally $\gamma \mid Y = (\hat{\phi} \mid Y^{\sim})^{-1} = \lambda$ is the original section for ϕ.

COROLLARY 3.2. *Let $X \xrightarrow{\phi} Y$ be a RIM extension.* (1) *If Y has an invariant measure so does X.* (2) *If X is uniquely ergodic then so is Y.*

Proof. By Proposition 3.1 there exists an affine isomorphism γ of $\mathcal{M}(Y)$ into $\mathcal{M}(X)$, and clearly (1) follows. If μ and ν are invariant measures on Y then by the unique ergodicity of X, $\gamma(\mu) = \gamma(\nu)$. Thus $\mu = \nu$ and Y is uniquely ergodic.

LEMMA 3.3. *Let $X \xrightarrow{\phi} Y$ be a RIM extension and $\lambda: y \to \mathcal{M}(X)$ a section for ϕ. If X is a metric space then there exists a residual set $\mathcal{O} \subseteq Y$ such that $y \in \mathcal{O}$ implies* $\mathrm{Supp}(\lambda_y) = \phi^{-1}(y)$.

Proof. Let 2^X denote the compact metric space of closed sub-sets of X, equipped with the Hansdorff topology. There is a natural action of T on 2^X induced by the action of T on X. The map $y \to \mathrm{Supp}\,(\lambda_y)$ from Y into 2^X is a lower-semi-continuous map and $t(\mathrm{Supp}(\lambda_y)) = \mathrm{Supp}(t\lambda_y) = \mathrm{Supp}(\lambda_{ty})$. Let $\mathcal{O} \subseteq Y$ be the set of points in Y at which the map $y \to \mathrm{Supp}(\lambda_y)$ is continuous, then $\mathcal{O}'$ is a residual sub-set of Y. Let $\mathscr{X} = \mathrm{cls}\{\mathrm{Supp}(\lambda_y) \mid y \in Y\}$ then $\mathscr{X}$ is a closed invariant sub-set of 2^X. If $A \in \mathscr{X}$ then $A = \lim \mathrm{Supp}(\lambda_{y_i})$ for some sequence $y_i \in Y$ and we can assume that $\lim y_i = y$ exists. Since $\mathrm{Supp}(\lambda_{y_i}) \subset \phi^{-1}(y_i)$ and since $\phi^{-1}: Y \to 2^X$ is an upper-semi-continuous map, it follows that $A \subseteq \phi^{-1}(y)$. Thus each element of $\mathscr{X}$ is contained in a fiber. Now if $y \in \mathcal{O}'$ then there is a unique element of $\mathscr{X}$ which is contained in $\phi^{-1}(y)$, namely $\mathrm{Supp}(\lambda_y)$.

Consider now the set $X' = \cup\{A \mid A \in \mathscr{X}\}$. Clearly this is a closed invariant sub-set of X, and since X is minimal $X' = X$. This implies that for $y \in \mathcal{O}$ $\mathrm{Supp}(\lambda_y) = \phi^{-1}(y)$.

LEMMA 3.4. *A strongly proximal homomorphism has a RIM iff it is an isomorphism.*

Proof. Let $X \xrightarrow{\phi} Y$ be a strongly proximal homomorphism with a RIM. By Proposition 3.1 (d) there exists an $\mathcal{M}(Y)$-irreducible affine sub-flow Q of $\mathcal{M}(X)$ such that $\hat{\phi}\,|\,\overline{\mathrm{ex}}(Q)\colon \overline{\mathrm{ex}}(Q) \to Y$ is an isomorphism. By Corollary 2.3 $\mathcal{M}(X)$ itself is $\mathcal{M}(Y)$-irreducible, hence $Q = \mathcal{M}(X)$ and $\overline{\mathrm{ex}}(Q) = X$.

This lemma showes that not every homomorphism of minimal flows has a RIM. For example every almost $1-1$ (almost automorphic) extension is strongly proximal and hence unless it is $1-1$, it does not possess a RIM. Of course any minimal flow without an invariant measure is an extension (of the trivial flow) without a RIM.

LEMMA 3.5. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "\chi"'] \arrow[dr, "\psi"] & \\ Y & & X \arrow[ll, "\phi"] \end{tikzcd}$$

be a commutative diagram of minimal flows.

(1) *If χ has a RIM so does ϕ.*

(2) *If ψ and ϕ have RIM then so does χ.*

(3) *If χ has a unique RIM and ψ has a RIM, then ϕ has a unique RIM.*

Proof. (1) Let $P\colon \mathscr{C}(Z) \to \mathscr{C}(Y)$ be a RIM for χ, then $f \to P(f \circ \psi)(f \in \mathscr{C}(X))$ is a RIM for ϕ.

(2) Clear.

(3) Let $P\colon \mathscr{C}(Z) \to \mathscr{C}(Y)$ be the unique RIM for χ and let $P_1\colon \mathscr{C}(Z) \to \mathscr{C}(X)$ be a RIM for ψ. By (1) there is also a RIM for ϕ, say $P_2\colon \mathscr{C}(X) \to \mathscr{C}(Y)$. Now by the uniqueness of P, $P_2 \circ P_1 = P$ and since $P_1(\mathscr{C}(Z)) = \mathscr{C}(X), P_2$ is unique.

DEFINITION. Let $(T, Z) \xrightarrow{\chi} (T, Y)$ be a homomorphism of minimal flows. We say that χ is a *group extension* if there exists a compact Hausdorff topological group K such that

(1) There is a jointly continuous action of K on Z, (Denoted by $(z, k) \to zk$, $z \in Z$, $k \in K$).

(2) For $z \in Z$, $t \in T$ and $k \in K$ $(tz)k = t(zk)$.

(3) For every $z \in Z$, $\chi^{-1}(\chi(z)) = zK$.

A homomorphism $(T,X)\xrightarrow{\phi}(T,Y)$ of minimal flows is called an *almost periodic* homomorphism (or extension), if there exists a group extension $(T,Z,K)\xrightarrow{\chi}(T,Y)$ and a homomorphism $(T,Z)\xrightarrow{\psi}(T,X)$ such that $\chi=\phi\circ\psi$. (Notice that ψ is also a group extension.)

LEMMA 3.6. *Let $X\xrightarrow{\phi}Y$ be an almost periodic extension. Denote*

$$\mathcal{N}=\{\mu\in\mathcal{M}(X)\,|\,\hat{\phi}(\mu)\text{ is a point mass on }Y\}.$$

Then $\mathcal{N}$ as a sub-flow of $\mathcal{M}$ (X), is pointwise almost periodic (i.e. $\mathcal{N}$ is a disjoint union of minimal sets), *and $\hat{\phi}\,|\mathcal{N}$: $\mathcal{N}\to Y$ is a distal extension.*

Proof. Let (T,Z,K), ψ and χ be as in the definition of almost periodic extension. We can assume that the action of K on Z is free. Define

$$\mathcal{L}=\{\mu\in\mathcal{M}(Z)\,|\,\hat{\chi}(\mu)\text{ is a point mass on Y}\}.$$

Fix a point $z_0\in Z$, let $y_0=\chi(x_0)$ and choose $\mu\in\mathcal{L}$ such that $\hat{\chi}(\mu)=y_0$. By our assumption the map $k\to z_0k$ is a homeomorphism of K onto $\chi^{-1}(y_0)$ and thus we can lift μ to a measure $\bar{\mu}$ on K:

$$\int_Z f(z)d\mu=\int_K f(z_0k)d\bar{\mu}:\qquad f\in\mathcal{C}(Z).$$

In this way we can define an action of K on $\hat{\chi}^{-1}(\delta_{y_0})$ namely $\overline{k\mu}=k\bar{\mu}$. (This depends of course on z_0).

Let $\mu,\theta\in\hat{\chi}^{-1}(\delta_{y_0})$, we shall show that θ is in the T-orbit closure of μ iff $\theta=k\mu$ for some $k\in K$. Indeed suppose there exists a net t_i in T such that $\lim t_i\mu=\theta$, without loss of generality we can assume that $\lim t_iz_0=z_1$, exists and it follows that $\chi(z_1)=y_0$. Hence there exists $k_1\in K$ such that $z_1=z_0k_1$. Now for every $f\in\mathcal{C}(Z)$ the net of functions $f^{t_i}(z_0k)$ converges uniformly in k to the function $f(z_1k)$. Hence

$$\begin{aligned}\int_Z f(z)d\theta&=\lim\int_Z f(z)dt_i\mu=\lim\int_Z f^{t_i}(z)d\mu\\&=\lim\int_K f^{t_i}(z_0k)d\bar{\mu}=\lim\int_K f(t_iz_0k)d\bar{\mu}\\&=\int_K f(z_0k_1k)d\bar{\mu}=\int_K f(z_0k)dk_1\bar{\mu}=\int_K f(z_0k)d\overline{k_1\mu}\\&=\int_Z f(z)dk_1\mu.\end{aligned}$$

Thus $k_1\mu=\theta$.

Conversely if $\theta = k_1\mu$ for some $k_1 \in K$, then there is a net t_i in T such that $\lim t_i z_0 = z_0 k_1$ and for $f \in \mathscr{C}(Z)$ we have

$$\begin{aligned}\int_Z f(z)d\theta &= \int_Z f(z)dk_1\mu = \int_K f(z_0k)d\overline{k_1\mu} \\ &= \int_K f(z_0k_1k)d\bar{\mu} = \lim \int_K f^{t_i}(z_0k)d\bar{\mu} \\ &= \lim \int_Z f^{t_i}(z)d\mu = \lim \int_Z f(z)dt_i\mu.\end{aligned}$$

Hence $\lim t_i\mu = \theta$.

It is now easy to see that $\mathscr{L}$ is pointwise almost periodic. Moreover if $\mu, \nu \in \mathscr{L}$ with $\hat{\chi}(\mu) = \hat{\chi}(\nu)$ and there exists a net t_i in T such that $\lim t_i\mu = \lim t_i\nu = \theta$. Then we can assume that $\hat{\chi}(\theta) = \hat{\chi}(\mu) = y_0$ and that $\lim t_i z_0 = z_1$ exists. Now as above there exists $k_1 \in K$ such that $z_0k_1 = z_1$ and $k_1\mu = \theta = k_1\nu$. Hence $\mu = \nu$ and $\hat{\chi} \mid \mathscr{L}: \mathscr{L} \to Y$ is distal. Finally since $\hat{\psi}(\mathscr{L}) = \mathscr{N}$ and since $\hat{\phi} \circ \hat{\psi} = \hat{\chi}$ it follows that $\mathscr{N}$ is point-wise almost-periodic and that $\hat{\phi} \mid \mathscr{N}: \mathscr{N} \to Y$ is distal.

REMARK. In the notations of Lemma 3.6, there is always an action of K on $\mathscr{M}(Z)$ namely $\mu \to \mu k$ where

$$\int_Z f(z)d(\mu k) = \int_Z f(zk)d\mu \qquad (f \in \mathscr{C}(Z), \mu \in \mathscr{M}(Z) \text{ and } k \in K).$$

When K is abelian the map $\mu \to \bar{\mu}$ of $\hat{\chi}^{-1}(y_0)$onto $\mathscr{M}(K)$, does not depend on the point $z_0 \in \chi^{-1}(y_0)$. Moreover $k\mu = \mu k$ and thus under the action $\mu \to \mu k$, $(T, \mathscr{L}_0, K) \xrightarrow{\hat{\chi}} (T, Y)$ is a group extension for every minimal set $\mathscr{L}_0 \subseteq \mathscr{L}$. Thus $\mathscr{N}_0 \xrightarrow{\hat{\phi}} Y$ is an almost periodic extension for every minimal set $\mathscr{N}_0 \subseteq \mathscr{N}$.

The following corollary was first proved by A. W. Knapp **[8]**. We include a proof which makes use of Lemma 3.6.

COROLLARY 3.7. *Let $X \xrightarrow{\phi} Y$ be an almost periodic extension then ϕ has a unique* RIM.

Proof. Let Q be an $\mathscr{M}(Y)$-irreducible affine sub-flow of $\mathscr{M}(X)$. By Proposition 2.2. $\overline{\mathrm{ex}}(Q) \xrightarrow{\hat{\phi}} Y$ is a strongly proximal extension of Y and by Lemma 3.6 $\overline{\mathrm{ex}}(Q) \xrightarrow{\hat{\phi}} Y$ a distal extension of Y. Hence $\hat{\phi} \mid \overline{\mathrm{ex}}(Q)$ is $1-1$ and by Proposition 3.1 (d), ϕ has a RIM.

Let (T, Z, K), χ and ψ be as in the definition of an almost-periodic extension. Assume again that the action of K on Z is free. The extension $Z \xrightarrow{\chi} Y$ is a group extension hence an almost periodic extension and by the first part of this proposition χ has section $\lambda: Y \to \mathcal{M}(Z)$. As in the proof of Lemma 3.6 one can show that for each $k \in K$ and $y \in Y$, $\bar{\lambda}_y = \overline{k\lambda_y}$. Hence $\bar{\lambda}_y$ is the Haar measure on K and λ is unique. The uniqueness of a RIM for ϕ now follows from Lemma 3.5 (3).

PROPOSITION 3.8. *Let X be a quasi-separable minimal flow* (i.e., X has sufficiently many metric factors) *and let $X \xrightarrow{\phi} Y$ be a distal homomorphism. Then ϕ has a* RIM.

Proof. By the Furstenberg-Ellis structure theorem for quasi-separable distal extensions, [1], there exist an ordinal η, a family of flows $\{X_\alpha \mid \alpha \leqq \eta\}$ and a family of homomorphism $\{X_{\alpha+1} \xrightarrow{\phi_\alpha} X_\alpha \mid \alpha < \eta\}$ such that $X_0 = Y, X_\eta = X, \phi_\alpha$ is an almost periodic extension and for limit ordinals $\beta \leqq \eta$ X_β is the invers limit of the system $\{X_\alpha, \phi_\alpha \mid \alpha < \beta\}$. Using Corollary 3.7 and Lemma 3.5 (2), and using the fact that for a limit ordinal $\beta \leqq \eta$ the union of the images of $\mathscr{C}(X_\alpha)$ in $\mathscr{C}(X_\beta)$ $(\alpha < \beta)$ is dense in $\mathscr{C}(X_\beta)$, one constructs inductively a RIM for ϕ.

REMARKS. (1) Since there exist distal flows which are not uniquely ergodic [3], it is clear that a RIM for a distal extension is not necessarily unique.

(2) Let $X \xrightarrow{\phi} Y$ be a distal homomorphism, is it true that for every minimal set $\mathcal{N}_0 \subseteq \hat{\phi}^{-1}(Y) \subseteq \mathcal{M}(X)$ the homomorphism $\mathcal{N}_0 \xrightarrow{\hat{\phi}} Y$ is distal? If this is true, and it can be proved without the use of the structure theorem, then the existence of a RIM will follow as in the proof of Corollary 3.7. In particular when Y is the trivial flow, a proof of the fact that a minimal set in $\mathcal{M}(X)$ is distal whenever X is distal will produce a new proof for the existence of an invariant measure for a distal flow. (In [9] there is an example of a distal flow on the torus X, such that $\mathcal{M}(X)$ is not distal.)

PROPOSITION 3.9. *Let $X \xrightarrow{\phi} Y$ be a* RIM *extension. If for some section $\lambda: Y \to \mathcal{M}(X)$, and some point $y_0 \in Y$,* $\mathrm{Supp}(\lambda_{y_0})$ *is a finite set, then ϕ is a finite to one almost periodic extension. In particular if for some $y_0 \in Y, \phi^{-1}(y_0)$ is finite then, ϕ is finite to one everywhere and is almost periodic.*

Proof. We show that ϕ is a finite to one distal extension and this implies that ϕ is almost periodic.

Let $\lambda_{y_0} = \Sigma_{l=1}^{n} a_l x_l$ where $0 < a_l < 1$, $\Sigma_{l=1}^{n} a_l = 1$, $x_l \in X$ satisfy $\phi(x_l) = y_0$ and if $k \neq l$ then $x_l \neq x_k$. Since λ is a section it follows that

$t\lambda_{y_0} = \lambda_{ty_0} = \Sigma_{l=1}^{n} a_l t x_l$. If t_α is a net in T then clearly $\lim t_\alpha \lambda_{y_0}$ exists and is equal to λ_y iff, in X^n, $\lim(t_\alpha x_1, \cdots, t_\alpha x_n) = (z_1, \cdots, z_n)$ exists and $\lambda_y = \Sigma_{l=1}^{n} a_l z_l$. Thus if we denote $Z = \mathrm{cls}\{(tx_1, \cdots, tx_n) \mid t \in T\}$ then Z as a sub-flow of X^n is isomorphic to Y via the map $y \to \lambda_y \to (z_1, \cdots, z_n)$ where $\lambda_y = \Sigma_{l=1}^{n} a_l z_l$ (If some of the $a_l - s$ are equal we can reorder them, if necessary, so that $z_l = \lim t_\alpha x_l$ whenever $\lim t_\alpha y_0 = y$.)

Since Y is a minimal flow we conclude that for each $y \in Y$, the point $(z_1, \cdots, z_n)$ of X^n where, $\lambda_y = \Sigma_{l=1}^{n} a_l z_l$, is an almost periodic point. In particular if $i \neq j$ then z_i and z_j are not proximal. Now $\cup\{z_i \mid z_i$ is some coordinate of some $z \in Z\} = \cup\{\mathrm{Supp}(\lambda_y) \mid y \in Y\}$ is clearly a closed invariant sub-set of X. Since X is minimal this set is equal to X and we can conclude that for each $y \in Y$ with $\lambda_y = \Sigma_{l=1} a_l z_l$, $\phi^{-1}(y) = \{z_1, \cdots, z_n\}$. Thus ϕ is a finite to one distal homomorphism. The proof is completed.

In a similar way one can show that a homomorphism $X \xrightarrow{\phi} Y$ of minimal flows, which is finite to one and open, is almost-periodic. Thus for a finite to one homomorphism of minimal flows the properties of having a RIM, of being open and of being almost-periodic are equivalent. Using this fact and a construction due to W. A. Veech **[12]** one can deduce the following proposition, which is probably well known.

PROPOSITION 3.10. *Let X be a minimal metric flow and $X \xrightarrow{\phi} Y$ a homomorphism. Suppose there exists a point $y_0 \in Y$ such that $\phi^{-1}(y_0)$ is a finite set. Then, either ϕ is almost $1-1$ or, there exists a commutative diagram*

$$\begin{tikzcd} X^* \arrow[r, "\theta^*"] \arrow[d, "\phi^*"'] & X \arrow[d, "\phi"] \\ Y^* \arrow[r, "\theta"'] & Y \end{tikzcd}$$

such that X^* is a minimal flow, θ and θ^* are almost $1-1$ extensions and θ^* is a nontrivial almost-periodic extension.

4. How to obtain a homomorphism with a RIM from a given homomorphism. Let $X \xrightarrow{\phi} Y$ be a homomorphism of minimal flows. By Proposition 2.2 we can find an $\mathcal{M}(Y)$-irreducible affine sub-flow Q in $\mathcal{M}(X)$, and then the map $\hat{\phi} \mid \overline{\mathrm{ex}}(Q)$: $\overline{\mathrm{ex}}(Q) \to Y$ is a strongly proximal homomorphism. Denote $\overline{\mathrm{ex}}(Q) = Y^\sim$ and $\hat{\phi} \mid Y^\sim = \theta$. Consider now the set $R \subseteq X \times Y^\sim$ defined by $R = \{(x, \nu) \mid \phi(x) = \theta(\nu)\}$. We have the following theorem.

THEOREM 4.1.

(1) *R contains a unique minimal set, $X^{\sim}$.*

(2) *The following diagram is connutative*

$$\begin{tikzcd} X^{\sim} \arrow[r, "\theta^{\sim}"] \arrow[d, "\phi^{\sim}"'] & X \arrow[d, "\phi"] \\ Y^{\sim} \arrow[r, "\theta"'] & Y \end{tikzcd}$$

Where $\theta^{\sim}$ and $\phi^{\sim}$ are the projections of $X^{\sim}$ onto X and $Y^{\sim}$ respectively.

(3) *θ and $\theta^{\sim}$ are strongly proximal homomorphisms.*

(4) *$\phi^{\sim}$ has a* RIM. *In fact for each $\nu \in Y^{\sim}$ the measure $\nu \times \delta_\nu$ on $X \times Y^{\sim}$ is supported in $X^{\sim}$ and the map $\lambda: Y^{\sim} \to \mathcal{M}(X^{\sim})$: $\lambda_\nu = \nu \times \delta_\nu$ is a section for $\phi^{\sim}$.*

Proof. (1) Let $X^{\sim}$ be an arbitrary minimal set in R, we shall prove that statements (2)-(4) holds for this particular choice of a minimal set and then it will follow from (4) that $X^{\sim}$ is the unique minimal set in R.

(2) Let $(x, \nu) \in X^{\sim}$ then $(\phi \circ \theta^{\sim})((x, \nu)) = \phi(x)$ and $(\theta \circ \phi^{\sim})((x, \nu)) = \theta(\nu)$. Since $X^{\sim} \subseteq R$, $\phi(x) = \theta(\nu)$ and $\phi \circ \theta^{\sim} = \theta \circ \phi^{\sim}$.

(3) We know already that θ is a strongly proximal homomorphism. Let ζ be a measure in $\mathcal{M}(X^{\sim})$ whose support is contained in a set of the form $(\theta^{\sim})^{-1}(x) = \{(x, \nu) \mid \nu \in Y^{\sim} \text{ and } (x, \nu) \in X^{\sim}\}$ for some $x \in X$. Then $(\phi^{\sim})^{\wedge}(\zeta)$ is supported in the set $\{\nu \in Y^{\sim} \mid (x, \nu) \in X^{\sim}\}$. But for $(x, \nu) \in X^{\sim}$, $\theta(\nu) = \phi(x)$ and thus $\operatorname{Supp}(\phi^{\sim})^{\wedge}(\zeta) \subseteq \theta^{-1}(\phi(x))$.

Now θ is a strongly proximal extension, hence there exists a net t_i in T such that $\lim t_i(\phi^{\sim})^{\wedge}(\zeta) = \delta_\nu \in \mathcal{M}(Y^{\sim})$, for some $\nu \in Y^{\sim}$. Since $\zeta = \delta_x \times (\phi^{\sim})^{\wedge}(\zeta)$ it is now clear that for a sub-net t_{i_j} of t_i, $\lim t_{i_j}\, \zeta$ is a point mass on $X^{\sim}$. Thus $\theta^{\sim}$ is also a strongly proximal homomorphism.

(4) The commutative diagram of minimal flows in (2), induces a commutative diagram of affine flows

$$\begin{tikzcd} \mathcal{M}(X^{\sim}) \arrow[r, "(\theta^{\sim})^{\wedge}"] \arrow[d, "(\phi^{\sim})^{\wedge}"'] & \mathcal{M}(X) \arrow[d, "\hat{\phi}"] \\ \mathcal{M}(Y^{\sim}) \arrow[r, "\hat{\theta}"'] & \mathcal{M}(Y). \end{tikzcd}$$

We recall that $Q = \overline{\mathrm{co}}(Y^{\sim}) \subseteq \mathcal{M}(X)$ is an $\mathcal{M}(Y)$-irreducible sub-set of $\mathcal{M}(X)$.

Denote by $Q^{\sim}$ the affine sub-flow, $((\theta^{\sim})^{\wedge})^{-1}(Q)$ of $\mathcal{M}(X^{\sim})$, then

$$\hat{\theta}((\phi^{\sim})^{\wedge}(Q^{\sim})) = \hat{\phi}(\theta^{\sim})^{\wedge}(Q^{\sim}) = \hat{\phi}(Q) = \mathcal{M}(Y).$$

But θ is a strongly proximal extension, and by Corollary 2.3, $\mathcal{M}(Y^{\sim})$ is $\mathcal{M}(Y)$-irreducible, hence $(\phi^{\sim})^{\wedge}(Q^{\sim}) = \mathcal{M}(Y^{\sim})$. Let $\nu \in Y^{\sim}$ be fixed and choose $\zeta \in Q^{\sim}$ such that $(\phi^{\sim})^{\wedge}(\zeta) = \delta_\nu \in \mathcal{M}(Y^{\sim})$. This implies that $\zeta = (\theta^{\sim})^{\wedge}(\zeta) \times \delta_\nu$. Denote $(\theta^{\sim})^{\wedge}(\zeta) = \eta$ and note that $\eta \in Q$.

By the commutativity of the above diagram $\hat{\phi}(\nu) = \theta(\nu) = \hat{\theta}(\delta_\nu) = (\hat{\theta} \circ (\phi^{\sim})^{\wedge})(\zeta) = (\hat{\phi} \circ (\theta^{\sim})^{\wedge})(\zeta) = \hat{\phi}(\eta)$, but $\nu \in Y^{\sim}$ hence $\hat{\phi}(\nu) = \hat{\phi}(\eta) =$ a point mass on Y.

It now follows from Theorem 2.1. (4) that ν and η are proximal points of Q. Therefore there exists a net t_i in T such that $\lim t_i\nu = \lim t_i\eta$. Since $Y^{\sim}$ is a minimal set and $\nu \in Y^{\sim}$, the common limit lies in $Y^{\sim}$ and we can assume that it is actually equal to ν. Now

$$\lim t_i\zeta = \lim t_i(\eta \times \delta_\nu) = \nu \times \delta_\nu \in Q^{\sim} \subseteq \mathcal{M}(X^{\sim}).$$

Thus for every $\nu \in Y^{\sim}$ the product measure $\nu \times \delta_\nu$ is supported in $X^{\sim}$, and it is now clear that the map $\lambda : Y^{\sim} \to \mathcal{M}(X^{\sim})$ defined by $\lambda_\nu = \nu \times \delta_\nu$ is a section for $\phi^{\sim}$. This completes the proof of Theorem 4.1.

Consider now a minimal flow Y and let M be the universal minimal flow. There exists a homomorphism $M \xrightarrow{\phi} Y$. Using the construction of Theorem 4.1 (with $X = M$) we obtain a commutative diagram

$$\begin{tikzcd}
 & M \arrow[dl, "\phi^{\sim}"'] \arrow[d, "\phi"] \\
Y^{\sim} \arrow[r, "\theta"'] & Y
\end{tikzcd}$$

where $Y^{\sim} \subseteq \mathcal{M}(M)$, θ is strongly proximal and $\phi^{\sim}$ has a RIM. (By the universality of $M, \theta^{\sim}$ is $1-1$).

If $Y' \xrightarrow{\psi} Y$ is a strongly proximal homomorphism then there exists a homomorphism η, such that the diagram

$$\begin{tikzcd} & M \arrow[dl, "\phi^{\sim}"'] \arrow[dr, "\eta"] \arrow[dd, "\phi"] & \\ Y^{\sim} \arrow[dr, "\theta"'] & & Y' \arrow[dl, "\psi"] \\ & Y & \end{tikzcd}$$

is commutative.

Let $Q = \overline{\mathrm{co}}(Y^{\sim}) \subseteq \mathcal{M}(M)$, $Q' = \hat{\eta}(Q)$; then $Q' \subseteq \mathcal{M}(Y')$ and since $\hat{\phi} = \hat{\psi} \circ \hat{\eta}$ we have $\hat{\psi}(Q') = \hat{\psi} \circ \hat{\eta}(Q) = \hat{\phi}(Q) = \mathcal{M}(Y)$. But ψ is strongly proximal, hence $\mathcal{M}(Y')$ is $\mathcal{M}(Y)$-irreducible (Corollary 2.2) and $Q' = \mathcal{M}(Y')$. In particular $\hat{\eta}(Y^{\sim}) = Y'$ and $\psi \circ (\hat{\eta} \mid Y^{\sim}) = \hat{\psi} \circ (\hat{\eta} \mid Y^{\sim}) = \hat{\theta} \circ ((\phi^{\sim})^{\wedge} \mid Y^{\sim}) = \theta \circ ((\phi^{\sim})^{\wedge} \mid Y^{\sim})$. Now the map $\lambda \colon Y^{\sim} \to \mathcal{M}(M)$: $\lambda_{\nu} = \nu$ is a section and $(\phi^{\sim})^{\wedge} \mid Y^{\sim} = \lambda^{-1}$. Thus for $\nu \in Y^{\sim} \subseteq \mathcal{M}(M)$, $(\phi^{\sim})^{\wedge}(\nu) = \delta_{\nu} \in \mathcal{M}(Y^{\sim})$ and $\psi \circ (\hat{\eta} \mid Y^{\sim}) = \theta$. This shows that $Y^{\sim}$ is a universal strongly proximal extension of Y.

If $Y' \xrightarrow{\text{s.p.}} Y$, is another universal strongly proximal extension of Y, then there exist two homomorphism η and ι such that the diagram

$$\begin{tikzcd} Y^{\sim} \arrow[rr, shift left, "\eta"] \arrow[dr, "\psi"'] & & Y' \arrow[ll, shift left, "\iota"] \arrow[dl, "\theta"] \\ & Y & \end{tikzcd}$$

is commutative.

Now for $y \in Y^{\sim}$, $\psi((\iota \circ \eta)(y)) = (\theta \circ \eta)(y) = \psi(y)$. Hence $(\iota \circ \eta)(y)$ and y are proximal points. But $\iota \circ \eta$ is an automorphism of the flow $Y^{\sim}$, hence this is possible only if $(\iota \circ \eta)(y) = y$ i.e. $\iota \circ \eta =$ identity. This shows that $Y^{\sim}$ is unique up to an isomorphism.

Incidentally this shows that for every minimal flow Y and a homomorphism $M \xrightarrow{\text{s.p.}} Y$, any two $\mathcal{M}(Y)$-irreducible affine sub-flows of $\mathcal{M}(M)$ are affinely isomorphic.

In particular when Y is the trivial flow $Y^{\sim} = \Pi_S$, the universal minimal strongly proximal flow (see [6]), and every irreducible affine sub-flow of $\mathcal{M}(M)$ is affinely isomorphic to $\mathcal{M}(\Pi_S)$.

The flow $Y^{\sim}$ has the property that every homomorphism $Z \xrightarrow{\psi} Y^{\sim}$ where Z is minimal, has a RIM. (Lemma 3.5 (1)).

The observation that any two $\mathcal{M}(Y)$ irreducible affine sub-flows of $\mathcal{M}(M)$ are isomorphic raises the following question. Given a homomorphism $X \xrightarrow{\phi} Y$ of minimal flows, is it true that any two $\mathcal{M}(Y)$-irreducible affine sub-flows of $\mathcal{M}(X)$ are affinely isomorphic? Taking Y to be the trivial flow the question is whether any two irreducible affine sub-flows of $\mathcal{M}(X)$ are necessarifiy isomorphic. In particular if X is a minimal flow with an invariant measure is it true that every irreducible affine sub-flow of $\mathcal{M}(X)$ is trivial?

A particular case in which the answer to the above questions is clearly affirmative, is the case in which there is a unique $\mathcal{M}(Y)$-irreducible affine sub-flow of $\mathcal{M}(X)$. This is the case iff in the construction of Theorem 4.1 the homomorphism $X^{\sim} \xrightarrow{\phi} Y^{\sim}$ has a unique RIM.

The following example of a minimal flow X such that $\mathcal{M}(X)$ contains a unique irreducible affine sub-flow (which is an invariant measure), is due to Professor H. Furstenberg.

Let G be a semi-simple connected Lie group with finite center, $G = KAN$ an Iwasawa decomposition for G and let M be the centralizer of A in K. Then $H = MAN$ is a closed amenable sub-group of G. Theorem 2.6 of **[4]** states that the action of H on any homogeneous space of G is uniquely ergodic. Let Γ be a discrete uniform sub-group of G and let $Q \subseteq \mathcal{M}(G/\Gamma)$, be an affine G-invariant irreducible sub-flow of $\mathcal{M}(G/\Gamma)$. In particular Q is H-invariant, and since H is amenable the unique H-invariant measure on G/Γ lies in Q. Thus Q is unique. Since G/Γ carries a unique G-invariant measure, m, it follows that $Q = \{m\}$.

This example can be generalized as follows

PROPOSITION 4.2. *Let (T,X) be a minimal flow. Suppose there exists a sub-group S of T which is amenable and such that (S,X) is a uniquely ergodic flow. Then, $\mathcal{M}(X)$ contains a unique T-invariant affine irreducible sub-flow.*

We conclude with the following question which, in fact, is the reason for our interest in relatively invariant measures.

Generalizing results of H. Furstenberg in **[5]** and H. Keynes and J. B. Robertson in **[7]**, R. Peleg proved the following theorem **[10**, Theorem 11, where Y is the trivial flow].

THEOREM. *Let (X,T) be a minimal metric flow with an invariant measure. Then X is topologically weakly mixing* (i.e. $X \times X$ *is* *topologicaly ergodic), iff the only almost periodic factor of X is the trivial one.*

We state the following conjecture.

CONJECTURE. Let $X \xrightarrow{\phi} Y$ be a RIM-extension and suppose X is metric. Let R be the sub-set of $X \times X$, defined by

$$R = \{(x_1, x_2) \mid \phi(x_1) = \phi(x_2)\}.$$

Then R is topologically ergodic iff the only almost periodic extension of Y, which is a factor of X, is Y itself.

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