PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

EXTENSION FUNCTIONS FOR RANK 2, TORSION FREE ABELIAN GROUPS

ERIC M. FRIEDLANDER

The set of isomorphism classes of rank 2, torsion free abelian groups with a pure subgroup isomorphic to a given rank 1 group is shown to be in natural 1−1 correspondence with the set of pairs consisting of a quotient type and a type of an extension function. In terms of these invariants, necessary and sufficient conditions are determined for such a group to be homogeneous or to admit a pure cyclic subgroup. Moreover, this 1−1 correspondence has an explicit inverse, so that examples are readily obtained.

Our method is to combine the Korosh-Malcev-Derry matrix classification (see [2] for this and other well known aspects of the theory of abelian groups which we employ) with some elementary observations converning abelian extensions of rank 1, torsion free abelian groups. We begin §1 by identifying $\mathrm{Ext}_Z(X, Y)$ for rank 1, torsion free abelian groups X and Y in terms of "extension functions." A Korosh-Malcev-Derry matrix sequence for the total group of such an extension is readily given in terms of an extension function. Moreover, an extension function explicitly determines a subgroup of $Q \oplus Q$. We obtain an explicit necessary condition for two extension functions to determine isomorphic total groups, as well as express the Korosh-Malcev-Derry matrix conditions in terms of extension functions.

In §2, we explicate the $1-1$ correspondence asserted above. We then "list" all homogeneous, rank 2, torsion free abelian groups of a given type. We also determine in terms of our invariants whether or not a rank 2, torsion free abelian group admits a pure cyclic subgroup.

We gratefully acknowledge many helpful conversations with C. Miller. Moreover, the referee's suggestion of generalizing an eariler version of this work proved most valuable. We refer the interested reader to ([1]) for a detailed study of rank 2, torsion free abelian groups up to quasi-isomorphism and to ([2]) for a survey of the literature on torsion free abelian groups.

1. Extension functions. A sequence $(a_1, \cdots, a_n, \cdots)$ with each a_i an extended nonnegative integer, $0 \leqq a_i \leqq \infty$, is called a characteristic. Two characteristics, $(a_1, \cdots, a_n, \cdots)$ and $(b_1, \cdots, b_n, \cdots)$, are said to have the same type if and only if $\Sigma_i (a_i - b_i)^2 < \infty$. If A is a torsion free abelian group, then the characteristic of any nonzero element x in A, $\mathrm{char}(x) = (a_1, \cdots, a_n, \cdots)$, is defined by

$a_i = \mathrm{char}_i(x) = \sup\{k : p_i^{-k}x$ in $A\}$, where p_i denotes the ith prime. Two torsion free, rank 1 abelian groups, X and Y, are isomorphic if and only if the type of $\mathrm{char}(x)$ equals the type of $\mathrm{char}(y)$ for any nonzero elements x and y of X and Y respectively.

We first explicitly compute $\mathrm{Ext}^1_{\mathbf{Z}}(X, Y)$ for any pair of rank 1, torsion free abelian groups X and Y.

PROPOSITION 1.1. *Let X and Y be rank* 1, *torsion free abelian groups and let x in X and y in Y be nonzero elements. Let* $(a_1, \cdots, a_n, \cdots) = \mathrm{char}(x)$ *and* $(b_1, \cdots, b_n, \cdots) = \mathrm{char}(y)$. *Then there exists an isomorphism*

$$\varphi(X, x; Y, y) \colon \mathrm{Ext}_{\mathbf{Z}}(X, Y) \to \mathrm{coker}(\theta_y \colon Y \to \Pi' \mathbf{Z}/p_i^{a_i})$$

where Π' denotes the product over all i with $b_i < \infty$, where $\mathbf{Z}/p_i^{a_i}$ denotes the p_i-adic integers $\mathbf{Z}_{p_i}$ if $a_i = \infty$, and where $\theta_y(m/n \cdot y) = (m/n \cdot p_1^{b_1}, \cdots, m/n \cdot p_n^{b_n}, \cdots)$. *Moreover,* $\varphi(X, x; Y, y)$ *is natural for maps* $(X', x') \to (X, x)$ *and maps* $(Y', y') \to (Y, y)$.

Proof. Let $Y \to Q$ be defined by sending y to 1. Since Q and Q/Y are divisible abelian groups, $\mathrm{Ext}_{\mathbf{Z}}(X, Y)$ is naturally isomorphic to $\mathrm{coker}(\mathrm{Hom}(X, Q) \to \mathrm{Hom}(X, Q/Y))$. Applying the serpent lemma to the following map of short exact sequences

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}(X/\mathbf{Z}\cdot x, Q) & \to & \mathrm{Hom}(X, Q) & \to & \mathrm{Hom}(\mathbf{Z}\cdot x, Q) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Hom}(X/\mathbf{Z}\cdot x, Q/Y) & \to & \mathrm{Hom}(X, Q/Y) & \to & \mathrm{Hom}(\mathbf{Z}\cdot x, Q/Y) & \to 0
\end{array}$$

we conclude that $\mathrm{Ext}_{\mathbf{Z}}(X, Y) \xrightarrow{\sim} \mathrm{coker}(Y \to \mathrm{Hom}(X/\mathbf{Z}\cdot x, Q/Y))$.

Let $\mu(p, a)$ denote the cyclic subgroup of $Q/\mathbf{Z}$ generated by $1/p^a$ (let $\mu(p, \infty) = \lim_n \mu(p, n)$). Then

$$X/\mathbf{Z}\cdot x \xrightarrow{\sim} \bigoplus \mu(p_i, a_i) \qquad \text{and} \qquad Q/Y \xrightarrow{\sim} \bigoplus{}' \mu(p_i, \infty)$$

where the latter sum is taken over all i with $b_i < \infty$. Thus

$$\mathrm{Hom}(X/\mathbf{Z}\cdot x, Q/Y) \xrightarrow{\sim} \mathrm{Hom}(\bigoplus \mu(p_i, a_i), \bigoplus{}' \mu(p_i, \infty)) \xrightarrow{\sim} \Pi' \mathbf{Z}/p_i^{a_i}.$$

The map $Y \to \mathrm{Hom}(X/\mathbf{Z}\cdot x, Q/Y)$ is easily checked to send y to the sequence $\{p_i^{b_i}\}$ under these identifications.

To check the naturality of $\varphi(X, x; Y, y)$ for a map $f \colon (X', x') \to (X, x)$, one must verify the commutativity of the following square:

$$\begin{array}{ccc}
\mathrm{Ext}_{\mathbf{Z}}(X, Y) & \xrightarrow{\varphi(X,x;Y,y)} & \mathrm{coker}(\theta_y\colon Y \to \Pi'\mathbf{Z}/p_i^{a_i}) \\
\downarrow \mathrm{Ext}_{\mathbf{Z}}(f, \mathbf{Z}) & & \downarrow \pi \\
\mathrm{Ext}_{z}(X', Y) & \xrightarrow{\varphi(Z',x';Y,y)} & \mathrm{coker}(\theta_y\colon Y \to \Pi'\mathbf{Z}/p_i^{a_i})
\end{array}$$

where π is induced by the factor by factor projection map. This verification, and the corresponding verification for naturality with respect to maps $g\colon (Y', y') \to (Y, y)$, are routine.

For a given (Y, y), we call f in $\Pi'\mathbf{Z}/p_i^{a_i}$ an "extension function" for the corresponding extension $\varphi(X, x; Y, y)^{-1}(\bar{f})$ in $\mathrm{Ext}_{\mathbf{Z}}(X, Y)$, where $\bar{f}$ is the image of f in $\mathrm{coker}(\theta_y)$ and (X, x) is any torsion free, rank 1 abelian group X with nonzero element x satisfying $\mathrm{char}(x) = (a_1, \cdots a_n, \cdots)$.

Observe that in the particular case that Y is free and y is a generator, Proposition 1.1 asserts that

$$\mathrm{Ext}_{\mathbf{Z}}(X, \mathbf{Z}) \to (\Pi\mathbf{Z}/p_i^{a_i})/\Delta(\mathbf{Z})$$

so that two extension functions determine isomorphic extensions if and only if they differ by a constant.

We recall that if A is a finite rank, torsion free abelian group, then $\Lambda \otimes \mathbf{Z}_p$ is a direct sum of $\mathbf{Z}_p$'s and Q_p's for every prime p, since $\mathrm{Ext}^1_{\mathbf{Z}}(Q_p, \mathbf{Z}_p) = 0$ (where $\mathbf{Z}_p$ denotes the p-adic integers and Q_p denotes the field of p-adic numbers). We say that $\{x_1, \cdots, x_n\}$ in $A \otimes \mathbf{Z}_p$ is a "basis" if $A \otimes \mathbf{Z}_p$ is the internal direct sum of the pure $\mathbf{Z}_p$ modules generated by the x_i's:

$$A \otimes \mathbf{Z}_p \xrightarrow{\sim} \mathbf{Z}_p \cdot x_{i_1} \oplus \cdots \oplus \mathbf{Z}_p \cdot x_{i_k} \oplus Q_p \cdot x_{i_{k+1}} \oplus \cdots \oplus Q_p \cdot x_{i_n}$$

A "matrix sequence" for a finite rank, torsion free abelian group A is a sequence of matrices $\{M_i\}$ expressing a given basis for $A \otimes Q$ in terms of bases for $A \otimes \mathbf{Z}_{p_i}$ thus, if A is rank 2, $\left\{M_i\begin{pmatrix}1\\0\end{pmatrix}, M_i\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is the given basis for $A \otimes Q$ expressed in terms of a basis for $A \otimes \mathbf{Z}_{p_i}$. Such a matrix sequence determines the isomorphism class of A.

In the following proposition, we determine matrix sequences in terms of extension functions.

PROPOSITION 1.2. *Let X and Y be rank* 1, *torsion free abelian groups, let x in X and y in Y be nonzero elements, and let* $\mathrm{char}(x) =$

$(a_1, \cdots, a_n, \cdots)$ *and* $\mathrm{char}(y) = (b_1, \cdots, b_n, \cdots)$. *For a given extension function* f *in* $\Pi' \mathbf{Z}/p_i^{a_i}$, *let*

$$0 \to Y \to A_f \to X \to 0$$

be the extension $\varphi(X, x; Y, y)^{-1}(\bar{f})$ *in* $\mathrm{Ext}^1_{\mathbf{Z}}(X, Y)$. *Then* A_f *has a matrix sequence*

$$\left\{ \begin{pmatrix} p_i^{b_i} & -f_i \\ 0 & p_i^{a_i} \end{pmatrix} \right\}$$

where $0 \leqq f_i < p_i^{a_i}$ *with* $f(i) = f_i$ *in* $\mathbf{Z}/p_i^{a_i}$ *if* $a_i, b_i < \infty$, *where* $f_i = 0$ *and* $p_i^{b_i}$ *designates* 1 *if* $b_i = \infty$, *and where* $f_i = f(i)$ *in* $\mathbf{Z}_{p_i}$ *and* $p_i^{a_i}$ *designates* 1 *if* $a_i = \infty$.

Proof. The extension $\varphi(X, x; Y, y)^{-1}(\bar{f})$ is obtained by pull-back from $0 \to Y \to Q \to Q/Y \to 0$ via the composition $X \to X/\mathbf{Z} \cdot x \xrightarrow{f} Q/Y$. For notational convenience, we view $A_f = Q \times_{Q/Y} X$ as a subgroup of $Q \oplus X$, containing $(1, 0) = y$ and $(0, x) = x$.

For each i with $b_i = \infty$, $\{y, p_i^{-a_i} y\}$ in A_f is a basis for $A_f \otimes \mathbf{Z}_{p_i}$. For each i with $b_i, a_i < \infty$, $\{p_i^{-b_i} y, f_i p_i^{-a_i - b_i} y + p_i^{-a_i} y\}$ in A_f is a basis for $A_f \otimes \mathbf{Z}_{p_i}$. For each i with $b_i < \infty$ and $a_i = \infty$ $f_{i,k} p_i^{-k-b_i} y + p_i^{-k} y$ is in A_f for all $k > 0$ where $f_i \equiv f_{i,k} \pmod{p_i^k}$, so that $\{p_i^{-b_i} y, f_i p_i^{-b_i} y + x\}$ is a basis for $A_f \otimes \mathbf{Z}_{p_i}$. Then $\begin{pmatrix} p_i^{b_i} & -f_i \\ 0 & p_i^{a_i} \end{pmatrix}$ is the matrix expressing the basis $\{y, x\}$ of $A_f \otimes Q$ in terms of the above basis for $A_f \otimes \mathbf{Z}_{p_i}$.

We next determine in terms of generators a standard model for the rank 2, torsion free abelian group A_f defined by the extension function f.

PROPOSITION 1.3. *Let* X *and* Y *be rank* 1, *torsion free abelian groups, let* x *in* X *and* y *in* Y *be nonzero elements, and let* $\mathrm{char}(x) = (a_1, \cdots, a_n, \cdots)$ *and* $\mathrm{char}(y) = (b_1, \cdots, b_n, \cdots)$. *For a given extension function* f *in* $\Pi' \mathbf{Z}/p_i^{a_i}$, *the map*

$$i(y, x)\colon A_f = Q \underset{Q/Y}{\times} X \to Q \oplus Q$$

sending y *to* $(1, 0)$ *and* x *to* $(0, 1)$ *is an isomorphism onto that subgroup* $A(f)$ *of* $Q \oplus Q$ *generated by* $(p_i^{b_i}, 0)$, $(f_i p_i^{-a_i - b_i}, p_i^{-a_i})$ *whenever* $a_i, b_i < \infty$; $(p_i^{-k}, 0)$, $(0, p_i^{-a_i})$ *for all* $k > 0$, *whenever* $a_i < \infty$, $b_i = \infty$; $(p_i^{-k}, 0)$, $(0, p_i^{-k})$ *for all* $k > 0$, *whenever* $a_i = \infty = b_i$. $(p_i^{-b_i}, 0)$, $(f_{i,k} p_i^{-k-b_i}, p_i^{-k})$ *where* $f_{i,k}$ *in* $\mathbf{Z}$ *satisfies* $f_{i,k} \equiv f_i \pmod{p_i^k}$ *for all* $k > 0$, *whenever* $a_i = \infty$, $b_i < \infty$.

Proof. Since A_f is torsion free and $\{y, x\}$ is a basis for $A_f \otimes Q$, $i(y, x)$ is an isomorphism of A_f onto its image $i(A_f)$. As checked in the proof of Proposition 1.2, $A(f) \subset i(A_f)$. Moreover, one readily checks that char(1, 0)) in $A(f)$ equals $(b_1, \cdots, b_n, \cdots)$ by determining the p-divisibility of (1, 0) in $A(f) \otimes \mathbf{Z}_p$ for all primes p. Therefore, $A(f) \cap (Q \otimes 0) = i(Y)$. Furthermore, the inclusion $A(f) \subset i(A_f)$ determines $A(f)/A(f) \cap (Q \oplus 0) \subset i(X)$ with $\overline{(0,1)} = i(y,x)(x)$. This is also an isomorphism, since char($\overline{(0,1)}$)) in $A(f)/A(f) \cap (Q \oplus 0)$ equals char(x) in X. Thus, $A(f)$ equals $i(A_f)$.

By employing the matrix classification for rank 2, torsion free abelian groups, we give below necessary and sufficient conditions for two extension functions f and g to determine isomorphic total groups A_f and A_g. Because of the very ineffective nature of the Korosh-Malcev-Derry matrix classification, the significance of Proposition 1.4 is the explicit necessary condition it provides.

PROPOSITION 1.4. *Let X and Y be rank 1, torsion free abelian grroups, let x in X and y in Y be nonzero elements, and let* char$(x) = (a_1, \cdots, a_n, \cdots)$ *and* char$(y) = (b_1, \cdots, b_n, \cdots)$. *Two extension functions f, g in $\Pi' \mathbf{Z}/p_i^{a_i}$ determine isomorphic total groups A_f and A_g if and only if there exists a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in* GL$(Q \oplus Q)$ *and matrices $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ in* GL$(Y \otimes \mathbf{Z}_{p_i} \oplus X \oplus \mathbf{Z}_{p_i})$ *satisfying*

$$(1.4.1)_i \qquad \begin{pmatrix} p_i^{b_i} & -f_i \\ 0 & p_i^{a_i} \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \cdot \begin{pmatrix} p_i^{b_i} & -g_i \\ 0 & p_i^{a_i} \end{pmatrix}$$

for all i. In particular, if A_f is isomorphic to A_g, then there exists $\alpha, \beta, \gamma, \delta$ in Q satisfying $\alpha\delta - \beta\gamma \neq 0$ such that f and g satisfy the following congruence condition for every i with $b_i < \infty$:

$$(1.4.2)_i \qquad p_i^{b_i} \cdot \beta - f_i \cdot \delta \equiv g_i(f_i p_i^{-b_i}\gamma - \alpha) \qquad (\mathrm{mod}\ p_i^{a_i})$$

Proof. The conditions $(1.4.1)_i$ express the relationship between matrix sequences for bases $\{x, y\}$ and $\{x', y'\}$ of $A_f \otimes Q \simeq Q \oplus Q \simeq A_g \otimes Q$ related by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in GL$(Q \oplus Q)$ in terms of bases $\{x_i, y_i\}$ and $\{x'_i, y'_i\}$ of $Y \otimes \mathbf{Z}_{p_i} \oplus X \otimes \mathbf{Z}_{p_i}$ related by $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ in GL$(Y \otimes \mathbf{Z}_{p_i} \oplus X \otimes \mathbf{Z}_{p_i})$. The necessity of conditions $(1.4.1)_i$ is clear; their sufficiency is well known (see [2]). If $b_i < \infty$ and $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ is in GL$(Y \otimes \mathbf{Z}_{p_i} \oplus X \otimes \mathbf{Z}_{p_i})$ then β_i is in $\mathbf{Z}_{p_i}$. With this in mind, conditions $(1.4.2)_i$ quickly follow from $(1.4.1)_i$.

Since the 0 function, $g = 0$, determines $A_g \simeq Y \oplus X$, conditions $(1.4.2)_i$ determine useful necessary conditions an extension function f must satisfy for A_f to be isomophic to $Y \oplus X$.

The following definition formulizes the relationship between extension functions f and g which determine isomorphic total groups A_f and A_g.

DEFINITION 1.5. Let characteristics $(a_1, \cdots, a_n, \cdots)$ and $(b_1, \cdots, b_n, \cdots)$ be given and let f, g be elements of $\Pi' \mathbf{Z}/p_i^{a_i}$ where Π' is the product over all i with $b_i < \infty$. Then f and g are said to be of the same type with respect to $(b_1, \cdots, b_n, \cdots)$ if there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathrm{GL}(Q \oplus Q)$ and $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ in $\mathrm{GL}(p_i^{-b_i}\mathbf{Z}_{p_i} \oplus p_i^{-a_i}\mathbf{Z}_{p_i})$ such that each of the equations $(1.4.1)_i$ holds, where $p_i^{-m}\mathbf{Z}_{p_i}$ is either $\mathbf{Z}_{p_i}$ or Q_{p_i} depending on whether $m < \infty$ or $m = \infty$.

2. Applications. As a first step toward classifying those rank 2, torsion free abelian groups A admitting a pure subgroup isomorphic to a given rak 1 group Y, we verify that the isomorphism class of A/H is independent of the choice of pure subgroup H isomorphic to Y. The proof of the following proposition was given to us by the referee.

PROPOSITION 2.1. *Let Y be a rank 1, torsion free abelian group, and let A be a rank 2, torsion free abelian group. For any two pure subgroups H, H' of A isomorphic to Y, A/H is isomorphic to A/H'.*

Proof. Let x and x' be nonzero elements of H and H' respectively. If there exists nonzero m/n, m'/n' in Q with $m/n \cdot x + m'/n' \cdot x' = 0$ in A, then H and H' are equal; namely, each is the pure subgroup generated by $n'mx = -nm'x'$. Consequently, we may assume $\{x, x'\}$ span $A \otimes Q$.

Define $H_0 = \{r \text{ in } Q \mid rx + sx' \text{ is in } A \text{ for some } s \text{ in } Q\}$. Then the map $A \to H_0$ sending $rx + sx'$ to r is surjective with kernel H'. Similarly, if $H_0' = \{s \text{ in } Q \mid rx + sx' \text{ is in } A \text{ for some } r \text{ in } Q\}$, then $A \to H_0'$ has kernel H. Therefore, it suffices to prove that H_0 is isomorphic to H_0'.

Consider the inclusions $H \subset H_0$ sending rx to r, and $H' \subset H_0'$ sending sx' to s. Then $H_0 \to H_0'/H'$ sending r to the class of some s such that $rx + sx'$ is in A is well defined: if $rx + s'x'$ is also in A, then $(s - s')x'$ is in H'. Clearly, this map induces an isomorphism $H_0/H \to H_0'/H'$. Since H_0 and H_0' are rank 1, torsion free with isomorphic subgroups H and H' respectively and isomorphic quotient groups H_0/H and H_0'/H', H_0 is isomorphic to H_0'.

The following theorem summarizes the discussion of §1 together with Proposition 2.1.

THEOREM 2.2. *Let Y be a rank 1, torsion free abelian group and let y in Y be a nonzero element. Then there is a natural $1-1$ correspondence with explicit inverse*

$$\Phi(Y, y): E(Y) \to T(Y, y)$$

between the set $E(Y)$ of isomorphism classes of rank 2, torsion free abelian groups with a pure subgroup isomorphic to Y and the set $T(Y, y)$ of pairs consisting of

(i) *the type of some characteristic $(a_1, \cdots, a_n, \cdots)$*

(ii) *the type with respect to* char(y) *of some extension function f in $\Pi'\mathbf{Z}/p_i^{a_i}$, which transforms to $\pm p_1^{c_1} \cdots p_n^{c_n} \cdots f$ in $\Pi'\mathbf{Z}/p_i^{a_i+c_i}$ if all $c_i \geqq 0$ and $\Sigma c_i < \infty$.*

Proof. We define $\Phi(Y, y)(A)$ for A in $E(Y)$ as follows. Choose a rank 1, pure subgroup H of A with $O \neq h$ in H such that $\operatorname{char}(h) = \operatorname{char}(y) = (b_1, \cdots, b_n, \cdots)$. Let x in A/H be any nonzero element with $\operatorname{char}(x) = (a_1, \cdots, a_n, \cdots)$. Let f in $\Pi'\mathbf{Z}/p_i^{a_i}$ be an extension function representing $\varphi(A/H, x; H, h)(0 \to H \to A \to A/H \to 0)$ as in Proposition 1.1. Define $\Phi(Y, y)(A) = \{\operatorname{char}(x), f\}$.

By Proposition 2.1, the type of $\operatorname{char}(x)$ is well defined, independent of the choice of H or x. By Proposition 1.4, type of f depends at most upon choices for $A/H, x, H, h$, since A is isomorphic to A_f. If H' is another rank 1, pure subgroup of A with $O \neq h'$ in H' such that $\operatorname{char}(h') = \operatorname{char}(y)$, if x' in A/H' is chosen with $\operatorname{char}(x') = \operatorname{char}(x)$, and if f' in $\Pi'\mathbf{Z}/p_i^{a_i}$ represents

$$\varphi(A/H', x'; H', h')(0 \to H' \to A \to A/H' \to 0),$$

then

$$\left\{\begin{pmatrix} p_i^{b_i} & -f_i \\ 0 & p_i^{a_i} \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} p_i^{b_i} & -f'_i \\ 0 & p_i^{a_i} \end{pmatrix}\right\}$$

are each matrix sequences for A by Proposition 1.2. Hence, by Proposition 1.4, f and f' have the same type. Finally, for a given rank 1, pure subgroup H of A with $O \neq h$ in H such that $\operatorname{char}(h) = \operatorname{char}(y)$, let $x' = mx$ be an integer nonzero multiple of x in A/H. As an extension function for $0 \to H \to A \to A/H \to 0$, f may be viewed as a classifying map $f: (A/H)/\mathbf{Z} \cdot x \to Q/H$; this induces the classifying map $f': (A/H)/\mathbf{Z} \cdot x' \to Q/H$. Under the identifications

$$\mathrm{Hom}((A/H)/\mathbf{Z}\cdot x, Q/H) = \Pi'\mathbf{Z}/p_i^{a_i},$$

$\mathrm{Hom}((A/H)/\mathbf{Z}\cdot x', Q/H) = \Pi'\mathbf{Z}/p_i^{a_i+c_i}$ with $c_i = \max\{c \mid p_i^c$ divides $m\}$, $f' = \pm p_1^{c_1}\cdots p_n^{c_n}\cdots f$ in $\Pi'\mathbf{Z}/p_i^{a_ic_i}$.

The naturality of $\Phi(Y, y)$ with respect to Y and y follows from the naturality of extension functions as given in Proposition 1.1. The fact that $\Phi(Y, y)$ is injective follows from the fact that A is isomorphic to A_f if $\Phi(Y, y)(A) = \{(a_1, \cdots a_n, \cdots), f\}$. The fact that $\Phi(Y, y)$ is surjective follows from Proposition 1.2, since $\Phi(Y, y)(A_f) = \{(a_1, \cdots, a_n, \cdots), f\}$. Finally, the explicit inverse of $\Phi(Y, y)$ is given in Proposition 1.3.

We say that a torsion free abelian group A is homogeneous of type $(b_1, \cdots, b_n, \cdots)$ if for every nonzero x in A, the type of $\mathrm{char}(x)$ equals the type of $(b_1, \cdots, b_n, \cdots)$. Using Theorem 2.2, we "list" all rank 2, torsion free abelian groups which are homogeneous of type

$$(b_1, \cdots, b_n, \cdots).$$

THEOREM 2.3. *Let $(b_1, \cdots, b_n, \cdots)$ be a characteristic and let Y be a rank 1, torsion free abelian group with $O \neq y$ in Y such that* $\mathrm{char}(y) = (b_1, \cdots, b_n, \cdots)$. *Then the set of all rank 2, torsion free abelian groups homogeneous of type $(b_1, \cdots, b_n, \cdots)$ is the subset of $E(Y)$ consisting of those A in $E(Y)$ such that $\Phi(Y, y)(A) = \{(a_1, \cdots, a_n, \cdots), f$ in $\Pi'\mathbf{Z}/p_i^{a_i}\}$ satisfies*

(i) *if $b_i = \infty$, then $a_i = \infty$;*

(ii) *if $b_i < \infty$ and $a_i = \infty$, then f_i in $\mathbf{Z}_{p_i}$ is not rational;*

(iii) *for all but finitely many i with $b_i < \infty$, $a_i \geqq b_i$ and $p_i^{b_i} \mid f_i$;*

(iv) *for every nonzero pair of integers $\{m, n\}$, there are only finitely many i with $a_i > b_i$, $p_i^{b_i} \mid f_i$, and $m - nf_i/p_i^{b_i}$ divisible by p_i.*

Proof. The set of isomorphism classes of rank 2, torsion free abelian groups homogeneous of type $(b_1, \cdots, b_n, \cdots)$ is clearly a subset of $E(Y)$. To prove the theorem, it suffices to prove the following for a given pair $\{(a_1, \cdots, a_n, \cdots), f$ in $\Pi'\mathbf{Z}/p_i^{a_i}\}$: the characteristic of every nonzero element (m, n) in $A(f) \cap \mathbf{Z} \oplus \mathbf{Z}$ as an element of $A(f)$ has the same type as $(b_1, \cdots, b_n, \cdots)$ iff conditions (i)–(iv) are satisfied (where $A(f)$ is given in Proposition 1.3.). We write

$$(m, n) = (mp_i^{b_i} - nf_i)(p_i^{-b_i}, 0) + np_i^{a_i}(f_i p_i^{-a_i-b_i}, p_i^{-a_i})$$

in $A(f) \otimes \mathbf{Z}_{p_i}$, where $p_i^{\infty} = 1$ and $f_i = 0$ if $b_i = \infty$. We observe for any $k > 0$ that (m, n) is divisible by p_i^k in $A(f)$ iff (m, n) is divisible by p_i^k in $A(f) \otimes \mathbf{Z}_{p_i}$.

If $b_i = \infty$, then every (m, n) in $A(f) \cap \mathbf{Z} \oplus \mathbf{Z}$ is infinitely p_i divisible (as an element of $A(f)$) iff $a_i = \infty$. If $b_i < \infty$, then (m, n) in

$A(f)\cap \mathbf{Z}\oplus\mathbf{Z}$ is not infinitely p_i divisible iff either $a_i<\infty$ or $a_i=\infty$ and $mp_i^{b_i}-nf_i\neq 0$. Consequently, (i) and (ii) are equivalent to the condition that one nonzero element of $A(f)$ is infinitely p_i divisible iff all nonzero elements of $A(f)$ are.

(m,n) in $A(f)\cap\mathbf{Z}\oplus\mathbf{Z}$ is divisible by $p_i^{b_i}$ for all but finitely many i with $b_i<\infty$ iff $n=0$ or for all but finitely many i with $b_i<\infty$, $a_i\geqq b_i$ and $p_i^{b_i}\mid f_i$. (m,n) is not divisible by $p_i^{b_i+1}$ for all but finitely many i with $b_i<\infty$ iff for all but finitely many i with $a_i>b_i$, $mp_i^{bi}-nf_i$ is not divisible by $p_i^{b_i+1}$. Consequently, (iii) and (iv) are equivalent to the condition that the characteristic of any nonzero element of $A(f)$ differs from $(b_1,\cdots,b_n,\cdots)$ at only finitely many i with b_i.

To show how explicit Theorem 2.3 is, we provide the following simple example.

EXAMPLE 2.4. Let $(b_1,\cdots,b_n,\cdots)$ be a characteristic such that $b_i<\infty$ for infinitely many i. Let Y be a rank 1, torsion free abelian group with nonzero element y with $\operatorname{char}(y)=(b_1,\cdots,b_n,\cdots)$. Then for any positive integer k,

$$\Phi(Y,y)^{-1}(\{b_1+k,\cdots,b_n+k,\cdots),f=i\cdot p_i^{b_i}\text{ in }\Pi'\mathbf{Z}/p_i^{b_i+k}\})$$

is an indecomposable, rank 2, torsion free abelian group homogeneous of type $(b_1,\cdots,b_n,\cdots)$.

Proof. If A in $E(Y)$ is homogeneous and decomposable, then $A\simeq Y\oplus Y$ so that the quotient type of A (i.e., the first invariant of Theorem 2.2) must be the type of $(b_1,\cdots,b_n,\cdots)$. Consequently, to check the example it suffices to verify conditions (i)–(iv) of Theorem 2.3 for $\{(b_1+k,\cdots,b_n+k,\cdots),f=i\cdot p_i^{b_i}\}$. Conditions (i), (ii), and (iii) are immediate. To verify condition (iv), we observe that for sufficiently large i $0<n\cdot i-m<p_i$ by the prime number theorem.

In particular, Example 2.4 gives examples of nonfree homogeneous groups A, with type $A=(0,\cdots,0\cdots)$, such that $A\otimes\mathbf{Z}_p$ is a free $\mathbf{Z}_p$ module for all primes p. These groups are called locally free.

As another application of Theorem 2.2, we determine the subset $E(\mathbf{Z})\cap E(Y)$ of $E(Y)$.

PROPOSITION 2.5. *Let $(b_1,\cdots,b_n,\cdots)$ be a characteristic and let Y be a rank 1, torsion free abelian group with $0\neq y$ in Y such that* $\operatorname{char}(y)=(b_1,\cdots,b_n,\cdots)$. *Then the subset $E(\mathbf{Z})\cap E(Y)$ of $E(Y)$ consists of those A in $E(Y)$ such that $\Phi(Y,y)(A)=\{(a_1,\cdots,a_n,\cdots),f$ in $\Pi'\mathbf{Z}/p_i^{a_i}\}$ satisfies:*

(i) *if $b_i=\infty$, then $a_i<\infty$;*

(ii) *for all but finitely many i with $a_i,b_i>0$, p_i does not divide f_i;*

(iii) *there exists a nonzero pair of integers* $\{m,n\}$ *such that* $mp_i^{b_i} - nf_i \neq 0$ *whenever* $a_i = \infty$, *and* $m - nf_i$ *is divisible by* p_i *for only finitely many* i *with* $b_i = 0$ *and* $a_i > 0$.

Proof. For Y cyclic, conditions (i), (ii), and (iii) are immediately satisfied for any pair of invariants $\{(a_1, \cdots, a_n, \cdots), f\}$. We may thus assume Y is not cyclic. As in the proof of Theorem 2.3, to prove the proposition it suffices to prove that (m,n) in $A(f) \cap \mathbf{Z} \oplus \mathbf{Z}$ generates a pure cyclic subgroup of $A(f)$ iff (i) and (ii) are satisfied and $\{m,n\}$ satisfies (iii). Since Y is assumed not to be cyclic, we need only consider (m,n) with $n \neq 0$. We write

$$(m,n) = (mp_i^{b_i} - nf_i)(p_i^{-b_i}, 0) + np_i^{a_i}(f_i p_i^{-a_i-b_i}, p_i^{-a_i}).$$

Now, (m,n) with $n \neq 0$ is not infinitely divisible in $A(f) \otimes \mathbf{Z}_{p_i}$ by p_i iff whenever $b_i = \infty$, then $a_i < \infty$; and whenever $a_i = \infty$, then $mp_i^{b_i} - nf_i \neq 0$. Furthermore, (m,n) with $n \neq 0$ is divisible by only finitely many p_i iff for only finitely many i with $b_i, a_i > 0$ f_i is divisible by p_i; and for only finitely many i with $b_i = 0$, $a_i > 0$ $m - nf_i$ is divisible by p_i.

References

1. R. A. Beaumont and R. S. Pierce, *Torsion free groups of rank two*, Mem. Amer. Math. Soc. Nr. **38**.
2. L. Fuchs, *Infinite abelian groups*, Vol. II, Academic Press, New York and London, 1973.

Received January 23, 1974. Partially supported by the U.S. — France Exchange of Scientists Program.

PRINCETON UNIVERSITY