PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

THREE QUESTIONS ON DUO RINGS

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A ring R (with unit element) is called a duo ring if every one-sided ideal is two-sided. This is equivalent with the existence of elements r' and r'' in R with $rs = sr'$, $sr = r''s$ for elements r, s in R. We will discuss in this note the following three problems: (A) Is the localization at a prime ideal P of a duo ring again a duo ring? (B) Is in a duo ring the P-component of zero equal to the right (left) P-component of zero? (C) Is in a noetherian duo domain the semi group of ideals (under multiplication) commutative?

The answer to all three questions is "no" in general, but "yes" for (A) and (B) in the noetherian case, and "yes" for (C) if R is integrally closed in its division ring of quotients.

1. Let R be a duo ring, P a prime ideal in R (necessarily completely prime). One defines the ideal $N = \{r \in R;\ s_1 r s_2 = 0$ for $s_1, s_2 \in S = R\backslash P\}$ as the P-component of zero. Since the image $\bar{S}$ of $S = R\backslash P$ in $\bar{R} = R/N$ is an Ore-system consisting of nonzero divisors, the ring of quotients $R_p = \bar{R}\bar{S}^{-1} = \{\bar{r}\bar{s}^{-1};\ \bar{r} \in \bar{R}, \bar{s} \in \bar{S}\}$ exists and is equal to $\bar{S}^{-1}\bar{R}$. We ask in problem (A) if this ring is again a duo ring. But the ring constructed in **[4]** provides us with a counter-example. We repeat the construction: Let F be the function field $\mathbf{Q}(t)$ in one variable t over the rational numbers $\mathbf{Q}$. The field F can be ordered by writing $(q_0 t^n + \cdots + q_0)(q'_m t^m + \cdots + q'_0)^{-1} > 0$ if and only if $q_n q_m{}' > 0$ for a typical element in F with $q_n \neq 0 \neq q'_m$. Let G be the group of all pairs (a, b), $a > 0$, a, b in F with $(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 a_2 + b_2)$ as operation. G is an ordered group if we use the lexicographic ordering and the unit element in G is $(1, 0)$. We then form the generalized power series ring $R = \mathbf{Q}\{\{G^+\}\}$ consisting of all series $\Sigma q_\alpha g_\alpha$ with q_α in $\mathbf{Q}$ and $\{g_\alpha\}$, $g_\alpha \geqq (1, 0)$ being a well ordered sequence of elements in G. (For details of this construction see **[3]**).

For $r = \Sigma q_\alpha g_\alpha \neq 0$ in R we define $v(r) = g_0$, where $g_0 = \{g_\alpha; g_\alpha \neq 0\}$ and $v(0) = \infty$ with the usual properties of this symbol. Using this notation we set $P = \{r \in R;\ v(r) > (1, q)$ for every q in $\mathbf{Q}\}$. This is a prime ideal in R and R itself is a duo ring. But $(t, 0)R_p$ is not a left ideal in R_p, since $(1, -1)(t, 0) = (t, -t) = (t, 0)(1, -t)$ is not contained in $(t, 0)R_p$.

We wrote in the above argument $g = 1g$ in R with 1 the unit element in $\mathbf{Q}$, g in G.

It is well known that by using generalized power series rings one can construct duo domains whose semigroup of principal ideals $\neq$ (0) is isomorphic to the positive cone of an arbitrary (fully) ordered group.

As a positive answer to a special case of problem (A) we prove the following:

LEMMA 1. *R_p is again a duo ring if R_p satisfies the maximum condition for principal right and left ideals.*

For a proof let us assume that not every right ideal in R_p is two-sided. Then there must exist an element s in S and an element r in R such that $s^{-1}rR_p \subsetneqq rR_p$. Since an element s' in R exists with $sr = rs'$, we have $s^{-1}rs' = r$ and obtain

$$rR_p \subsetneqq s^{-1}rR_p \subsetneqq s^{-2}rR_p \subsetneqq \cdots \subsetneqq \cdots,$$

a contradiction. That every left ideal is a right ideal is proved in the same fashion.

We wrote r, s instead of $\bar{r}, \bar{s}$ for elements in R_p as it is done in the commutative case.

COROLLARY 1. *Let R be a noetherian duo ring with a prime ideal P. For r in R, not in N, the P-component of zero, and s in $S = R\backslash P$ there exist elements s', s'' in S with $sr = rs'$, $rs = s''r$.*

R noetherian implies R_p noetherian and $s^{-1}rR_p = rR_p$ follows. This means that $s^{-1}r = rat^{-1}$ for elements a in R and t in S. This leads to $rt = rs'a$ for some s' in R and $r(t - s'a) = 0$. But R_p is a local ring, and s' in P implies that $(t - s'a)$ is a unit in R_p and $r = 0$ in R_p follows contrary to our assumption r not in N.

2. *Let* R be a duo ring with a prime ideal P. We defined the P-component of zero above, and the left P-component of zero is defined as $N_l = \{r \in R;\ sr = 0 \text{ for some } s \in S\}$. N_r, the right component of zero, is defined in the corresponding fashion, and we ask in question (B) if $N = N_l = N_r$ is true, which is the same as if we ask for $N_r = N_l$.

The ring R in 1. can be used to obtain a negative answer to question (B) as well: R contains the two-sided ideal $(t, t)R = I$. We put $T = R/I$ and write P_1 for the image of P in T. It follows that $sr = (1, 1)(t, 0) = (t, t)$ in R, but $\overline{(t, 0)}\bar{s} \neq \bar{0}$ for all s in $R\backslash P$ if we write $\bar{a}$ for the image in T of an element a in R. This means that $\overline{(t, 0)}$ is contained in the left P_1-component of zero, but not in the right P_1-component of zero in T.

In any duo ring R a primary ideal Q is defined as an ideal with the property that ab in Q implies either a or b^n in Q for some n or b or a^m in Q for some m. The radical $\sqrt{A} = \{r \in R;\ r^n \in A \text{ for some } n\}$ of an ideal A in R is defined as in the commutative case and the radical of a primary ideal Q is a prime ideal P; Q is then called P-primary or belonging to P. If R is a noetherian duo ring then every ideal can be written as the finite intersection of primary ideals belonging to different prime ideals. This means that the assumption of the following Lemma is certainly satisfied if R is a noetherian duo ring.

LEMMA 2. *Let R be a duo ring in which the zero ideal is the intersection of finitely many primary ideals, P a prime ideal in R. Then $N = N_r = D = \cap Q_i$, $Q_i \subset P$, if $\cap_{i=1}^n Q_i = (0)$ is the primary decomposition of* (0).

Proof. Let $D = \cap_{i=1}^k Q_i$ and it is clear that $1 \leqq k \leqq n$. For r in N elements s_1, s_2 in S do exist with $s_1 r s_2 = 0 \in Q_i$ for $i = 1, \cdots, k$. This implies r in D and $N \subset D$ follows. For every Q_j, $j = k+1, \cdots, n$ there exists an element $s_j \in S \cap Q_j$ and $s = \Pi s_j$ is still not contained in P. But sD and Ds are contained in every Q_i, $i = 1, \cdots, n$ and $sD = Ds = (0)$ follows. This shows $D \subset N_r$, $D \subset N_1$ and gives the result.

3. The examples listed in [5] all have the property

$$(*)\ aRbR = bRaR.$$

The duo rings constructed as generalized power series rings show that (*) is certainly not satisfied by all duo rings. But if R is duo and noetherian the method just mentioned leads also to a ring in which the ideal multiplication is commutative. Our next example (generalizing a construction used in commutative ring theory, see for example [2]) will show that nevertheless property (*) is not necessarily satisfied in a noetherian duo domain.

Let F be the splitting field of $p(x) = x^3 - 2$ over the rational numbers $\mathbf{Q}$. Let $\sqrt[3]{2}$ be the real root of $p(x)$ and $\sqrt[3]{2}\omega$ be one of the complex roots. Then there exists an automorphism σ of F with $\sigma(\sqrt[3]{2}) = \sqrt[3]{2}\omega$ that fixes $\mathbf{Q}$. We consider the twisted power series ring $K = F[[x, \sigma]]$ in one variable over F with elements $\Sigma_{i=0}^{\infty} x^i d_i$ where $d_i \in F$ and $dx = x\sigma(d)$. Let $R = \mathbf{Q} + xK$ be the subring of K consisting of all those elements whose constant term is contained in $\mathbf{Q}$.

Every element in R can be written as $x^n d_n (1 + \Sigma_{i=1}^{\infty} x^i d'_i)$ and $1 + \Sigma x^i d'_i$ is a unit in R.

Using this representation of elements in R one can show that R is a duo domain.

To prove that R is noetherian, let, I be any right ideal in R. Let $n = \min\{m, d_m \neq 0 \text{ for } \Sigma x^i d_i \in I\}$. (We may assume $I \neq (0)$) All coefficients d_n of elements in I for this minimal n form a finite dimensional vectorspace over $\mathbf{Q}$ with basis $d_{n,1}, d_{n,2}, \cdots, d_{n,k}$, say. Then there exist elements r_i, $i = 1, \cdots, k$, of the form $r_i = x^n d_{n,i} + \cdots$ in I and they generate I.

We observe finally that $xRx\sqrt[3]{2}R \neq x\sqrt[3]{2}RxR$

Even though we proved in 1. that every localization at a multiplicatively closed system S of a duo ring R is again a duo ring in case R is noetherian, the above ring R is contained in the ring $D = (\mathbf{R} \cap F) + xK$ which is contained in the quotient ring of R, but is not duo. ($\mathbf{R}$ is the field of real numbers)

The following result will be proved:

THEOREM. *Let R be noetherian integrally closed duo domain. Then $aRbR = bRaR$ for all a, b in R and ideal multiplication is commutative.*

Before proving the theorem we must explain what ik meant by R being integrally closed. We use the following definition which is equivalent to the usual one in the commutative case: The duo domain R is integrally closed if and only if $\mathrm{End}_R(M_R) = R$ for every finitely generated ideal $M \neq (0)$ of R.

For a proof of the theorem let $a \neq 0$, not a unit, be an element in R and let P be one of the prime ideals associated with aR. We will show, using essentially the commutative theory, see for example [7], that R_p is a ring satisfying property (*) and that R is the intersection of such rings.

With the help of Corollary 1 one generalizes Theorem 11 on p.214, Vol I in [7] und obtains an element b in R, not in aR, such that $bP \subset aR$. The ideal P is finitely generated and our assumption $\mathrm{End}_R(P, P) = R$ implies that $a^{-1}bP \not\subseteq P$. But then there exist elements p in P and c in R with

$$bp = ac, \text{ where } c \text{ is not contained in } P.$$

The ring R_p exists and $aR_p = acR_p = bpR_p \subset bPR_p \subset aR_p$ follows. This means that bPR_p is equal to aR_p and further $PR_p = pR_p$.

The local ring R_p has therefore a principal maximal ideal pR_p and is a noetherian duo domain. This implies that the intersection $\cap (pR_p)^n$ is zero (the Krull intersection theorem holds for noetherian duo domains) and that the powers $p^n R_P$ of pR_P are the only ideals $\neq (0)$ of

R_p. We can conclude further that P is a prime ideal of height 1 in R. Especially property (*) is satisfied in R_p. It remains to show that $R = \cap R_p$, where the intersection is taken over all prime ideals with height 1 in R. Let $s^{-1}r$ be an element in this intersection. Then $r \in \cap(sR_p \cap R)$ follows. But $sR_p \cap R = Q$ is the P-primary component of sR or R if $s \notin P$. We obtain $r \in \cap(sR_p \cap R) = sR$ and $s^{-1}r \in R$, as desired. Finally $aRbR = abR = \cap abR_p = \cap baR_p = baR = bRaR$. Ideal multiplication in R is commutative, since the product AB, A, B arbitrary ideals, is the sum of the ideals abR, for elements a in A and b in B.

COROLLARY 2. *Every ring between a noetherian integrally closed duo domain and its division ring of quotients is again a duo domain.*

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Received January, 1974.

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