PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

PRE-PRÜFER RINGS

MONTE B. BOISEN, JR. AND PHILIP B. SHELDON

The purpose of this paper is to investigate the class of pre-Prüfer rings. A ring is defined to be in this class in case each of its proper homomorphic images is a Prüfer ring. It is shown for a domain D that if D is a pre-Prüfer ring, then the prime spectrum of D forms a tree and every finitely generated ideal of D containing a bounded element is invertible. If every finitely generated regularizable ideal of a ring R is invertible, then R is a pre-Prüfer ring. Examples are presented to show that the converse of each of the two results stated above is false.

Introduction. Throughout this paper the term "ring" will denote a commutative ring with unity. Griffin [4] has generalized the notion of a Prüfer domain to rings with zero divisors by defining a *Prüfer ring* to be a ring in which every finitely generated regular ideal is invertible. (Here we use the terminology that a *regular element* is an element which is not a divisor of zero, and a *regular ideal* is an ideal that contains a regular element.) In addition to the obvious containment relation between the classes of Prüfer rings and Prüfer domains, these classes are also related by the fact that every proper homomorphic image of a Prüfer domain is a Prüfer ring [1]. In this paper we will study those rings and, in particular, those domains which share this latter property with Prüfer domains. Thus we define a ring R to be a *pre-Prüfer ring* if every proper homomorphic image of R is a Prüfer ring and we define R to be a *pre-Prüfer domain* if, in addition, R is an integral domain. While, as mentioned earlier, all Prüfer domains are included in the class of pre-Prüfer domains, they are not the only such domains. In particular, every one-dimensional domain is a pre-Prüfer domain, since each of its proper homomorphic images, being a zero-dimensional ring, is a total quotient ring [6, Proposition 1, p. 1120], and total quotient rings are trivial examples of Prüfer rings. However, we will show that the ideal-theoretic structure of a pre-Prüfer domain is similar in some ways to that of a Prüfer domain. But we will see that this similarity does not extend to the ring case, and that the class of Prüfer rings is not even contained in the class of pre-Prüfer rings.

Our first main result (Theorem 1.2) shows that the prime ideal structure of a pre-Prüfer domain is much like that of a Prüfer domain in that the set of prime ideals contained in a given prime ideal is linearly ordered by inclusion. Furthermore, our second main result (Theorem 1.5) shows that the invertibility aspect of the ideal structure of a Prüfer domain is partially duplicated in every pre-Prüfer domain. Specifically, in a pre-Prüfer domain a finitely generated ideal is necessarily invertible if it contains an element a such that $\cap_{i=1}^{\infty}(a^i) \neq (0)$. While the two properties given in the two results mentioned above are necessary conditions for a domain to be a pre-Prüfer domain, we show by an example (1.6) that they are not sufficient conditions even when taken together. We also show by examples that the ideal structure of a pre-Prüfer ring is far less restricted than in the domain case. In particular, we show in Examples 3.1 and 3.2 that neither of the necessary conditions for domains are necessary for a ring to be a pre-Prüfer ring. In Theorem 2.4 we present a condition that is sufficient in general for a ring to be a pre-Prüfer ring, namely that every finitely generated ideal that is regularizable (Definition 2.1) be invertible. However, this condition is shown by an example (2.5) not to be a necessary one, even in the domain case.

A detailed treatment of the theory of Prüfer rings can be found in Larsen and McCarthy **[5]**. We will call an ideal a *proper ideal* in case it is unequal to zero and unequal to the whole ring, and a homomorphism is said to be a *proper homomorphism* in case its kernel is a proper ideal. Otherwise, our terminology is essentially that of **[5]**. In particular, by a "local ring" we mean a ring with a unique maximal ideal but not necessarily a Noetherian ring. Where it is convenient and unambiguous to do so, we will denote the homomorphic image of an element or a set by placing a bar over its symbol.

1. The domain case: some necessary conditions.

In this section we will present two necessary conditions for an integral domain D to be a pre-Prüfer domain. While this section is primarily concerned with pre-Prüfer domains, our first result is true for pre-Prüfer rings in general and hence will be stated and proved in this more general setting.

1.1. THEOREM. *Let R be a pre-Prüfer ring and let S be a multiplicative system of R. Then the ring of quotients R_S is a pre-Prüfer ring. In case R is also a Prüfer ring, then R_S is a Prüfer ring.*

Proof. We begin by proving the second assertion. Suppose that R is a Prüfer pre-Prüfer ring. Let E be the ideal $\{x \in R \mid xs = 0$ for some $s \in S\}$. When $E = (0)$, $R/E = R$ which was assumed to be a Prüfer ring. When $E \neq (0)$, R/E is again a Prüfer ring since it is a proper homomorphic image of the pre-Prüfer ring R. Thus the ring of quotients R_S, which by definition equals $(R/E)_{S/E}$, is an overring of the Prüfer ring R/E, and hence is itself a Prüfer ring **[4**, apply part 5 of Theorem 13, p. 61**]**.

Now to prove the first assertion we assume only that R is a pre-Prüfer ring. Let B be a nonzero ideal of $R_{\mathscr{S}}$. Then $B = AR_S$ for some nonzero ideal A of R. Since R is a pre-Prüfer ring, R/A is a Prüfer ring, as well as a pre-Prüfer ring. Hence, by the first part of this proof, $(R/A)_{S/A}$ is a Prüfer ring. Since R_S/B, which equals R_S/AR_S, is isomorphic to the Prüfer ring $(R/A)_{S/A}$, we have shown that every proper homomorphic image of R_S is a Prüfer ring as required.

Our next result concerns the *prime spectrum* of a ring R, by which we mean the partially ordered set of prime ideals of R, ordered by inclusion. Of special interest is the case when the prime spectrum forms a *tree*, that is, a partially ordered set in which no two incomparable elements share a common upper bound. It is immediate from the properties of rings of quotients that the prime spectrum of R forms a tree if and only if the prime spectrum of R_M is linearly ordered for each maximal ideal M of R.

1.2. THEOREM. *The prime spectrum of a pre-Prüfer domain forms a tree.*

Proof. In view of Theorem 1.1 and the remark immediately preceding the statement of this theorem, it is sufficient to show that in a local pre-Prüfer domain, the prime spectrum is linearly ordered. Let D be a local pre-Prüfer domain with maximal ideal M, and let P_1 and P_2 be prime ideals of D. We want to show P_1 and P_2 are comparable. Assume not. Let $A = P_1 \cap P_2$. If $x \notin P_1 \cup P_2$, then in D/A, denoted by $\bar{D}, \bar{x}\bar{y} = \bar{0}$ implies that $xy \in P_1 \cap P_2$ and so $y \in P_1 \cap P_2$. Consequently $\overline{(P_1 \cup P_2)}$ contains all of the zero divisors of $\bar{D}$. Conversely, assume that $x \in P_1 \cup P_2$, say $x \in P_1$. We note that $P_2 \backslash P_1$ is nonempty and that for every $y \in \overline{P_2 \backslash P_1}$, both $\bar{x}\bar{y} = \bar{0}$ and $\bar{y} \neq \bar{0}$. Hence $\bar{x}$ is a zero divisor. In short, $\overline{(P_1 \cup P_2)}$ is precisely the set of all zero divisors of $\bar{D}$.

Next we establish an alternate characterization of the set of zero divisors of $\bar{D}$. Since D is a local pre-Prüfer domain and since $A \neq (0)$, $\bar{D}$ is a local Prüfer ring. Moreover, $\overline{P_1 \cup P_2}$ cannot equal $\bar{M}$, so $\bar{M}$ must contain a regular element of $\bar{D}$. In other words, $\bar{D}$ is a Prüfer ring with a unique regular maximal ideal, and therefore $(\bar{D}, \bar{M})$ is a valuation pair [**2**, Theorem 2.3, p. 8]. Let $\bar{I}$ denote the elements of $\bar{D}$ with infinite value under the valuation associated with this pair. We note that $\bar{I}$ consists entirely of zero divisors since an element with infinite value cannot have an inverse in the total quotient ring. On the other hand, if $z \in \bar{D} \backslash \bar{I}$, then z has finite value, and hence there is an element y in the total quotient ring such that zy has zero value. Since $\bar{M}$ is the positive prime of this valuation, $zy \in \bar{D} \backslash \bar{M}$, and hence zy is a unit of $\bar{D}$. Thus z

cannot be a zero divisor in $\bar{D}$, and we conclude that $\bar{I}$ is the set of zero divisors of $\bar{D}$.

We have shown that $\overline{(P_1 \cup P_2)} = \bar{I}$. But this is a contradiction because $\bar{I}$, which is clearly an ideal, cannot equal the union of the two incomparable ideals $\bar{P}_1$ and $\bar{P}_2$.

As mentioned in the introduction, if a domain has Krull dimension one, it is a pre-Prüfer domain. The converse to this statement is clearly false, since all Prüfer domains are pre-Prüfer domains. However, within the class of Noetherian domains, the pre-Prüfer domains and the one-dimensional domains coincide, as the following corollary shows.

1.3. COROLLARY. *Let D be a Noetherian domain. Then D is a pre-Prüfer domain if and only if D is one-dimensional.*

Proof. We need only show that a Noetherian pre-Prüfer domain is one-dimensional; by Theorem 1.1 and the properties of localizations, it suffices to consider local Noetherian pre-Prüfer domains. By Theorem 1.2, such a domain must have linearly ordered prime ideals, and by Krull's Principal Ideal Theorem [5, Theorem 7.6, p. 159], every nonunit must lie in the unique height-one prime. Hence the maximal ideal must have height one. In other words, the domain is one dimensional.

As we remarked earlier, a sufficient condition for a domain to be a pre-Prüfer domain is that all of its finitely generated ideals are invertible — that is, that it is a Prüfer domain — but this condition is not necessary. A necessary condition in this direction involves the concept of boundedness.

1.4. DEFINITION. If r is an element of the ring R, we say that r is *bounded* in case $\bigcap_{i=1}^{\infty} r^i R \neq (0)$. Otherwise we say that r is *unbounded.*

1.5. THEOREM. *In a pre-Prüfer domain, every finitely generated ideal containing a bounded element is invertible.*

Proof. Let D be a pre-Prüfer domain and let A be a finitely generated ideal of D containing a bounded element a. To show A is invertible, it is sufficient to show that AD_M is invertible in D_M for each maximal ideal M of D. We note that by Theorem 1.1, D_M is itself a pre-Prüfer domain. Moreover, $(0) \neq \bigcap_{i=1}^{\infty} a^i D \subseteq \bigcap_{i=1}^{\infty} a^i D_M$, and so AD_M contains a bounded element. Thus it is sufficient to prove this theorem in the case where D is a local domain.

We consider two cases. The first case is where D contains a nonzero unbounded element. Since the set of bounded elements of D

forms a saturated multiplicative system and since, by Theorem 1.2, a local pre-Prüfer domain has linearly ordered prime ideals, we see that the set of unbounded elements of D forms a nonzero prime ideal Q. Since A contains a bounded element, it follows that $A \not\subseteq Q$. The second case is where D has no nonzero unbounded elements. In this case D does not have a minimal prime ideal since the elements of a minimal prime ideal are all unbounded [**7**, Corollary 1.4, p. 323]. Using once again the fact that the prime ideals of D are linearly ordered, we conclude that there exists a nonzero prime ideal Q of D such that $A \not\subseteq Q$. In either case we have a nonzero prime ideal Q of D that contains all of the unbounded elements of D and does not contain A.

Since Q is a nonzero prime ideal of the pre-Prüfer domain D, then D/Q is a local Prüfer domain, hence a valuation domain. Therefore, the image of A in D/Q is a nonzero principal ideal, generated by the image of some element of A in D/Q, say b. Since $b \notin Q, b$ is bounded. Let J denote the nonzero ideal $(\cap_{i=1}^{\infty}(B^i)) \cap Q$. Suppose now that $bx \in J$. Then since $bx \in Q$, and $b \notin Q$, we have that $x \in Q$. On the other hand, since $bx \in \cap_{i=1}^{\infty}(b^i)$ and D is a domain, we conclude that $x \in \cap_{i=1}^{\infty}(b^{i-1}) = \cap_{i=1}^{\infty}(b^i)$. Therefore $x \in J$, and hence $b + J$ is a regular element of D/J. But D/J is a local Prüfer ring with maximal ideal M/J. Thus [**4**, Theorem 13, part 3, p. 61] $(D/J, M/J)$ has the regular total order property, which means that the images in $(D/J)_{M/J}$ of every pair of ideals of D/J, at least one of which is regular, are totally ordered by inclusion. In this case, $(D/J)_{M/J} = D/J$, so the regular ideal $((b) + J)/J$ and the ideal $(Q + J)/J$ are comparable ideals of D/J. Since $J \subseteq (b) \cap Q$, we can see that (b) and Q are comparable. But $b \notin Q$, so $Q \subseteq (b)$. Now b was chosen so that the image of (b) in D/Q equals the image of A in D/Q. Hence $Q + A = Q + (b) = (b)$, and consequently, $A \subseteq (b)$. Thus $A = (b)$, and so A is invertible.

The two preceding results have shown that in a pre-Prüfer domain, the prime spectrum forms a tree and every finitely generated ideal containing a bounded element is invertible. We will now present an example that shows that a domain satisfying both of these conditions need not be a pre-Prüfer domain. First we will develop some notation.

Let F be a field and let X and Y be indeterminates. Let T_1 be the set $\{X^n \mid n = 0 \text{ or } n \in Z^+\} \cup \{Y^k X^m \mid k \in Z^+, m \in Z\}$, and let D_1 be the set of all linear combinations over F of elements of T_1. The fact that D_1, when considered as a subset of the field $F(X, Y)$, is a subring of $F(X, Y)$ follows at once from the fact that T_1 is closed under multiplication. In fact, D_1 could be equivalently characterized as the semigroup ring of T_1 over F. By a "monomial" of D_1 we will mean the product of an element of F with an element of T_1, and a monomial of the form αX^0,

$\alpha \in F$, will be called a "constant term". Let M_1 denote the set of all elements of $D_1$with zero constant term. Since M_1 is clearly a maximal ideal of D_1, we can form the localization $(D_1)_{M_1}$, which we will denote by V.

Next define T_0 to be the subsemigroup $\{X^n \mid n = 0 \text{ or } n \in Z^+\} \cup \{Y^k X^m \mid k \in Z^+, m \geqq -k^2\}$ of T_1. Let D_0 denote the subring of D_1 consisting of all linear combinations over F of elements of T_0, and let $M_0 = M_1 \cap D_0$, which is clearly a maximal ideal of D_0.

1.6. EXAMPLE. *Let D_0 and M_0 be as defined above, and let D denote $(D_0)_{M_0}$. Then we will show that D has the following properties.*

(i) *The prime spectrum of D forms a tree; specifically, it forms a chain of length two.* (Proposition 1.9)

(ii) *Every finitely generated ideal of D containing a bounded element is invertible.* (Proposition 1.10)

(iii) *D is not a pre-Prüfer domain.* (Proposition 1.11)

1.7. LEMMA. *V is a rank-two valuation domain.*

Proof. We first observe that for each pair of monomials in D_1, one must divide the other, and so every element of D_1 is a product of a monomial in D_1 with an element of $D_1 \backslash M_1$. So in $V = (D_1)_{M_1}$, every element is a product of a monomial and a unit. Since these monomials already divide each other in D_1, it follows that for every pair of elements of V, one must divide the other. Hence V is a valuation domain. To show that V has rank two, we show that $P_V = (\{YX^{-n} \mid n \in Z^+\})$ and $M_V = (X) + P_V$ are the only proper prime ideals of V. The proof that they are each prime ideals is straightforward and will be omitted. That P_V is minimal follows at once from the fact that every element of P_V is in the radical of every nonzero ideal contained in P_V. Similarly, each element of $M_V \backslash P_V$ is in the radical of every ideal contained in M_V but not P_V, and therefore there are no prime ideals properly between M_V and P_V. Thus the proof is complete.

1.8. LEMMA. *V is integral over D.*

Proof. First we consider a monomial that is in D_1 but not in D_0. Then it must be of the form $\alpha Y^k X^m$ with $\alpha \in F, k \geqq 1$, and $m < -k^2$. We note that m is negative and, since $k \geqq 1$, that $m^2 \leqq (km)^2$. Thus $(\alpha Y^k X^m)^{-m} = \alpha^{-m} Y^{k(-m)} X^{-m^2}$ which is in D_0, since $-m^2 \geqq -(k(-m))^2$. Hence the monomial $\alpha Y^k X^m$ is integral over D_0. Since D_1 is generated over D_0 by such monomials, it follows that D_1 is integral over D_o.

Next we let S_0 be the multiplicative system $D_0 \backslash M_0$, and we observe that $(D_1)_{S_0}$ is integral over $(D_0)_{S_0}$ [5, Proposition 4.5, p. 84]. But M_1 is the only prime ideal of D_1 lying over M_0. Hence [3, Theorem 11.11, p. 107], $(D_1)_{S_0} = (D_1)_{M_1}$. In other words, $V = (D_1)_{M_1}$ is integral over $(D_0)_{S_0} = (D_0)_{M_0} = D$.

1.9. PROPOSITION. *The prime spectrum of D is* $(0) \subsetneqq P \subsetneqq M$, *where* $P = (\{Y^n X^{-n^2}\}_{n=1}^{\infty})$ *and* $M = (X) + P$.

Proof. Since V is integral over D, the prime ideals D are precisely the contractions of the prime ideals of V. These contractions are clearly the ideals P and M described in the statement of the proposition.

1.10. PROPOSITION. *Every finitely generated ideal of D containing a bounded element is invertible.*

Proof. We will show that the only ideal of D that contains a bounded element is D itself, or equivalently, that the maximal ideal M consists entirely of unbounded elements. Since the bounded elements of D form a saturated multiplicative system, the set of unbounded elements must be a union of prime ideals. Hence we need only demonstrate the existence of an unbounded element in $M \backslash P$. We will show that X is such an element. Suppose d is in $(\cap_{i=1}^{\infty} X^i D) \backslash (0)$. We may ignore the denominator and take d to be an element of D_0, a linear combination over F of elements in T_0. Let $\alpha Y^{k_0} X^{m_0}, \alpha \in F$, be a nonzero monomial that appears in the expansion for d. Choose $n_0 > k_0^2 + m_0$. Then by our choice of d, we know that $d \in X^{n_0} D$ and thus that there exists $d_0 \in D_0$ and $s_0 \in D_0 \backslash M_0$ such that $X^{n_0}(d_0/s_o) = d$. In other words $d_0 = ds_0 X^{-n_0}$. But the expansion of d includes the nonzero monomial $\alpha Y^{k_0} X^{m_0}$. Also since $s_0 \notin M_0$, it has nonzero constant term. Thus the expansion of $ds_0 X^{-n_0}$ includes a nonzero monomial of the form $\beta Y^{k_0} X^{m_0 - n_0}$, $\beta \in F$. But $m_0 - n_0 < -k_0^2$, so the expression $Y^{k_0} X^{m_0 - n_0}$ is not in T_0 and hence $\beta^{k_0} X^{m_0 - n_0}$ cannot appear as a monomial in the expansion of d_0. Thus we have contradicted our assumption that $d \in \cap_{i=1}^{\infty} X^i D \backslash (0)$, and conclude that $\cap_{i=1}^{\infty} X^i D = (0)$. End of proof.

1.11. PROPOSITION. *D is not a pre-Prüfer domain.*

Proof. Let A_0 be the ideal $(\{Y^n X^{-n^2} \mid n \geqq 2\})$ of D_0, and let A be the extension of A_0 to D. If D/A were a Prüfer ring, then it would be a local Prüfer ring. Hence, by the same argument used in the proof of Theorem 1.5 involving the regular total order property, it follows that every regular element of D/A must divide every zero divisor of D/A. But we will show that D/A does not have this condition by considering the elements $X+A$ and $YX^{-1}+A$ of D/A. Note that, by the way we defined A_0, X multiplies all elements of $D_0\backslash A_0$ into $D_0\backslash A_0$. A straightforward argument involving rings of quotients shows that in $D=(D_0)_{M_0}$, X multiplies elements of $D\backslash A$ into $D\backslash A$. In other words, $X+A$ is a regular element of D/A. Moreover YX^{-1} is a zero divisor in D/A since $Y(YX^{-1})\in A$. Suppose now that $X+A$ divides $YX^{-1}+A$. Then $(X+A)(d+A)=YX^{-1}+A$ for some $d\in D$, or equivalently, $Xd-YX^{-1}\in A$. Since $A=(A_0)_{M_0}$, there exists an element $s_0\in D_0\backslash M_0$ such that both $ds_0\in D_0$ and $(Xd-YX^{-1})s_0\in A_0$. But $YX^{-1}s_0$ includes in its expansion a nonzero monomial of the form $\alpha YX^{-1}, \alpha\in F$. In order for $Xds_0-YX^{-1}s_0$ to be in A_0, the term αYX^{-1} has to be cancelled out by a term in the expansion of Xds_0. But a nonzero monomial of the form $\beta YX^{-2}, \beta\in F$, cannot appear in the expansion of ds_0 since $ds_0\in D_0$ and so this cancellation cannot take place. Hence $X+A$ does not divide $YX^{-1}+A$ in D/A and so D/A cannot be a Prüfer ring. Hence D is not a pre-Prüfer ring.

2. The general case: a sufficient condition.

In this section we present a sufficient condition for a ring R to be a pre-Prüfer ring and we give an example to show that even when R is a domain this condition is not a necessary one. The sufficient condition involves the concept of a regularizable ideal which we define next.

2.1. DEFINITION. An ideal I of the ring R is said to be *regularizable* if there is a proper homomorphic image of R in which the image of I is a regular proper ideal.

While the definition indicates that to check whether an ideal I is regularizable, one may have to check the regularity of the image of I in all proper homomorphic images of R, the following result shows that it is enough to make this check only in the domains that are proper homomorphic images of R. Moreover, in this same result, we show that the condition that I be regularizable can be expressed entirely in terms of the ideal structure of R.

2.2. PROPOSITION. *Let I be an ideal of the ring R. The following statements are equivalent.*

(i) *I is regularizable.*

(ii) *There exists a domain D that is a proper homomorphic image of R and in which the image of I is a proper ideal.*

(iii) *There exists a nonzero prime ideal P such that I and P are not comaximal and $I\not\subseteq P$.*

Proof. (iii)$\Rightarrow$(ii): Let $D = R/P$. (ii) $\Rightarrow$ (i): Trivial. (i)$\Rightarrow$(iii): Let $\bar{R} = R/B$ be a proper homomorphic image of R in which $\bar{I}$ is a regular proper ideal. Let $\bar{M}$ be a maximal ideal of $\bar{R}$ containing $\bar{I}$. Then $\bar{I}_{\bar{M}}$ is a regular ideal of $\bar{R}_{\bar{M}}$ [3, apply Lemma 4.1, p. 34]. Since the set of zero divisors of a ring is a union of prime ideals and since $\bar{I}_{\bar{M}}$ is a regular ideal, there exists a prime ideal in $\bar{R}_{\bar{M}}$ not containing $\bar{I}_{\bar{M}}$. This prime ideal is the extension of a prime ideal of $\bar{R}$ which is contained in $\bar{M}$ and which is the image of a nonzero prime ideal P of R. From the fact that $\bar{I}_{\bar{M}} \not\subseteq \bar{P}_{\bar{M}}$, it follows that $\bar{I} \not\subseteq \bar{P}$. Consequently, $I \not\subseteq P$. Moreover, I and P are both contained in M and hence are not comaximal.

2.3. Proposition. *Let I be an ideal of the ring R and let $\bar{R}$ be a homomorphic image of R. If I is invertible and if $\bar{I}$, the image of I in $\bar{R}$, is regular, then $\bar{I}$ is invertible.*

Proof. Assume I is invertible. Then there exist $a_1, \ldots, a_n \in I$ and $x_1, \ldots, x_n$ in the total quotient ring of R such that $\Sigma x_i a_i = 1$ and $x_i I \subseteq R$ for $1 \leqq i \leqq n$. Let r be an element of I such that $\bar{r}$ is regular, and let y_i denote $x_i r$ for $1 \leqq i \leqq n$. Then by our choice of the x_i's each $y_i \in R$ and $\Sigma y_i a_i = r$. Hence $\Sigma \bar{y}_i \bar{a}_i = \bar{r}$, and since $\bar{r}$ is regular, $\Sigma(\bar{y}_i/\bar{r})\bar{a}_i = \bar{1}$. To show $\bar{I}$ is invertible, we need only show $(\bar{y}_i/\bar{r})\bar{I} \subseteq \bar{R}$ for each i. But if $z \in I$,

$$(\bar{y}_i/\bar{r})\bar{z} = (\bar{y}_i\bar{z})/\bar{r} = \overline{((x_i r)z)}/\bar{r} = \overline{(x_i r z)}/\bar{r} = \overline{((x_i z)r)}/\bar{r} = \overline{(x_i z)} \in \bar{R},$$

and the proof is complete.

Now we present the sufficient condition mentioned earlier for a ring to be a pre-Prüfer ring.

2.4. Theorem. *If R is a ring in which every finitely generated regularizable ideal is invertible, then R is a pre-Prüfer ring.*

Proof. Let $\bar{R}$ be a proper homomorphic image of R, and consider the finitely generated regular proper ideal $(\bar{a}_1, \ldots, \bar{a}_n)$ of $\bar{R}$. Then the ideal $(a_1, \ldots, a_n)$ of R is regularizable, since its image in $\bar{R}$ is $(\bar{a}_1, \ldots, \bar{a}_n)$; thus by hypothesis, $(a_1, \ldots, a_n)$ is invertible. Hence, by Proposition 2.3, $(\bar{a}_1, \ldots, \bar{a}_n)$ is invertible, and so $\bar{R}$ is a Prüfer ring. Therefore, R is a pre-Prüfer ring, as required.

Next we consider whether the condition that every finitely generated regularizable ideal be invertible is necessary in a pre-Prüfer ring. In the nondomain case, the requirement that an ideal be invertible of course implies that it must be a regular ideal. Thus a ring with a regularizable ideal consisting entirely of zero divisors cannot possibly satisfy the condition in Theorem 2.4. In particular, any pre-Prüfer ring with a prime ideal containing only zero divisors and having height greater than zero provides a counterexample to the necessity of the condition; such a ring is found in Example 3.2.

Since the above considerations obviously do not apply in the domain case, we are left with the question of whether every finitely generated regularizable ideal in a pre-Prüfer domain must be invertible. The remainder of this section is devoted to constructing a counterexample which answers this question in the negative.

Let F be a field with a rank-two valuation v defined on it. Let D_1 be the same ring as defined in the construction of Example 1.6 with respect to this field F. Let T_2 denote the subsemigroup $\{X^n \mid n = 0$ or $n \in Z^+\} \cup \{Y^k X^m \mid k \geqq 2, m \in Z\}$ of T_1, and let D_2 denote the subring of D_1 consisting of all linear combinations over F of elements of T_2.

Let $M_2 = M_1 \cap D_2$ and let D_3 denote the localization $(D_2)_{M_2}$. By an argument similar to that used in Lemma 1.8, it follows that the valuation ring $V = (D_1)_{M_1}$ is integral over D_3, and consequently that the prime spectrum of D_3 is $(0) \subsetneqq P_3 \subsetneqq M_3$, where $P_3 = (\{Y^k X^m \mid k \geqq 2, m \in Z\})$ and $M_3 = (X) + P_3$.

Next let w denote the extension of the valuation v to $F(X,Y)$ defined on $F[X,Y]$ by $w(\Sigma a_{ij} X^i Y^j) = \min \{v(a_{ij})\}$, and extended to the quotient field of $F[X,Y]$ in the standard fashion. Let W denote the valuation ring of w, a rank-two valuation ring with quotient field $F(X,Y)$ and with prime spectrum denoted by $(0) \subsetneqq P_W \subsetneqq M_W$.

2.5. EXAMPLE. *Let E denote the intersection of the domains D_3 and W defined above. We will show that E has the following properties.*

(i) *E is a pre-Prüfer ring.* (Proposition 2.7)

(ii) *E has a finitely generated regularizable ideal which is not invertible.* (Proposition 2.8)

2.6. LEMMA. *The maximal ideals of E are $M_3 \cap E$ and $M_W \cap E$. Moreover, $E_{(M_3 \cap E)} = D_3$ and $E_{(M_W \cap E)} = W$.*

Proof. Since E is the intersection of the local rings D_3 and W, the elements of $E \backslash (M_3 \cup M_W)$, being invertible in both D_3 and W, are invertible in E. Hence every nonunit of E lies inside $(M_3 \cap E) \cup (M_W \cap E)$, and consequently the maximal ideals of E must be precisely $M_3 \cap E$ and $M_W \cap E$.

Next consider $E_{(M_3 \cap E)}$. It is necessarily contained in D_3, so we need only show the reverse containment. Pick $d \in D_3$. Since the value group of w equals the value group of w restricted to F, we can find $\alpha \in F$ such that $w(\alpha) \geqq 0$ and $w(\alpha) + w(d) \geqq 0$. Hence α and αd are in W, and since $F \subseteq D_3$, both α and αd are in D_3. Therefore, α and $\alpha d \in E$. Moreover, since α is a unit in D_3, $\alpha \in E \backslash M_3$. In short, d is in $E_{(M_3 \cap E)}$, since $d = \alpha d/\alpha$, and so $D_3 = E_{(M_3 \cap E)}$.

Next we show $E_{(M_W \cap E)} = W$, and as before, we need only show $W \subseteq E_{(M_W \cap E)}$. Pick an element of W, say f/g where $f, g \in F[X, Y]$ and $w(f) \geqq w(g)$. Choose $\beta \in F$ with $w(\beta) = -w(g)$, and rewrite f/g as $(\beta Y^2 f)/(\beta Y^2 g)$. Note first that $w(\beta Y^2 g) = w(\beta) + w(g) = 0$, and hence $\beta Y^2 g \in W \backslash M_W$. Also $w(\beta Y^2 f) = w(\beta) + w(f) \geqq w(\beta) + w(g) \geqq 0$, so $\beta Y^2 f \in W$. Moreover, both $\beta Y^2 f$ and $\beta Y^2 g$ are in D_2, hence in D_3. So $\beta Y^2 f \in E$ and $\beta Y^2 g \in E \backslash M_W$. Since $f/g = (\beta Y^2 f)/(\beta Y^2 g)$ was an arbitrary element of W, $W \subseteq E_{(M_W \cap E)}$, as required.

2.7. PROPOSITION. *E is a pre-Prüfer domain.*

Proof. Let A be a proper ideal of E and let $\bar{E}$ denote the ring E/A. Choose a finitely generated regular ideal in $\bar{E}$. By taking a preimage of each of the generators, we obtain a finitely generated ideal J of E whose image $\bar{J}$ is the ideal originally chosen in $\bar{E}$. We wish to show that $\bar{J}$ is invertible.

We consider two cases. *Case I*: $A \subseteq M_3 \cap E$. In this case we will show J is invertible. It suffices to show $J_{(M_W \cap E)}$ and $J_{(M_3 \cap E)}$ are each principal. First $J_{(M_W \cap E)}$ is a finitely generated ideal of the valuation domain $W = E_{(M_W \cap E)}$,and hence is principal. If $J \not\subseteq M_3 \cap E$, then $J_{(M_3 \cap E)} = E_{(M_3 \cap E)}$ and we are done. So we now restrict our attention to the case where $J \subseteq M_3 \cap E$. Note that $P_3 \cap E$ is the only proper prime ideal of E properly contained in $M_3 \cap E$ since the only proper prime ideals of $D_3 = E_{(M_3 \cap E)}$ are P_3 and M_3. Thus $A \subseteq P_3 \cap E$, for otherwise $\bar{J}$ would be contained in the minimal prime ideal $\overline{M_3 \cap E}$, and hence would not be a regular ideal of $\bar{E}$[6, p. 1120]. Thus $\overline{P_3 \cap E}$ is a minimal prime ideal of $\bar{E}$. Hence J cannot be contained in $P_3 \cap E$, since if it were, $\bar{J}$ again would not be a regular ideal. So $J_{(M_3 \cap E)}$ is a finitely generated ideal of D_3 which is contained in M_3 but not in P_3. It is straightforward, from the way in which D_3 was constructed, that each element of $M_3 \backslash P_3$ is of the form uX^n where n is a positive integer and u is a unit of D_3 and that each such element divides every element of P_3. So among the generators of $J_{(M_3 \cap E)}$ there are some of the form uX^n. That generator for which the power of X is the smallest obviously divides all of the other generators, and hence generates the ideal. So $J_{(M_3 \cap E)}$ is principal, and hence J is invertible. Therefore by Proposition 2.3, $\bar{J}$ is invertible, as required.

Case II: $A \not\subseteq M_3 \cap E$. Then we pick a in $A \backslash (M_3 \cap E)$. Consider the ideal $J' = J + (a)$. Then J' is clearly invertible since $J'_{(M_3 \cap E)} =$

$E_{(M_3 \cap E)}$ and $J'_{(M_W \cap E)}$ is a finitely generated ideal of the valuation ring W, and hence principal. But $\bar{J}$ is the image of the invertible ideal J', and again we conclude that $\bar{J}$ is invertible.

2.8. PROPOSITION. *If α is an element of F which is in $M_W \backslash P_W$, then the ideal $(\alpha Y^2, \alpha Y^3)$ of E is regularizable but not invertible.*

Proof. Let I denote the ideal $(\alpha Y^2, \alpha Y^3)$ of E. Since $\alpha \in M_W$ and since $w(Y^2) = w(Y^3) = 0$, it follows that $I \subseteq M_W \cap E$, and so I and $P_W \cap E$ are not comaximal. Also $\alpha Y^2 \not\in P_W$ since $\alpha \not\in P_W$, and so we have that $I \not\subseteq P_W \cap E$. Hence by Proposition 2.2, I is regularizable.

To show I is not invertible, it is sufficient to show $I_{(M_3 \cap E)}$ is not principal. Since $E_{(M_3 \cap E)} = D_3$, and since α is a unit of D_3, we may rewrite $I_{(M_3 \cap E)}$ as $(Y^2, Y^3)D_3$. Suppose $(Y^2, Y^3)D_3$ is generated by a single element, say d. Since $D_3 = (D_2)_{M_2}$, we may take $d \in D_2$; in this case, d must be a linear combination over F of elements from T_2. Since V is a valuation overring of D_3 and $dD_3 = (Y^2, Y^3)D_3$, it follows that $dV = (Y^2, Y^3)V$ which equals Y^2V. Thus the value of d equals that of Y^2. The only way this can happen is when every monomial in the expression for d has an exponent on Y of at least two. By our choice of d, $Y^3 \in dD_3$. In other words, $s_2Y^3 = dd_2$ for some $d_2 \in D_2$ and $s_2 \in D_2 \setminus M_2$. By our choice of s_2, the expression s_2Y^3 must have a term $\beta Y^3, \beta \in F$. But the product dd_2 can have no such term, since no nonzero terms of d_2 have exponent one on Y. We have reached a contradiction of our assumption that $(Y^2, Y^3)D_3$ is principal, and thus conclude that I is not invertible.

3. The nondomain case: some counterexamples. In this section we will present three examples. The first and second are pre-Prüfer rings that do not satisfy the conditions that were shown to be necessary for pre-Prüfer domains in Theorems 1.2 and 1.5 respectively. The third is an example of a Prüfer ring which is not a pre-Prüfer ring, showing that another result in the domain case — namely that all Prüfer domains are pre-Prüfer domains — cannot be extended to the general case.

3.1. EXAMPLE. *Let F be a field and let R denote the ring $F[X,Y]_{(X,Y)}/(XY)$ where X and Y are indeterminates. Then R is a pre-Prüfer ring and its prime spectrum is not a tree.*

Proof. In $R, (\bar{X})$ and $(\bar{Y})$ are incomparable prime ideals both contained in the maximal ideal $(\bar{X}, \bar{Y})$ and so the prime spectrum of R is not a tree. To show that R is a pre-Prüfer ring we will denote by A the kernel of a proper homomorphism and examine the possible cases for

A. If A equals $(\bar{X})$ or $(\bar{Y})$, then R/A is naturally isomorphic to $F[Y]_{(Y)}$ or $F[X]_{(X)}$ respectively, and these, of course, are both Prüfer domains. If A contains either $(\bar{X})$ or $(\bar{Y})$ properly, then R/A is zero-dimensional since in this case $(\bar{X},\bar{Y})$ is the only prime ideal of R containing A. Therefore R/A is a total quotient ring, hence a Prüfer ring. For the final case suppose that A is a nonzero ideal of R containing neither $\bar{X}$ nor $\bar{Y}$. Pick a nonzero element of A of the form $\overline{p(X,Y)}$ where $p(X,Y) \in (X,Y)F[X,Y]$. Since $\bar{X}\bar{Y} = \bar{0}$, we can write

$$\overline{p(X,Y)} = p(\bar{X},\bar{Y}) = a_n\bar{X}^n + a_{n+1}\bar{X}^{n+1} + \cdots + a_t\bar{X}^t + b_m\bar{Y}^m + b_{m+1}\bar{Y}^{m+1} + \cdots + b_s\bar{Y}^s.$$

But then $p(\bar{X},\bar{Y}) = \bar{X}^n u_1 + \bar{Y}^m u_2$ where u_1 and u_2 are units or zero. If $u_1 = 0$, then $u_2 \neq 0$ and hence $\bar{Y}^m \in A$. If $u_1 \neq 0$, then $\bar{X}(\bar{X}^n u_1 + \bar{Y}^m u_2) = \bar{X}^{n+1}u_1 \in A$ and so $\bar{X}^{n+1} \in A$. In either case, we see that some power of $\bar{X}$ or of $\bar{Y}$ is in A. Assume that some power of $\bar{X}$ is in A and let k be the smallest positive integer such that $\bar{X}^k \in A$. Since $\bar{X} \notin A, k > 1$. Hence $\bar{X}^{k-1}(\bar{X},\bar{Y}) = (\bar{X}^k) \subseteq A$. Therefore, since $\bar{X}^{k-1} \notin A$, the ideal $(\bar{X},\bar{Y})/A$ consists entirely of zero divisors in R/A. But $(\bar{X},\bar{Y})/A$ is the unique maximal ideal of R/A, and consequently, R/A is again a total quotient ring. This completes the proof.

3.2. EXAMPLE. *Let v be a valuation on a field F with value group $Z \oplus Z$ with the lexicographic ordering. Let V be the valuation ring of v and let A be the ideal $\{a \in V \mid v(a) \geqq (1,1)\}$ of V. Then V/A is a pre-Prüfer ring in which there exists a noninvertible finitely generated ideal containing a bounded element.*

Proof. Let $\bar{V}$ denote V/A. Then $\bar{V}$, being the homomorphic image of a Prüfer domain, is a pre-Prüfer ring. Let M and P be the maximal and minimal primes in the rank-two valuation domain V. Any element of $\bar{V}$ in $\bar{M}\backslash\bar{P}$ is bounded since all of the powers of the principal ideal of such an element contain the nonzero ideal $\bar{P}$. An element x in V with value (1,0) is not in A but x multiplies each nonunit of V into A. Hence $\bar{V}$ is a total quotient ring. If $\bar{B}$ is any finitely generated proper ideal of $\bar{V}$ not contained in $\bar{P}$, then $\bar{B}$ contains a bounded element but is not invertible since it is not regular.

3.3. EXAMPLE. *Let F be a field and let*

$$R = F[X, Y, Z]_{(X,Y,Z)}/(XZ, YZ, Z^2)$$

where X, Y, and Z are indeterminates. Then R is a Prüfer ring which is not a pre-Prüfer ring.

Proof. In the ring R, the image of Z is a nonzero element which annihilates every nonunit of R. So R is a total quotient ring and hence a Prüfer ring. Since (Z) contains (XZ, YZ, Z^2), $F[X, Y]_{(X,Y)}$ is a proper homomorphic image of R. But $F[X, Y]_{(X,Y)}$ is not a Prüfer ring and so R is not a pre-Prüfer ring.

REFERENCES

1. Monte B. Boisen, Jr. and Max D. Larsen, *On Prüfer rings as images of Prüfer domains*, Proc. Amer. Math. Soc., **40** (1973), 87–90.
2. ———, *Prüfer and valuation rings with zero divisors*, Pacific J. Math., **40** (1972), 7–12.
3. Robert Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.
4. Malcolm Griffin, *Prüfer rings with zero divisors*, J. Reine Angew. Math., **239/240** (1969), 55–67. MR41 # 188.
5. Max D. Larsen and Paul J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, 1971.
6. Joseph Lipman, *Some properties of normalization and localization*, Proc. Amer. Math. Soc., **16** (1965), 1120–1122.
7. Jack Ohm, *Some counterexamples related to integral closure in* $D[[x]]$, Trans. Amer. Math. Soc., **122** (1966), 321–333.

Received January 29, 1974.

VIRGINIA POLYTECHNIC INSTITUTE
AND
STATE UNIVERSITY