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A CLASS OF SYMMETRIC DIFFERENTIAL OPERATORS WITH DEFICIENCY INDICES (1, 1)

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Let $\mathscr{H}$ denote the Hilbert space of analytic functions on the unit disk which are square summable with respect to the usual area measure. In this paper we show that every symmetric differential operator of order two or more having the form $L = \sum_{i=0}^{n} (a_{i+1}(i)z^{i+1} + a_{i-1}(i)z^{i-1})D^i$, $a_{-1}(0) = 0$, has defect indices (1, 1) and hence has self-adjont extensions in $\mathscr{H}$. We are also able to show that $L + M$ has defect indices (1, 1) where M is a symmetric Euler operator of order $n - 1$, $M = \sum_{i=0}^{n-1} b_i z^i D^i$, provided that $|b_{n-1}| < (n-1)|a_{n+1}(n)|$.

In what follows $\mathscr{H}$ denotes the square summable analytic functions in $|z| < 1$, with inner product $(f, g) = \iint_{|z|<1} f(z)\overline{g(z)}dxdy$. A complete orthonormal set for $\mathscr{H}$ is provided by the functions $\phi_n(z) = ((n+1)/\pi)^{1/2}z^n$, $n = 0, 1, \cdots$, from which it follows that $\sum_{n=0}^{\infty} a_n z^n$ represents an element of $\mathscr{H}$ if and only if $\sum_{n=0}^{\infty} |a_n|^2/(n+1)$ converges. The formal differential operator $L = \sum_{i=0}^{n} p_i D^i$, $p_i \in \mathscr{H}$, is said to be formally symmetric if $L\phi_n \in \mathscr{H}$ for all n and $(L\phi_n, \phi_m) = (\phi_n, L\phi_m)$. Let $T_0 f = Lf$ for f in the span of the ϕ_n. Then the closure of T_0, S, is a symmetric operator and its defect indices $m^+(m^-)$ are just the number of linearly independent solutions of $L\phi = \lambda^+\phi(L\phi = \lambda^-\phi)$ in $\mathscr{H}$, where $\mathrm{Im}(\lambda^+) > 0$ $(\mathrm{Im}(\lambda^-) < 0)$. [2]

It is known, [1], that if L is an nth order formally symmetric operator then the p_i are polynomials of degree at most $n + i$, $p_i(z) = \sum_{k=0}^{n+1} a_k(i)z^k$, where the $a_k(i)$ satisfy the following linear systems for $p = 0, 1, \cdots, n$

$$(1) \qquad S_p\colon \sum_{k=0}^{n} a_{k+p}(k)B(i, k) = \sum_{k=p}^{n} \bar{a}_{k-p}(k)B(i+p, k)A^2(i+p, i) \qquad i = 0, 1, 2, \cdots .$$

Where,

$$(2) \qquad \begin{aligned} A(i, j) &= [(i+1)/(j+1)]^{1/2} \\ B(i, j) &= i!/(i-j)! \quad i \geqq j \\ &= 0 \qquad\qquad i < j . \end{aligned}$$

Setting $a_{k+p}(k) = a_{k-p}(k) = 0$ for $p \neq 1$, L has the form

$$(3) \qquad \sum_{i=0}^{n} (a_{i+1}(i)z^{i+1} + a_{i-1}(i)z^{i-1})D^i , \qquad a_{-1}(0) = 0 .$$

In this paper we show that such L's give rise to operators with defect indices (1, 1). Before doing so it is necessary to determine the nature of the relationships among the $a_{i\pm1}(i)$ implied by S_1.

LEMMA. *The $a_{i\pm1}(i)$ satisfy S_1 if and only if*

$$\begin{aligned} a_{i+1}(i) &= (i+2)\bar{a}_i(i+1) + \bar{a}_{i-1}(i) \quad i = 0, 1, \cdots, n\,, \\ a_{-1}(0) &= a_n(n+1) = 0\,. \end{aligned} \tag{4}$$

Proof. The proof hinges on the algorithm provided by Theorem 2.3 of [1] which states that the system S_p is satisfied if and only if

$$j!\, a_{j+p}(j) = R_0^j \qquad j = 0, 1, \cdots, n\,, \tag{5}$$

where $R_i^0 = \sum_{k=p}^n \bar{a}_{k-p}(k)B(i+p,k)A^2(i+p,i)$, and the R_i^j are defined recurrsively by

$$R_i^j = R_{i+1}^{j-1} - R_i^{j-1}\,. \tag{6}$$

For $p = 1$, $R_i^0 = \sum_{k=1}^n \bar{a}_{k-1}(k)((i+2)i!)/(i+1-k)!$, where we agree to set the terms involving $i+1-k<0$ equal to 0. Setting $i = 0$, (5) becomes

$$a_1(0) = \sum_{k=1}^n \bar{a}_{k-1}(k)2/(1-k)! = 2\bar{a}_0(1)\,.$$

We now show that for $j \geqq 1$ R_i^j is given by

$$R_i^j = \sum_{k=1}^n \bar{a}_{k-1}(k)[i!/(i+j+1-k)!](ik+2k-j)P_j(k)\,, \tag{7}$$

where

$$P_1(k) = 1 \quad \text{and} \quad P_j(k) = (k-1)\cdots(k-j+1)\,, \quad j > 1\,,$$

and

$$[i!/(i+j+1-k)!] = 0\,, \qquad i+j+1-k < 0\,.$$

A simple calculation yields

$$R_i^1 = R_{i+1}^0 - R_i^0 = \sum_{k=1}^n \bar{a}_{k-1}(k)[i!/(i+2-k)!](ik+2k-1)\,,$$

so that (7) holds for $j = 1$. Assumming that (7) holds for j we obtain

$$R_i^{j+1} = \sum_{k=1}^n \bar{a}_{k-1}(k)[i!/(i+j+2-k)!]P_j(k)Q\,,$$

where

$$\begin{aligned} Q &= (i+1)(ik+3k-j) - (i+j+2-k)(ik+2k-j) \\ &= (ik+2k-j-1)(k-j) \,. \end{aligned}$$

Hence we have $R_i^{j+1} = \sum_{k=1}^{n} \bar{a}_{k-1}(k)[i!/(i+j+2-k)!](ik+2k-j-1)P_{j+1}(k)$. For $j = n$, (5) and (7) yield

$$n!\, a_{n+1}(n) = \sum_{k=1}^{n} \bar{a}_{k-1}(k)(2k-n)P_n(k)/(n+1-k)!$$

Since $P_n(k) = 0$ for $k = 1, \cdots, n-1$, the series reduces to $\bar{a}_{n-1}(n)[n(n-1)\cdots 1]$, from which it follows that $a_{n+1}(n) = \bar{a}_{n-1}(n)$. For $1 \leqq j < n$, (5) and (7) yield

$$\begin{aligned} j!\, a_{j+1}(j) &= \sum_{k=1}^{n} \bar{a}_{k-1}(k)(2k-j)P_j(k)/(j+1-k)! \\ &= \sum_{k=j}^{n} \bar{a}_{k-1}(k)(2k-j)P_j(k)/(j+1-k)! \,, \end{aligned}$$

since $P_j(k) = 0$ for $k = 1, \cdots j-1$. On the other hand, the terms for $j+1-k < 0$ vanish leaving us with

$$j!\, a_{j+1}(j) = \bar{a}_{j-1}(j) j P_j(j) + \bar{a}_j(j+1)(j+2)P_j(j+1) \,.$$

Since $P_j(j) = (j-1)!$ and $P_j(j+1) = j!$, we have

$$a_{j+1}(j) = \bar{a}_{j-1}(j) + (j+2)\bar{a}_j(j+1) \,.$$

THEOREM. *Let L be the operator of* (3) *then S has defect indices* $m^+ = m^- = 1$.

Proof. The idea of the proof is to show that the equation $L\phi = \lambda\phi (\operatorname{Im}\lambda \neq 0)$ has exactly one power series solution $\phi(z) = \sum_{j=0}^{\infty} \alpha_j z^j$ and that $|\alpha_j|$ is $O(j^{-1/p})$ for some positive integer p. This implies that $\sum_{j=0}^{\infty} |\alpha_j|^2/(j+1)$ converges and $\phi \in \mathscr{H}$, thus $m^+ = m^- = 1$.

Let $\phi(z) = \sum_{j=0}^{\infty} \alpha_j z^j$ be a formal power series solution of $L\phi = \lambda\phi$. Substituting this series into $L\phi = \lambda\phi$ we obtain

$$L\phi(z) = \sum_{j=0}^{\infty} [\alpha_j C_j z^{j+1} + \alpha_j D_j z^{j-1}] = \sum_{j=0}^{\infty} \lambda \alpha_j z^j \,,$$

where

$$\begin{aligned} C_j &= \sum_{i=0}^{n} a_{i+1}(i)\pi_i(j) \\ D_j &= \sum_{i=1}^{n} a_{i-1}(i)\pi_i(j) \,, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\pi_0(x) = 1 \\ &\pi_i(x) = x(x-1)\cdots(x-i+1) \qquad i = 1, 2, \cdots, n \,. \end{aligned}$$

Since D_0 vanishes, we have

$$\alpha_1 D_1 + \sum_{j=1}^{\infty} (\alpha_{j+1} D_{j+1} + \alpha_{j-1} C_{j-1}) z^j = \sum_{j=0}^{\infty} \lambda \alpha_j z^j \,, \tag{9}$$

whence

$$\begin{aligned} &\alpha_1 D_1 = \lambda \alpha_0 \qquad (\operatorname{Im} \lambda \neq 0) \\ &\alpha_{j+1} D_{j+1} + \alpha_{j-1} C_{j-1} = \lambda \alpha_j \qquad\qquad j = 1, 2, \cdots . \end{aligned} \tag{10}$$

If $D_1 \neq 0$ we have $\alpha_1 = \lambda \alpha_0 / D_1$ and (10) can be solved recurrsively for $\alpha_2, \alpha_3, \cdots$, in terms of α_0, provided that D_j never vanishes for $j = 2, 3, \cdots$. Thus we have the single formal power series solution $\phi(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots$. If $D_1 = 0$, let ρ be the smallest positive integer for which $D_\rho \neq 0$, then $\alpha_j = 0$ for $j < \rho - 1$ and $\alpha_\rho = \lambda \alpha_{\rho-1}/D_\rho$, and (10) can be solved recurrsively for $\alpha_{\rho+1}, \alpha_{\rho+2}, \cdots$, in terms of $\alpha_{\rho-1}$, provided that D_j never vanishes for $j > \rho$. In this case we have the single formal power series solution $\phi(z) = z^{\rho-1} + \alpha_\rho z^\rho + \cdots$. The case when D_j vanishes for some $j > \rho$ presents some complications and will be considered later in the proof.

It is not difficult to see that D_1 thru D_n are not all zero. From (8) we have

$$\begin{aligned} D_1 &= a_0(1) \\ D_2 &= a_0(1)\pi_1(2) + a_1(2)\pi_2(2) \\ &\vdots \\ D_n &= a_0(1)\pi_1(n) + \cdots + a_{n-1}(n)\pi_n(n) \,. \end{aligned}$$

Since the $\pi_i(j) \neq 0$ for $i \leqq j = 1, 2, \cdots, n$, it follows that $D_1 = \cdots = D_n = 0$ implies $a_0(1) = a_1(2) = \cdots = a_{n-1}(n) = 0$. But $a_{n+1}(n) = \overline{a_{n-1}(n)} = 0$, (4), contradicting the fact that L is of order n.

Suppose then that D_ρ, $1 \leqq \rho \leqq n$, is the first nonvanishing D_j and that $D_j \neq 0$ for $j > \rho$. We then have at most one analytic solution of the form $\phi(z) = z^{\rho-1} + \alpha_\rho z^\rho + \cdots$.

Solving (10) for α_{j+1} and estimating we obtain

$$|\alpha_{j+1}| \leqq \left| \frac{\lambda}{D_{j+1}} \right| |\alpha_j| + \left| \frac{C_{j-1}}{D_{j+1}} \right| |\alpha_{j-1}| \,. \tag{11}$$

We now estimate the coefficients of $|\alpha_j|$ and $|\alpha_{j-1}|$ for large j. To do this we first investigate the nature of C_{j-1} and D_{j+1} as polynomials in j.

From (8) and the fact that $\pi_k(x) = x^k - k/2(k-1)x^{k-1} + \cdots$, it follows that D_{j+1} is a polynomial in j of degree n,

$$\begin{aligned} D_{j+1} = a_{n-1}(n) j^n + [a_{n-1}(n)\{n - (n-1)n/2\} + a_{n-2}(n-1)] j^{n-1} \\ + \text{lower powers of } j \,. \end{aligned} \tag{12}$$

Similarly,

$$(13)\quad C_{j-1} = a_{n+1}(n)j^n + [a_{n+1}(n)\{-n - (n-1)n/2\} + a_n(n-1)]j^{n-1} + \text{lower powers of } j.$$

From the lemma we know that $\bar{a}_{n-1}(n) = a_{n+1}(n)$ and that

$$\begin{aligned} a_n(n-1) &= \bar{a}_{n-2}(n-1) + (n+1)\bar{a}_{n-1}(n) \\ &= \bar{a}_{n-2}(n-1) + (n+1)a_{n+1}(n) . \end{aligned}$$

Hence, (12) and (13) become

$$(14)\quad \begin{aligned} D_{j+1} &= \bar{a}_{n+1}(n)j^n \\ &+ [\bar{a}_{n+1}(n)\{n - (n-1)n/2\} + a_{n-2}(n-1)]j^{n-1} + \cdots \end{aligned}$$

$$(15)\quad \begin{aligned} C_{j-1} &= a_{n+1}(n)j^n \\ &+ [a_{n+1}(n)\{1 - (n-1)n/2\} + \bar{a}_{n-2}(n-1)]j^{n-1} + \cdots . \end{aligned}$$

Thus we obtain, for $j > \rho$,

$$(16)\quad \frac{C_{j-i}}{D_{j+1}} = \omega\theta(j) = \omega\frac{j^n + [(\varDelta + 1) + \theta]j^{n-1} + \cdots}{j^n + [(\varDelta + n) + \bar{\theta}]j^{n-1} + \cdots} ,$$

where $|\omega| = 1$, $\varDelta = -(n-1)n/2 < 0$, $\theta = \bar{a}_{n-2}(n-1)/a_{n+1}(n)$.

Concerning $|\theta(j)|$ we obtain, upon dividing,

$$\theta(j) = 1 - \frac{n-1}{j} + i\frac{2\,\mathrm{Im}\,(\theta)}{j} + O\,(j^{-2})$$

Thus $|\theta(j)|^2 = 1 - (2(n-1))/j + O\,(j^{-2})$, from which it follows that

$$|C_{j-1}/D_{j+1}| = 1 - \frac{(n-1)}{j} + O\,(j^{-2}) .$$

For $\xi > 0$ we note that $|C_{j-1}/D_{j+1}| \leqq 1 - \xi j^{-1}$ for j sufficiently large if and only if $-(n-1) < -\xi$, or $\xi < n-1$. Hence we have

$$(17)\quad \begin{aligned} &|C_{j-1}/D_{j+1}| \leqq 1 - \frac{\xi}{j} , \quad \text{for } j \text{ sufficiently large and} \\ &0 < \xi < n-1 . \end{aligned}$$

Using (11), (17) and the fact that $|D_{j+1}^{-1}|$ is $O\,(j^{-n})$ we have, for j sufficiently large,

$$(18)\quad |\alpha_{j+1}| \leqq (1 - \gamma j^{-1}) \max\{|\alpha_j|, |\alpha_{j-1}|\} ,$$

where $0 < \gamma < \xi < n-1$.

Using the arguments given in [3], p 3-4 it follows from (18), that there exists a $K > 0$ and a positive integer p such that $|\alpha_j| \leqq$

$Kj^{-1/p}$ for j sufficiently large. Hence $\sum_{j=0}^{\infty} |\alpha_j|^2/(j+1)$ converges and $\phi \in \mathscr{H}$. To complete the proof we have only to deal with the case where D_j vanishes for some $j > \rho$.

Suppose $D_k = 0$ for some integer $k > \rho$. Since D_j is a polynomial in j of degree n, there is a largest integer k such that $D_k = 0$. The power series solution $\hat{\phi}$ obtained from (10) by taking $\alpha_j = 0$, $j = 0, 1, \cdots, k-1$, and solving recurrsively for α_j, $j > k$, in terms of α_k is, as we have seen, in $\mathscr{H}$. If there are other power series solutions for which all the α_j, $j = 0, \cdots, k-1$, are not zero, these solutions would be in $\mathscr{H}$ as well, hence $m_+(m_-) \geqq 2$. We now show that this is not the case by demonstrating the existence of λ, $\operatorname{Im}(\lambda) \neq 0$, for which $\hat{\phi}$ is the only power series solution possible.

If $D_k = 0$, we obtain the following homogeneous system of equations in $\alpha_0, \alpha_1, \cdots, \alpha_{k-1}$.

$$
\begin{aligned}
-\lambda\alpha_0 - D_1\alpha_1 &= 0 \\
C_{j-1}\alpha_{j-1} - \lambda\alpha_j + D_{j+1}\alpha_{j+1} &= 0 \qquad j = 1, 2, \cdots, k-2 \\
C_{k-2}\alpha_{k-2} - \lambda\alpha_{k-1} &= 0
\end{aligned}
\tag{19}
$$

For the system determinant we have $\Delta_k(\lambda) = (-1)^k\lambda^k + \cdots$, and hence $\Delta_k(\lambda)$ vanishes at only a finite number of points in the complex plane. Thus we can find λ, $\operatorname{Im}(\lambda) \neq 0$, for which $\Delta_k(\lambda) \neq 0$, which implies that $\alpha_0 = \cdots = \alpha_{k-1} = 0$. Hence there is only one analytic solution of $L\phi = \lambda\phi$, namely $\hat{\phi}$.

The defect indices of S are not changed if we add to L the formally symmetric Euler operator M of order $n-1$, provided the leading coefficient of M is not too large. The proof of this follows directly from the proof of the theorem. Let $M = \sum_{i=0}^{n-1} b_i z^i D^i$, b_i real, and take $L_1 = L + M$. Since $M(z^j) = p(j)z^j$, where $p(j) = b_0 + b_1 j + \cdots + b_{n-1} j(j-1) \cdots (j-n+2)$, equation (9) becomes

$$
\begin{aligned}
\alpha_1 D_1 + \sum_{j=1}^{\infty} (\alpha_{j+1}D_{j+1} + \alpha_{j-1}C_{j-1})z^j &= \sum_{j=0}^{\infty} (\lambda - p(j))\alpha_j z^j \\
&= \sum_{j=0}^{\infty} \lambda_j \alpha_j z^j \,, \quad \lambda_j = \lambda - p(j) \,,
\end{aligned}
\tag{20}
$$

where $\operatorname{Im}(\lambda_j) \neq 0$ $j = 0, 1, \cdots$, since $p(j)$ is always real. Hence,

$$
\begin{aligned}
&\alpha_1 D_1 = \lambda_0 \alpha_0 \\
&\alpha_{j+1}D_{j+1} + \alpha_{j-1}C_{j-1} = \lambda_j \alpha_j \qquad j = 1, 2, \cdots .
\end{aligned}
$$

Just as in the proof of the theorem, we have the single power series solution $\phi(z) = z^{\rho-1} + \alpha_\rho z^\rho + \cdots$, where D_ρ is the first nonvanishing D_j and $D_j \neq 0$, $j > \rho$. Moreover, for j sufficiently large,

$$
|\alpha_{j+1}| \leqq \left|\frac{\lambda_j}{D_{j+1}}\right| |\alpha_j| + \left(1 - \frac{\xi}{j}\right) |\alpha_{j-1}| \,,
\tag{21}
$$

where $0 < \xi < n - 1$.

The estimates on the growth of the $|\alpha_j|$, [3], will go through if we can show that

$$|\alpha_{j+1}| \leqq (1 - \gamma j^{-1}) \max \{|\alpha_j| \, |\alpha_{j-1}|\} , \tag{22}$$

for j sufficiently large and $\gamma > 0$. Using (4), (12), and the fact that $\lambda_j = b_{n-1} j^{n-1} + \cdots$, we have

$$|\lambda_j / D_{j+1}| = \varepsilon j^{-1} + O(j^{-2}) , \tag{23}$$

where

$$\varepsilon = |b_{n-1}/a_{n+1}(n)| .$$

From (21) and (23) it follows that (22) holds provided $\xi - \varepsilon > \gamma > 0$ or $\xi > \varepsilon$. But $\xi < n - 1$, so we must have $|b_{n-1}| < (n-1)|a_{n+1}(n)|$.

The case when $D_j = 0$ for $j > \rho$ is handled in the same manner as before, by showing that there exist λ, $\operatorname{Im}(\lambda) \neq 0$, such that $\hat{\phi}$ is the only power series solution of $L_1\phi = \lambda\phi$.

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