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SOME GENERALIZATIONS OF SCHAUDER'S THEOREM IN LOCALLY CONVEX SPACES

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Let E_1, E_2, E_3, E_4 be four locally convex Hausdorff spaces (l.c.s.); denote by $\mathscr{L}_b(E_i, E_k)$ the set of all continuous linear operators from E_i into E_k with the topology of uniform convergence on bounded subsets of E_i. Given two linear operators $f \in \mathscr{L}(E_1, E_2)$ and $g \in \mathscr{L}(E_3, E_4)$, consider the generalized adjoint operator $\mathrm{Hom}\,(f, g): \mathscr{L}_b(E_2, E_3 \to \mathscr{L}_b(E_1, E_4)$ defined by $u \to \mathrm{Hom}\,(f, g)u = g \circ u \circ f$. This paper deals with transformation properties of $\mathrm{Hom}\,(f, g)$ and their interactions with those of f and g. This purpose may be illustrated by a result due to K. Vala which generalizes Schauder's well-known theorem concerning precompact operators and their adjoints on normed spaces: Let all spaces under consideration be normed, let f and g both be nonzero. Then $\mathrm{Hom}\,(f, g)$ is a precompact operator if and only if f and g are precompact operators. In the present paper bounded and precompact operators on l.c.s. are investigated.

1. Introduction. Unless otherwise stated notation used throughout this paper will be that of Schaefer [6]. E_i ($i \in N$) will always denote a l.c.s. over the field of complex numbers C, $\mathscr{P}_i$ denotes the set of all continuous seminorms, and $\mathscr{B}(E_i)$ a fundamental system of bounded subsets of E_i. By U_{p_i} denote the closed, convex, circled neighborhood of zero $\{x \in E_i: P_i(x) \leqq 1\}$ in E_i. If B_i runs through $\mathscr{B}(E_i)$, p_k through $\mathscr{P}_k$, the family

$$W(B_i, U_{p_k}) = \{u \in \mathscr{L}(E_i, E_k): u(B_i) \subset U_{p_k}\}$$

is a neighborhood base of zero in $\mathscr{L}_b(E_i, E_k)$.

DEFINITION 1.1. A linear operator $u: E_i \to E_k$ is said to be *precompact* (*compact* or *bounded*) if there exists a neighborhood of zero U_{p_i} in E_i such that $u(U_{p_i})$ is a precompact (relatively compact or bounded) subset of E_k. A linear operator $u: E_i \to E_k$ is said to be *semi-precompact* if u maps bounded subsets of E_i into precompact subsets of E_k.

Of course bounded linear operators are continuous, whereas semi-precompact operators need not. Let $u \in \mathscr{L}(E_1, E_2)$, $w \in \mathscr{L}(E_3, E_4)$, and let $v \in \mathscr{L}(E_2, E_3)$ be a semi-precompact (precompact, compact, or bounded) operator. Then $v \circ u \in \mathscr{L}(E_1, E_3)$ and $w \circ v \mathscr{L}(E_2, E_4)$ are semi-precompact (precompact, compact, or bounded) operators.

A subset $\mathscr{H} \subset \mathscr{L}(E_i, E_k)$ is said to be *collectively precompact* if there exists a neighborhood of zero U_{p_i} in E_i such that $\mathscr{H}(U_{p_i}) = \{u(x): u \in \mathscr{H}, x \in U_{p_i}\}$ is a precompact subset of E_k. Let $f \in \mathscr{L}(E_1, E_2)$ and $g \in \mathscr{L}(E_3, E_4)$ both be nonzero. Then the linear mapping $\operatorname{Hom}(f, g): \mathscr{L}_b(E_2, E_3) \to \mathscr{L}_b(E_1, E_4)$ given by $u \to \operatorname{Hom}(f, g)u = g \circ u \circ f$ is continuous, as an easy calculation shows. By setting $E_3 = E_4 = C$ and $g = \mathrm{id}_C$ (identity map on C) $\operatorname{Hom}(f, g)$ becomes the adjoint operator f' of f. That is why one may call $\operatorname{Hom}(f, g)$ a *generalized adjoint operator*. In generalizing Schauder's well-known theorem Vala [7] has obtained the following result.

THEOREM 1.1. *Let E_1, E_2, E_3, E_4 denote four normed linear spaces, let $f \in \mathscr{L}(E_1, E_2)$ and $g \in \mathscr{L}(E_3, E_4)$ both be nonzero. Then the mapping*

$$\operatorname{Hom}(f, g): \mathscr{L}_b(E_2, E_3) \longrightarrow \mathscr{L}_b(E_1, E_4)$$

given by $u \to \operatorname{Hom}(f, g)u = g \circ u \circ f$ is precompact if and only if both f and g are precompact.

We shall be interested in similar results in case of l.c.s.. For that purpose the notion of an *infrabarrelled* l.c.s. is needed. The definition given below is not entirely standard but is easily proved equivalent to the standard one (see Horváth [4, Definition 2, p. 217]). Just this definition gives more insight in why infrabarrelled spaces are involved.

DEFINITION 1.1. A l.c.s. E_1 is said to be *infrabarrelled* if each bounded subset of $\mathscr{L}_b(E_1, E_2)$ is equicontinuous for all l.c.s. E_2.

As in normed linear spaces semi-precompact operators are precompact, the following result due to Apiola [1] and Geue [3] is a generalization of Vala's theorem.

THEOREM 1.2. *Let E_1, E_2, E_3, E_4 denote four* l.c.s. *and assume in addition E_2 is infrabarrelled. Let $f \in \mathscr{L}(E_1, E_2)$ and $g \in \mathscr{L}(E_3, E_4)$ both be nonzero. Then*

$$\operatorname{Hom}(f, g): \mathscr{L}_b(E_2, E_3) \longrightarrow \mathscr{L}_b(E_1, E_4)$$

given by $u \to \operatorname{Hom}(f, g)u = g \circ u \circ f$ is a semi-precompact continuous linear operator if and only if both f and g are semi-precompact.

REMARK. If one drops the assumption of E_2 being infrabarrelled, the implication 'Hom (f, g) semi-precompact $\curvearrowright$ both f and g semi-precompact' remains true, whereas the converse implication in general becomes false (see Apiola [1, Theorem 3.3.]).

In this paper we shall be interested in what happens if we consider precompact or bounded linear operators instead of semi-precompact continuous linear operators. Considering the precompact situation the following result due to Floret [2, § 13] is rather instructive.

THEOREM 1.3. *There exists a reflexive Fréchet space* E_1, *a Banach space* E_2, *and a semi-precompact operator* $f \in \mathscr{L}(E_1, E_2)$ *such that the adjoint operator* $f': E'_{2b} \rightarrow E'_{1b}$ *of* f *is precompact without* f *being precompact itself.*

REMARK. The theorem demonstrates that if Hom (f, g) is precompact, f in general is not. But as E_1 is a reflexive l.c.s. this result also shows that Hom (f, g) in general will not be precompact even if both f and g are.

2. Transformation properties of bounded operators. In dealing with bounded operators we have

THEOREM 2.1. *Let* E_1, E_2, E_3, E_4 *denote four* l.c.s., *and let* $f \in \mathscr{L}(E_1, E_2)$, $g \in \mathscr{L}(E_3, E_4)$. *Then*

(i) *If* f *and* g *both are bounded linear operators, then* Hom (f, g) *is bounded.*

(ii) *Let* f *and* g *both be nonzero and assume in addition* E_1 *is infrabarrelled. Then* Hom (f, g) *is bounded if and only if both* f *and* g *are bounded operators. If* Hom (f, g) *is precompact, then* g *is a precompact operator.*

The following lemma gives more insight into the proof of the theorem.

LEMMA 2.1. *Let* E_1 *and* E_2 *denote two* l.c.s., *let* $x_0 \in B_1 \in \mathscr{B}(E_1)$ *and* $p_1 \in \mathscr{P}_1$ *such that* $p_1(B_1) = \sup_{x \in B_1} p_1(x) = p_1(x_0) = 1$. *Then we have*

$$\{u(x_0): u \in W(B_1, U_{p_2}) \text{ and } \dim u(E_1) = 1\} = U_{p_2}$$

for all $p_2 \in \mathscr{P}_2$.

Proof. The inclusion $\subset$ is evident. For the converse inclusion let $y_0 \in U_{p_2}$ be given. By the theorem of Hahn-Banach there exists a $\varphi \in E_1'$ such that $\varphi(x_0) = 1$ and $|\varphi(x)| \leqq p_1(x)$ for all $x \in E_1$. Now the mapping $x \longrightarrow u(x) = \varphi(x)y_0$ is in $\mathscr{L}(E_1, E_2)$. Furthermore $u(x_0) = y_0$ and $p_2(u(B_1)) = |\varphi(B_1)|\, p_2(y_0) \leqq 1$, therefore $u \in W(B_1, U_{p_2})$. This completes the proof.

Proof of Theorem 2.1. For the demonstration of (i) let $p_1 \in \mathscr{P}_1$ and $p_3 \in \mathscr{P}_3$ denote two semi-norms such that $f(U_{p_1})$ and $g(U_{p_3})$ are bounded sets in E_2 and E_4. Then we get the following factorization:

$$\begin{array}{ccccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{u} & E_3 & \xrightarrow{g} & E_4 \\ & \hat{f}\searrow & \uparrow i_2 & & & \hat{g}\searrow & \uparrow i_4 \\ & & [\![f(U_{p_1})]\!] & & & & [\![g(U_{p_3})]\!] \end{array}$$

where for a convex, circled, bounded subset $A \subset E_i$, $[\![A]\!]$ denotes the vector spaces spanned by A provided with the norm defined by the Minkowski functional of A. The mapping $\mathrm{Hom}\,(i_2, \hat{g})\colon \mathscr{L}_b(E_2, E_3) \longrightarrow \mathscr{L}_b([\![f(U_{p_1})]\!], [\![g(U_{p_3})]\!])$ is bounded, because it is a continuous linear mapping into a normed space. Hence the composed mapping $\mathrm{Hom}\,(f, g) = \mathrm{Hom}\,(\hat{f}, i_4) \circ \mathrm{Hom}\,(i_2, \hat{g})$ is bounded. For the demonstration of (ii) let $W(B_2, U_{p_3})$ be a neighborhood of zero in $\mathscr{L}_b(E_2, E_3)$ such that $\mathrm{Hom}\,(f, g)\, W(B_2, U_{p_3})$ is a bounded (precompact) subset of $\mathscr{L}_b(E_1, E_4)$. As E_1 is infrabarrelled, this set is equicontinuous. Now we find a semi-norm $p_2 \in \mathscr{P}_2$ and a $x_0 \in E_1$ such that $p_2(f(x_0)) = 1$ and $p_2(B_2) \leqq 1$. By taking $B_2 \cup \{f(x_0)\}$ instead of B_2, we may assume $f(x_0) \in B_2$ Then by Lemma 2.1. we have $U_{p_3} = \{u(f(x_0))\colon u \in W(B_2, U_{p_3})$ and $\dim u(E_2) = 1\}$. Hence $g(U_{p_3}) = \{g \circ u \circ f(x_0)\colon u \in W(B_2, U_{p_3})$ and $\dim u(E_2) = 1\} \subset \mathrm{Hom}\,(f, g)\, W(B_2, U_{p_3})(x_0)$ is a bounded (precompact) subset of E_4. Thus g is a bounded (precompact) linear operator. To show that f is also bounded, let $z_0 \in E_3$ such that $g(z_0) \neq 0$. Now obviously $\mathscr{L}_b(E_2, Cz_0)$ and $\mathscr{L}_b(E_1, Cg(z_0))$ are complemented subspaces of $\mathscr{L}_b(E_2, E_3)$ and $\mathscr{L}_b(E_1, E_4)$. Furthermore $\varphi \longrightarrow \varphi \otimes z_0$ and $\psi \longrightarrow \psi \otimes g(z_0)$ are canonical isomorphisms from $\mathscr{L}_b(E_2, C)$ and $\mathscr{L}_b(E_1, C)$ onto $L_b(E_2, Cz_0)$ and $\mathscr{L}_b(E_1, Cg(z_0))$. Thus we are in the situation of the following diagram:

$$\begin{array}{ccccc} \mathscr{L}_b(E_2, Cz_0) & \hookrightarrow & \mathscr{L}_b(E_2, E_3) & \xrightarrow{\mathrm{Hom}\,(f, g)} & \mathscr{L}_b(E_1, E_4) \\ \uparrow & & & & \downarrow \pi \\ \mathscr{L}_b(E_2, C) & & & & \mathscr{L}_b(E_1, Cg(z_0)) \\ & \searrow^{f'} & & & \downarrow \\ & & & \longrightarrow & \mathscr{L}_b(E_1, C) \end{array}$$

The composed mapping

$$\varphi \longrightarrow \varphi \otimes z_0 \longrightarrow g \circ (\varphi \otimes z_0) \circ f \longrightarrow \varphi \circ f \otimes g(z_0) \longrightarrow \varphi \circ f$$

from $\mathscr{L}_b(E_2, C)$ into $\mathscr{L}_b(E_1, C)$ is the adjoint operator f' of f. As Hom (f, g) is a bounded operator, so is f'. Hence there is a bounded subset B_2 of E_2 such that $f'(B_2^0)$ is bounded in $E'_{1b} = \mathscr{L}_b(E_1, C)$. Because of $(f'(B_2^0))^0 = (f'')^{-1}(B_2^{00})$ we get $f''((f'(B_2^0))^0) \subset B_2^{00}$. As E_1 is infrabarrelled E''_{1b} induces the given topology on E_1. Thus $f = f''|_{E_1}$ is a bounded operator. This completes the proof.

Especially, the adjoint operator $f'\colon E'_{2b} \to E'_{1b}$ of a bounded operator $f \in \mathscr{L}(E_1, E_2)$ is bounded, and if E_1 is infrabarrelled also the converse holds. Without E_1 being infrabarrelled, the last statement in general no longer remains true. To give an example consider the mapping $id_E\colon E_\sigma \to E_\sigma$ where E denotes an infinite dimensional normed linear space and σ denotes the $\sigma(E, E')$-topology on E. Then $(id_E)' = id_{E'}\colon E'_b \to E'_b$ is a bounded operator, whereas $id_E\colon E_\sigma \to E_\sigma$ is not.

3. Transformation properties of precompact operators. As we have learned from Theorem 1.3 there exists no precompact version of Theorem 2.1 without further assumption put on the l.c.s. or the operators involved. In general only the following holds

THEOREM 3.1. *Let* E_1, E_2, E_3, E_4 *denote four* l.c.s., *and let* $f \in \mathscr{L}(E_1, E_2)$, $g \in \mathscr{L}(E_3, E_4)$ *both be bounded operators. Then*

(i) Hom (f, g) *is a bounded operator. If in addition* E_2 *is infrabarrelled, f and g both are precompact* (semi-precompact would be sufficient), *then* Hom (f, g) *is bounded and semi-precompact.*

(ii) *If g is a precompact operator, then there exists a neighborhood of zero* $W(B_2, U_{p_3})$ *in* $\mathscr{L}_b(E_2, E_3)$ *such that* Hom $(f, g) W(B_2, U_{p_3})$ *is a collectively precompact subset of* $\mathscr{L}(E_1, E_4)$.

Proof. Only (ii) is to be shown. For this purpose let $B_2 \in \mathscr{B}(E_2)$ and U_{p_1}, U_{p_3} be given such that $f(U_{p_1}) \subset B_2$, and $g(U_{p_3})$ is precompact. Then the following inclusions hold

$$\mathrm{Hom}\,(f, g)\, W(B_2, U_{p_3})(U_{p_1}) \subset \{g \circ u(y)\colon y \in B_2,\ u \in W(B_2, U_{p_3})\} \subset g(U_{p_3})\,.$$

Since $g(U_{p_3})$ is precompact by assumption, we are done.

Now we shall give conditions making sure that Hom (f, g) becomes a precompact operator. The following theorem is a generalization of a result due to Ringrose [5].

THEOREM 3.2. *Let* E_1, E_2, E_3, E_4, E_5 *denote five* l.c.s., *and let* $f_1 \in \mathscr{L}(E_1, E_2)$ *and* $g \in \mathscr{L}(E_4, E_5)$ *both be precompact, let* $f_2 \in \mathscr{L}(E_2, E_3)$

be bounded. Then the mapping

$$\mathrm{Hom}\,(f_2\circ f_1,\, g)\colon \mathscr{L}_b(E_3,\, E_4) \longrightarrow \mathscr{L}_b(E_1,\, E_5)$$

defined by $u \longrightarrow \mathrm{Hom}\,(f_2\circ f_1,\, g)u = g\circ u\circ f_2\circ f_1$ *is precompact.*

Proof. By E_{p_i} we denote the normed space $(E_i/p_i^{-1}(o),\, p_i)$. Let U_{p_1}, U_{p_2}, and U_{p_4} denote three neighborhoods of zero such that $f_1(U_{p_1})$ and $g(U_{p_4})$ both are precompact, and $f_2(U_{p_2})$ is bounded. Then we get the following factorization

$$\begin{array}{ccccccccc}
E_1 & \xrightarrow{f_1} & E_2 & \xrightarrow{f_2} & E_3 & \xrightarrow{u} & E_4 & \xrightarrow{g} & E_5 \\
\downarrow{\scriptstyle \pi_1} & \nearrow{\scriptstyle \tilde{f}_1} & \downarrow{\scriptstyle \pi_2} & \nearrow{\scriptstyle \tilde{f}_2} & & & \downarrow{\scriptstyle \pi_4} & \nearrow{\scriptstyle \tilde{g}} & \\
E_{p_1} & \xrightarrow[\hat{f}_1]{} & E_{p_2} & & \xrightarrow[v]{} & & E_{p_4} & &
\end{array}$$

Now the following mappings are continuous:

$$\mathrm{Hom}\,(\hat{f}_1,\, \tilde{g})\colon \mathscr{L}_b(E_{p_2},\, E_{p_4}) \longrightarrow \mathscr{L}_b(E_{p_1},\, E_5)$$
$$\mathrm{Hom}\,(\tilde{f}_2,\, \pi_4)\colon \mathscr{L}_b(E_3,\, E_4) \longrightarrow \mathscr{L}_b(E_{p_2},\, E_{p_4})$$

$w \longrightarrow \gamma(w) = w\circ\pi_1$ from $\mathscr{L}_b(E_{p_1},\, E_5)$ into $\mathscr{L}_b(E_1,\, E_5)$. But by Theorem 1.2 the first mapping is semi-precompact, hence precompact, as $\mathscr{L}_b(E_{p_2},\, E_{p_4})$ is a normed space. Hence the composed mapping $\mathrm{Hom}\,(f_2\circ f_1,\, g) = \gamma\circ\mathrm{Hom}\,(\hat{f}_1,\, \tilde{g})\circ\mathrm{Hom}\,(\tilde{f}_2,\, \pi_4)$ is a precompact operator. This completes the proof.

Now we shall give some applications of this theorem.

COROLLARY 1. *Let* E_1, E_2, E_3 *denote three* l.c.s., *let* $f_1\in\mathscr{L}(E_1,\, E_2)$ *be a precompact operator, and let* $f_2\in\mathscr{L}(E_2,\, E_3)$ *be a bounded operator. Then the adjoint operator*

$$(f_2\circ f_1)' = f_1'\circ f_2'\colon E_{3b}' \longrightarrow E_{1b}'$$

is precompact.

Especially. Let f *be a precompact endomorphism of* E_1. *Then* $(f')^2\colon E_{1b}'\rightarrow E_{1b}'$ *is precompact.*

A proof of this corollary is immediately obtained from Theorem 3.2 by setting $E_4 = E_5 = C$ and $g = \mathrm{id}_C$. The corollary is originally due to Ringrose [5]. The assumptions put on f_1 may be weakened: Let E_2 be infrabarrelled and let f_1 be semi-precompact instead of being precompact. Then by Theorem 1.2 $f_1'\colon E_{2b}'\rightarrow E_{1b}'$ is semi-precompact, and $f_2'\colon E_{3b}'\rightarrow E_{2b}'$ is bounded by Theorem 2.1. Hence the composed operator $f_1'\circ f_2'$ is precompact. Indeed, by using the same arguments as Ringrose in [5], one can show, that $f_1' \circ f_2'$ is even compact.

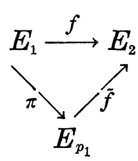
Corollary 2. *Let* E_1, E_2, E_3, E_4 *denote four* l.c.s., *let* $f \in \mathscr{L}(E_1, E_2)$ *and* $g \in \mathscr{L}(E_3, E_4)$.

(i) *Let* E_1 *be a Schwartz space* (for a definition and further properties of these spaces see Horváth [4]), *let* f *be bounded, and let* g *be precompact. The* Hom (f, g) *is a precompact operator.*

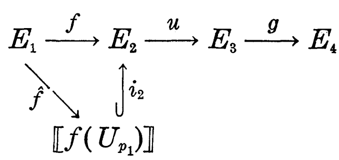
(ii) *Let* E_1 *denote an infrabarrelled Schwartz space, let* f *and* g *both be nonzero. Then* Hom (f, g) *is precompact if and only if* f *and* g *both are precompact.*

(iii) *Let* E_2 *be a Fréchet space, let* f *and* g *both be precompact. Then* Hom (f, g) *is a precompact operator.*

Proof. Let $f \in \mathscr{L}(E_1, E_2)$ be a bounded operator. Then there exists a suitable neighborhood of zero U_{p_1} in E_1 such that we have the following factorization



By Horváth [4, Proposition 3, p. 275] we may assume that π is precompact if E_1 is a Schwartz space. Thus we are in the situation of the following diagram



where $\hat{f}$ is a precompact operator. Hence Hom $(f, g) =$ Hom $(i_2 \circ \hat{f}, g)$ is precompact by Theorem 3.2. Now assume Hom (f, g) is precompact. Then by Theorem 2.1 f is bounded and g is precompact. As E_1 is a Schwartz space f is precompact by the first diagram.

To end this proof a result concerning compact subsets of a Fréchet space is needed, which can easily be deduced from Schaefer [5, Corollary 1, p. 151]:

If B *is a convex, circled, compact subset of a Fréchet space* E, *then there exists another convex, circled, compact subset* $A \subset E$ *containing* B *such that the embedding* $[\![B]\!] \hookrightarrow [\![A]\!]$ *is a compact mapping.*

Hence in (iii) we are in the situation of the following diagram

$$\begin{array}{ccccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{u} & E_3 & \xrightarrow{g} & E_4 \\ \downarrow{\scriptstyle \hat{f}} & & \uparrow{\scriptstyle i_2} & & & & \\ [\![\overline{f(U_{p_1})}^{E_2}]\!] & \underset{i}{\hookrightarrow} & [\![A]\!] & & & & \end{array}$$

Where $[\![\overline{f(U_{p_1})}^{E_2}]\!] \underset{i}{\hookrightarrow} [\![A]\!]$ is compact (A a convex, circled, compact subset of E_2). Then by Theorem 3.2. $\mathrm{Hom}(f, g) = \mathrm{Hom}(i_2 \circ (i \circ \hat{f}), g)$ is precompact. This completes the proof.

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