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PROPERTIES OF MARTINGALE-LIKE SEQUENCES

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The purpose of this paper is to define a new type of stochastic sequence and to explore its properties. These new sequences of random variables, called eventual martingales, generalize the concept of a martingale.

Several known results concerning the almost sure limiting behavior of martingales are shown to remain valid for eventual martingales. In addition, eventual martingales are compared with three other martingale-like sequences.

Consider a probability space $(\Omega, \mathscr{F}, P)$. A stochastic sequence $(X_n, \mathscr{F}_n, n \geqq 1)$ will be called an *eventual martingale* if and only if (iff)

(1) $P[E(X_n|\mathscr{F}_{n-1}) \neq X_{n-1}$ infinitely often (i.o.)$] = 0$.

This says, in effect, that, except on an event of probability zero, the martingale property $E(X_n|\mathscr{F}_{n-1}) = X_{n-1}$ holds for all sufficiently large n. In view of the Borel-Cantelli lemma, $(X_n, \mathscr{F}_n, n \geqq 1)$ is an eventual martingale if $\sum_{n=1}^{\infty} P[E(X_n|\mathscr{F}_{n-1}) \neq X_{n-1}] < \infty$; in particular, every martingale is an eventual martingale.

In §2, a decomposition theorem for eventual martingales will be established and used to generalize some known martingale results. Section 3 will explore the relationship among eventual martingales and three other generalizations of martingales.

Assume throughout that $\mathscr{F}_0$ is the trivial sigma-field. Let $I(A)$ denote the indicator function of an event $A \in \mathscr{F}$.

2. A decomposition theorem.

Crucial to the considerations of this section is the following result.

THEOREM 1. *Let* $(X_n, \mathscr{F}_n, n \geqq 1)$ *be an eventual martingale. Then there exist stochastic sequences* $(M_n, \mathscr{F}_n, n \geqq 1)$ *and* $(Z_n, \mathscr{F}_n, n \geqq 1)$ *such that* (i) $X_n = M_n + Z_n$ *for all* $n \geqq 1$, (ii) $(M_n, \mathscr{F}_n, n \geqq 1)$ *is a martingale, and* (iii) $P[Z_{n+1} \neq Z_n \text{ i.o.}] = 0$.

Proof. Let $d_1 = X_1$ and, for $n \geqq 1$, let $d_{n+1} = X_{n+1} - X_n$. If $n \geqq 1$, let $M_n = \sum_{k=1}^{n} d_k I(E(d_k|\mathscr{F}_{k-1}) = 0)$ and $Z_n = X_n - M_n$. Then (i) and (ii) are obvious. Moreover, $Z_{n+1} - Z_n = d_{n+1} I(E(d_{n+1}|\mathscr{F}_n) \neq 0)$ so $[Z_{n+1} \neq Z_n] \subseteqq [E(d_{n+1}|\mathscr{F}_n) \neq 0]$. Hence $0 \leqq P[Z_{n+1} \neq Z_n \text{ i.o.}] \leqq P[E(d_{n+1}|\mathscr{F}_n) \neq 0 \text{ i.o.}] = 0$ by (1).

REMARK. Let $B = [Z_{n+1} \neq Z_n \text{ i.o.}]$. Theorem 1 (iii) says that, except for $\omega \in B$, each (real) sequence $\{Z_n(\omega)\}$ is constant from some point onward. Therefore, $\{Z_n\}$ is an almost surely (a.s.) convergent sequence of random variables (rv). Thus it is evident that X_n converges a.s. iff M_n converges a.s. Moreover, for any positive real sequence $c_n \to \infty$, the sequences $\{X_n/c_n\}$ and $\{M_n/c_n\}$ have the same limiting behavior, since $\lim_{n\to\infty} Z_n/c_n = 0$ a.s.

These facts allow several properties of martingales to remain valid for eventual martingales, as the next theorem shows.

THEOREM 2. *Let* $(X_n = \sum_{k=1}^n d_k, \mathscr{F}_n, n \geqq 1)$ *be an eventual martingale.*

(i) (cf. Chow [4]). *If* $\sum_{n=1}^\infty E|d_n|^{2r}/n^{1+r} < \infty$ *for some* $r \geqq 1$, *then* $\lim_{n\to\infty} X_n/n = 0$ a.s.

(ii) (cf. Burkholder [3]). *If* $E((\sum_{n=1}^\infty d_n^2)^{1/2}) < \infty$, *then* X_n *converges* a.s.

(iii) (cf. Stout [7]). *If* $|d_n| \leqq M$ a.s. *for some* $M < \infty$, *and if* $s_n^2 \equiv \sum_{k=1}^n E(d_k^2 | \mathscr{F}_{k-1}) \to \infty$ a.s., *then* $\limsup_{n\to\infty} X_n/(2s_n^2 \log\log s_n^2)^{1/2} = 1$ a.s.

Proof. $|M_n - M_{n-1}| = |d_n I(E(d_n|\mathscr{F}_{n-1}) = 0)| \leqq |d_n|$. Thus the hypothesis of (i) and the theorem in [4] imply $M_n/n \to 0$ a.s. and, hence, $X_n/n \to 0$ a.s. Furthermore, under (ii), the hypothesis of Theorem 2 of [3] holds for $(M_n, \mathscr{F}_n, n \geqq 1)$ so M_n converges a.s. which is tantamount to (ii).

Finally, let $v_n^2 \equiv \sum_{k=1}^n E(d_k^2 I(E(d_k|\mathscr{F}_{k-1}) = 0)|\mathscr{F}_{k-1})$. Now $0 \leqq P[E(d_k^2|\mathscr{F}_{k-1}) \neq E(d_k^2 I(E(d_k|\mathscr{F}_{k-1}) = 0)|\mathscr{F}_{k-1}$ i.o.$] \leqq P[E(d_k|\mathscr{F}_{k-1}) \neq 0$ i.o.$] = 0$ by (1) so

(2) $v_n/s_n \to 1$ a.s.

But $s_n \to \infty$ a.s. so $v_n \to \infty$ a.s. Hence $(M_n, \mathscr{F}_n, n \geqq 1)$ obeys the conditions in Theorem 1 and 2 of [7] so $\limsup_{n\to\infty} M_n/(2v_n^2 \log\log v_n^2)^{1/2} = 1$ a.s. The remark preceding the theorem and (2) now imply (iii).

REMARK. Let $(X_n = \sum_{k=1}^n d_k, \mathscr{F}_n, n \geqq 1)$ be an eventual martingale. Writing

$$X_n = \sum_{k=1}^n (d_k - E(d_k|\mathscr{F}_{n-1})) + \sum_{k=1}^n E(d_k|\mathscr{F}_{n-1})$$

yields another decomposition of X_n which satisfies (i), (ii) and (iii) of Theorem 1. The next result uses this new decomposition to extend another result of Burkholder [3].

THEOREM 3. *Let* $(X_n = \sum_{k=1}^n d_k, \mathscr{F}_n, n \geqq 1)$ *be an eventual martingale such that* $\sup_{n\geqq 1} E|X_n - \sum_{k=1}^n E(d_k|\mathscr{F}_{k-1})| < \infty$. *For* $n \geqq 1$, *let* ν_n *be an* $\mathscr{F}_{n-1}$*measurable* rv. *Then* $\sum_{k=1}^n \nu_k d_k$ *converges* a.s. *on the* *event* $[\sup_{n\geqq 1}|\nu_n| < \infty]$. *In particular,* X_n *converges* a.s. *as* $n \to \infty$.

Proof. By hypothesis, $(\sum_{k=1}^n (d_k - E(d_k|\mathscr{F}_{k-1}))$ is an $\mathscr{L}_1$-bounded martingale. So, by Theorem 1 of Burkholder [3],

(3) $\sum_{k=1}^n \nu_k(d_k - E(d_k|\mathscr{F}_{k-1}))$ converges a.s. on $[\sup_{n\geqq 1}|\nu_n| < \infty]$.

Let $C = [E(d_n|\mathscr{F}_{n-1}) \neq 0 \text{ i.o.}]$; then $P(C) = 0$ by (1). Hence, if $\omega \notin C$, there exists an integer $N = N(\omega)$ such that $E(d_n|\mathscr{F}_{n-1}) = 0$ for $n \geqq N$. Therefore, $\sum_{k=1}^n \nu_k E(d_k|\mathscr{F}_{k-1})$ converges a.s. This fact, together with (3), yields the result. Of course, the special case results when $\nu_n \equiv 1$ for all $n \geqq 1$.

3. On various generalizations of martingales. Alloin [1] calls $(X_n, \mathscr{F}_n, n \geqq 1)$ a *progressive martingale* iff $[E(X_n|\mathscr{F}_{n-1}) = X_{n-1}] \subseteqq [E(X_{n+1}|\mathscr{F}_n) = X_n]$ for all $n \geqq 1$ and $\lim_{n\to\infty} P[E(X_n|\mathscr{F}_{n-1}) = X_{n-1}] = 1$. Mucci [6] calls $(X_n, \mathscr{F}_n, n \geqq 1)$ a *martingale in the limit* iff $\lim_{n\geqq m\to\infty}(E(X_n|\mathscr{F}_m) - X_m) = 0$ a.s. According to Blake [2], $(X_n, \mathscr{F}_n, n \geqq 1)$ is *fairer with time* iff $\lim_{n\geqq m\to\infty} P[|E(X_n|\mathscr{F}_m) - X_m| > \varepsilon] = 0$ for all $\varepsilon > 0$.

The final theorem indicates some relationships involving these three concepts and eventual martingales.

THEOREM 4. (i) *Every progressive martingale is an eventual martingale.*

(ii) *Every progressive martingale is a martingale in the limit.*

(iii) *Every uniformly integrable eventual martingale* $(X_n = \sum_{k=1}^n d_k, \mathscr{F}_n, n \geqq 1)$ *with* $\sup_{n\geqq 1} E|\sum_{k=1}^n E(d_k|\mathscr{F}_{k-1})| < \infty$ *is fairer with time.*

Proof. If $(X_n, \mathscr{F}_n, n \geqq 1)$ is a progressive martingale, then $P[E(X_n|\mathscr{F}_{n-1}) \neq X_{n-1} \text{ i.o.}] = \lim_{n\to\infty} P\{\bigcup_{k=n}^\infty [E(X_k|\mathscr{F}_{k-1}) \neq X_{k-1}] = \lim_{n\to\infty} P[E(X_n|\mathscr{F}_{n-1}) \neq X_{n-1}] = 0$ so (i) is true.

Let $t = \inf\{n \geqq 1: E(X_n|\mathscr{F}_{n-1}) = X_{n-1}\}$. Since $(X_n, \mathscr{F}_n, n \geqq 1)$ is a progressive martingale, t is a stopping rule; i.e. $t \in \{1, 2, \cdots, \infty\}$, $P[t < \infty] = 1$ and $[t = n] \in \mathscr{F}_n$ for all $n \geqq 1$. Now if $t \leqq m$, where $m \geqq 1$, then $E(X_k|\mathscr{F}_{k-1}) = X_{k-1}$ for $k \geqq m$. But $[t \leqq m] \in \mathscr{F}_m$, so, for

$$n > m\,, \quad (E(X_n|\mathscr{F}_m) - X_m)I(t \leqq m)$$
$$= \sum_{k=m+1}^{n} E(I(t \leqq m)E(X_k - X_{k-1}|\mathscr{F}_{k-1})|\mathscr{F}_m) = 0\,.$$

For each $\omega \in [t < \infty]$, there exists $m_0 = m_0(\omega)$ such that $\omega \in [t \leqq m_0]$ so $E(X_n|\mathscr{F}_m) - X_m = 0$ for all $n \geqq m \geqq m_0$, proving (ii).

(iii) is a consequence of a result on page 162 of [5], Theorem 3 above and Theorem 1 of Mucci [6].

REMARK. None of the statements in Theorem 4 have valid converses. The converses of (i) and (ii) are both shown to be false by letting $d_1, d_2, \cdots$ be independent rv with $E(d_3) = 1$, $E(d_n) = 0$ if $n \neq 3$, defining $X_n = \sum_{k=1}^n d_k$ and taking $\mathscr{F}_n$ to be the sigma-field generated by $d_1, d_2, \cdots, d_n$. The converse to (iii) is contradicted by the following example, the first of two examples which show that no general relationship exists between sequence fairer with time and eventual martingales.

EXAMPLE 1. A martingale in the limit need not be an eventual martingale, even if it is uniformly integrable. Let $d_1, d_2, \cdots$ be independent rv such that $P[d_n = 1] = n^{-2}$ whereas $P[d_n = 0] = 1 - n^{-2}$ for $n \geqq 1$. Let $\mathscr{F}_n$ be the sigma-field generated by $d_1, \cdots, d_n$ and set $X_n = \sum_{k=1}^n d_k$. Since $E(\sum_{k=1}^\infty |d_k|) = \sum_{k=1}^\infty k^{-2} < \infty$ and $|X_n| \leqq \sum_{k=1}^\infty |d_k|$ for all $n \geqq 1$, $\{X_n\}$ is uniformly integrable. Moreover, $E(X_n | \mathscr{F}_m) - X_m = \sum_{k=m+1}^n k^{-2}$ for $n > m \geqq 1$ so $(X_n, \mathscr{F}_n, n - 1)$ is a martingale in the limit. But $E(X_n | \mathscr{F}_{n-1}) = X_{n-1} + n^{-2} \neq X_{n-1}$ for all $n \geqq 2$, so it is not an eventual martingale.

EXAMPLE 2. An eventual martingale need not be fairer with time and, hence, need not be a martingale in the limit. Let $U_1, U_2, \cdots$ be independent rv such that $P[U_n = -1] = 2^{-n}$ and $P[U_n = 1] = 1 - 2^{-n}$ for $n \geqq 1$. Let $\mathscr{F}_n$ be the sigma-field generated by $U_1, U_2, \cdots, U_n$, $d_1 = U_1$, $d_{n+1} = 2^n U_{n+1} I(U_n = -1)$ and $X_n = \sum_{k=1}^n d_k$ for $n \geqq 1$. For $k > 1$,

$$\begin{aligned} E(d_k | \mathscr{F}_{k-1}) &= 2^{k-1} I(U_{k-1} = -1) E(U_k | \mathscr{F}_{k-1}) \\ &= (2^{k-1} - 1) I(U_{k-1} = -1) . \end{aligned}$$

Hence $\sum_{k=2}^\infty P[E(X_n | \mathscr{F}_{n-1}) \neq X_{n-1}] = \sum_{k=2}^\infty P[U_{k-1} = -1] = \sum_{k=2}^\infty 2^{-k+1} < \infty$. Thus $(X_n, \mathscr{F}_n, n \geqq 1)$ is an eventual martingale.

But, if $m \geqq 2$, $E(X_{2m} - X_m | \mathscr{F}_m) = \sum_{k=m+1}^{2m} E(E(d_k | \mathscr{F}_{k-1}) | \mathscr{F}_m) = \sum_{k=m+2}^{2m} (2^{k-1} - 1) P[U_{k-1} = -1] + (2^m - 1) I(U_m = -1) \geqq \sum_{k=m+2}^{2m} (1 - 2^{-k+1}) > (1 - 2^{-2m+1}) > 1/2$. Hence, if $\varepsilon < 1/2$,

$$P[|E(X_{2m} | \mathscr{F}_m) - X_m| > \varepsilon] = 1$$

for all $m > 1$, so $(X_n, \mathscr{F}_m, n \geqq 1)$ is not fairer with time.

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