PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 2, 1975

REMOVABLE DISCONTINUITIES OF A-HOLOMORPHIC FUNCTIONS

I. GLICKSBERG

Under certain conditions a function continuous and A-holomorphic off a peak set extends continuously to the peak set if its restriction to the Šilov boundary does.

1. Let A be a uniform algebra with spectrum $M = M_A$ and Šilov boundary ∂_A. A function f is called A-holomorphic on an open subset U of M if it is locally uniformly approximable there by elements of A.

For K a zero set of A lying in the Choquet boundary, it was shown in [2, Th. 3] that any $f \in C(M\backslash K)$, A-holomorphic on $M\backslash\partial_A$, has a countinuous extension to M if $f \mid (\partial_A\backslash K)$ has an extension in $C(\partial_A)$. Only at a late stage in the proof did it come into play that K was in the Choquet boundary, in the guise of the uniqueness of representing measures for its points[1], and in some special situations this use of uniqueness in the proof is unnecessary. Here we want to note a variant of this kind which exploits the setting in which the abstract Radó theorem [1] holds, but features only a peak set K in M rather than a zero set. As was actually the case in [1, 2], M need only be a closed boundary for A for which local maximum modulus holds relative to another boundary ∂ (i.e., for any open U in $M\backslash\partial$ and $a \in A$, $|a(U)| \leqq \sup |a(\partial U)$, where ∂U is the boundary of U in M). Our main assumption about A is that

(1) *A is an intersection of closed subalgebras A_α of $C(M)$, each satisfying local maximum modulus relative to ∂ and maximal with respect to this property.*

In particular ∂ is a boundary for A_α, and A itself then satisfies local maximum modulus relative to ∂; moreover any A with this last property lies in maximal algebras of the above kind by the proof of [1, 3, 4], while [1, 3.3] (applied to each A_α) shows that Radó's theorem holds for an algebra A satisfying (1): $f \in C(M)$, A-holomorphic on $M\backslash(f^{-1}(0) \cup \partial)$ is necessarily in A. (We should also note that an A_α which satisfies local maximum modulus and is relatively maximal in $C(M)$ (i.e., for which properly larger subalgebras of $C(M)$ have properly large Šilov boundaries) is maximal in the required sense when $\partial = \partial_{A_\alpha}$ for all α.)

[1] K could have been any nowhere dense zero set in M for which all elements have unique Jensen measure on ∂, as is easily seen (cf. [2, p. 405, midpage]).

Now suppose $K \subset M$ is a nowhere dense peak set and $f \in C(M\backslash K)$ is A-holomorphic on $M\backslash K$ (or just on $M\backslash(\partial \cup K \cup f^{-1}(0))$), while $f \mid (\partial\backslash K)$ extends continuously to an element of $C(\partial)$. As we shall see later by an example, (1) is not sufficient to imply f extends to an element of $C(M)$, but there are simple conditions insuring this: for example, suppose there is a w^* continuous "section" t from M into the representing measures on ∂ for which $t(\phi)$ represents ϕ on A. Then if $g \in A$ peaks on K and we set

$$(2) \qquad f_\varepsilon = \begin{cases} f \cdot \exp(-\varepsilon/(1-g)^{1/2}) & \text{on } M\backslash K \\ 0 & \text{on } K \end{cases}$$

(where $|\arg(1-g)^{1/2}| \leqq \pi/4$) then $f_\varepsilon \in C(M)$ and is also A-holomorphic on $(M\backslash(\partial \cup K \cup f^{-1}(0)) = M\backslash(\partial \cup f_\varepsilon^{-1}(0))$, so $f_\varepsilon \in A$ by Radó's theorem, whence $t(\phi)(f_\varepsilon) = f_\varepsilon(\phi)$. Now for any $\lambda \geqq 0$ representing $\phi \in M\backslash K$, $\lambda(K) = \lim \lambda(g^n) = \lim g(\phi)^n = 0$, so the restriction $\lambda_K = 0$, and λ is carried by M/K; in particular this is true for $t(\phi)$, and since $f_\varepsilon \to f$ on $M\backslash K$ as $\varepsilon \to 0$, while $|f_\varepsilon| \leqq |f|$, by dominated convergence $t(\phi)(f_\varepsilon) = f_\varepsilon(\phi)$ implies $t(\phi)(f) = f(\phi)$, $\phi \in M\backslash K$. But by hypothesis $f \mid (\partial\backslash K)$ has a continuous extension h in $C(\partial)$ while $\phi \to t(\phi)$ is w^* continuous into the space of measures on ∂, so that $\phi \to t(\phi)(h)$ is continuous on M, and coincides with f on M/K since

$$t(\phi)(h) = t(\phi)_{M\backslash K}(h) = t(\phi)(f) = f(\phi) \text{ for } \phi \in M\backslash K\,.$$

It is precisely the fact that $f_\varepsilon \in A$ which allows us to improve on [2, Th. 3] in our special setting.

Call a representing measure λ on ∂ for $\phi \in K$ *accessible from* $M\backslash K$ if it is the w^* limit of a net $\{\lambda_\delta\}$ of representing measures on ∂ for points in $M\backslash K$. (We can in fact allow the λ_δ to be complex representing measures if we insist they be carried by $\partial\backslash K$, which is automatic for $\lambda_\delta \geqq 0$ as we just saw.) Finally, let M^ϕ denote the set of representing measures on ∂ for ϕ.

THEOREM 1. *Suppose A satisfies* (1), *$K \subset M$ is a nowhere dense peak set, and*

(3) *$f \in C(M\backslash K)$ is A-holomorphic on $M\backslash(\partial \cup K \cup f^{-1}(0))$ while $f \mid (\partial\backslash K)$ has an extension in $C(\partial)$.*

Then f extends continuously at each $\phi \in K$ for which the subset of M^ϕ accessible from $M\backslash K$ is convex[2]. *In particular, if the latter holds for all $\phi \in K$, f has a continuous extension to all of M which* (by

[2] We could equally well only assume that for any pair λ, λ' of Jensen measures in M_α^ϕ we have elements $\lambda_1 = \lambda, \lambda_2, \cdots, \lambda_n = \lambda'$ in M_α^ϕ for which the open segment joining λ_i nd λ_{i+1} contains an element of M_α^ϕ (so in particular if λ, λ' can be joined by a finite chain of convex subsets of M_α^ϕ), as is easily seen from the final step in the proof.

Radó's theorem) *is in* A.

Alternatively, we could replace representing measures by Jensen measures everywhere, including the definition of accessibility. Moreover, when $\partial = \partial_A$, accessibility for all ϕ in K is a priori a weaker condition than one which is fully equivalent to the existence of a continuous extension to M for each f satisfying (3). Indeed let $A^\perp$ denote the measures on ∂ orthogonal to A, and $A^\perp_{K'}$ the set of restriction of such measures to the complement K' of K. ($A^\perp_{K'} \subset A^\perp$ of course since $\mu \in A^\perp$ implies $\mu_K \in A^\perp$.)

THEOREM 2. *If A satisfies* (1) *for $\partial = \partial_A$ the following are equivalent*:

(a) *Every f satisfying* (3) *has a continuous extension to M.*

(b) *$A^\perp_{K'}$ is w^* dense in $A^\perp$.*

(c) *The real elements of $A^\perp$ lie in the w^* closure of $A^\perp_{K'}$.*

COROLLARY 3. *If A satisfies* (1) *for $\partial = \partial_A$ then the set of accessible elements of M^ϕ is convex for each $\phi \in K$ only if $A^\perp_{K'}$ is w^* dense in $A^\perp$; moreover this w^* density implies each element of M^ϕ is accessible from $M\backslash K$ using complex representing measures.*

We shall first prove Theorem 2 and Corollary 3, and then return to Theorem 1.

Proof of Theorem 2. Evidently (b) implies (c). To see (c) implies (a), suppose f satisfies (3) and in fact is extended to $\partial \cup (M/K)$ so $f \mid \partial$ is in $C(\partial)$, while the real elements of $A^\perp$ lie in the w^* closure of $A^\perp_{K'}$. As earlier, f_ε defined in (2) lies in A by Radó's theorem, while $f_\varepsilon \to f$ on $M\backslash K$ and $|f_\varepsilon| \leqq |f|$ there. So since $f_\varepsilon \perp A^\perp_{K'} \subset A^\perp$, by dominated convergence $f \perp A^\perp_{K'}$; but $f \mid \partial \in C(\partial)$ and the w^* closure of $A^\perp_{K'}$ contains the real elements of $A^\perp$, while all our measures are carried by ∂, so f (as extended) is orthogonal to those real measures.

Now for any $\phi \in M$, if $\lambda_1, \lambda_2 \in M^\phi$ then $f \perp (\lambda_1 - \lambda_2)$ so that f is constant on M^ϕ: thus $\lambda \to \lambda(f)$ is a continuous function on the w^* compact space S of all multiplicative probability measures on ∂ which is constant on each set of constancy of the natural (continuous) map of S onto M, so setting $\tilde{f}(\phi) = \lambda(f)$ for any $\lambda \in M^\phi$ defines an $\tilde{f} \in C(M)$. On the other hand the fact that $\lambda(f_\varepsilon) = f_\varepsilon(\phi)$ for $\lambda \in M^\phi$ and $\phi \in M\backslash K$ implies $\lambda(f) = f(\phi)$ by dominated convergence again, so $\tilde{f} = f$ on $M\backslash K$, and $\tilde{f}$ is the desired extension of f.

It remains to prove (a) implies (b), and we argue by contradiction. Suppose that $A^\perp_{K'}$ is not w^* dense in $A^\perp$, so we have an $h \in C(\partial)\backslash(A \mid \partial)$, $h \perp A^\perp_{K'}$. Since $A^\perp_{K'}$ is an A-module, $h \perp AA^\perp_{K'} \subset A^\perp_{K'}$, or $A \perp hA^{\perp}_{K'}$, so $(hA^{\perp}_{K'}) \subset A^{\perp}$ and thus $hA^{\perp}_{K'} \subset A^{\perp}_{K'}$, which h annihilates. So $h \perp hA^{\perp}_{K'}$ or $h^2 \perp A^{\perp}_{K'}$, whence $Ah^2 \perp A^{\perp}_{K'}$; continuing we see that $A^{\perp}_{K'}$ is orthogonal to the closed subalgebra B of $C(\partial)$ generated by A and h. Moreover for $\phi \in M\backslash K$, the elements of M^{ϕ} all coincide on any $b \in B$ exactly as before: $\lambda_1, \lambda_2 \in M^{\phi}$ implies $\lambda_1 - \lambda_2 \in A^{\perp}_{K'}$. So we can view B as a subspace of the space of continuous functions on $M\backslash K$ as in the preceding paragraph (although here our space of measures and $M\backslash K$ are only locally compact): we set $\tilde{b}(\phi) = \lambda(b)$, $\lambda \in M^{\phi}$. Finally if we now define h_{ε} on ∂ as in (2) to be 0 on $K \cap \partial$ and $h \cdot \exp(-\varepsilon/(1-g)^{1/2})$ on $\partial\backslash K$ then $h_{\varepsilon} \in h(A \mid \partial)$ (since the exponential, defined to be zero on K, lies in A by Radó's theorem); thus $h_{\varepsilon} \perp A^{\perp}_{K'}$ and since it vanishes on K, $h_{\varepsilon} \perp A^{\perp}$. Consequently $h_{\varepsilon} \in A \mid \partial$ and for $\phi \in M\backslash K$ and $\lambda \in M^{\phi}$, $\lambda(h^n_{\varepsilon} a) = \lambda(h_{\varepsilon})^n \lambda(a)$, $a \in A$. So by dominated convergence $\lambda(h^n a) = \lambda(h)^n \lambda(a)$, which shows λ is multiplicative on B, and thus that our injection $b \to \tilde{b}$ of B into $C(M\backslash K)$ is an algebra homomorphism. Now for $\phi \in \partial\backslash K$, $\tilde{b}(\phi) = b(\phi)$ since we may use the point mass for λ; in particular then $\tilde{h} \in C(M\backslash K)$ has

$$\tilde{h} \mid (\partial\backslash K) = h \mid (\partial\backslash K) \in C(\partial) \mid (\partial\backslash K) .$$

In fact $\tilde{h}$ is A-holomorphic on $M\backslash K$, for $\tilde{h}_{\varepsilon} = \hat{h}_{\varepsilon} \in A$, since $h_{\varepsilon} \in A \mid \partial$; thus dominated convergence and Mazur's theorem guarantee that $\tilde{h}$ is uniformly approximable by elements of A on any compact subset of $M\backslash K$. Now our function $\tilde{h}$ satisfies (3), and if it had a continuous extension k to all of M then by [2, 3.5] (a variant of the Radó theorem) $k \in A$ since it is A-holomorphic except on a nowhere dense zero set K. But $k = h$ on $\partial\backslash K$, hence on $(\partial\backslash K)^{-}$, and that set is precisely ∂ (otherwise $K \cap \partial$ contains a nonvoid subset open in $\partial = \partial_A$, so contains a peak point, whence the zero set K has nonvoid interior in M by the basic lemma 2.1 of [1]). So we conclude $h = k$ on ∂ and $h \in A \mid \partial$ despite that fact that $h \notin A \mid \partial$. Our assumption that (b) failed has implied (a) fails, so (a) implies (b), completing our proof of Theorem 2.

At this point we should note that Corollary 3 follows directly from Theorems 1 and 2. First if the accessible elements of M^{ϕ} are convex for each $\phi \in K$ then Theorem 1 guarantees (a) of Theorem 2 holds, so (b) follows. On the other hand if $A^{\perp}_{K'}$ is w^* dense in $A^{\perp}$ and $\phi \in K$ then $\phi = \lim_{\Delta} \phi_{\delta}$ for some net $\{\phi_{\delta}\}$ in $M\backslash K$; if $\lambda \in M^{\phi_{\delta}}$ then $\{\lambda_{\delta}\}$ has a w^* cluster point λ_0 in M^{ϕ}. For any other $\lambda \in M^{\phi}$, $\lambda - \lambda_0 = \lim_{\Gamma} \nu_{\gamma}$, $\nu_{\gamma} \in A^{\perp}_{K'}$, so since we can assume $\lambda_0 = \lim \lambda_{\delta}$, passing to a confinal subnet, we have $\lambda = \lim_{\Delta \times \Gamma} (\lambda_{\delta} + \nu_{\gamma})$ (where $\Delta \times \Gamma$ is the product directed system). Consequently each element of M^{ϕ} is accessible from $M\backslash K$ using complex representing measures, as asserted.

Note that in contrast to the relatively trivial proof of the second half of Corollary 3, the first half uses the more complicated part of Theorem 2 as well as Theorem 1.

Our proof of Theorem 1 uses most of the proof of [2, Th. 3], which we reproduce here because of the notational changes required, along with some of the argument already used. To begin let f be extended to $M_0 = (\partial\backslash K)^- \cup (M\backslash K) = M\backslash(K\backslash(\partial\backslash K)^-)$ so $f[(\partial\backslash K)^- \in C(\partial\backslash K)^-]$ and let B be the uniformly closed algebra of bounded functions on M_0 generated by A and f. Let X be the closure in M_B of M_0; X is of course a boundary for B, and we shall identify B and $B^\wedge \mid X$. Since f and the elements of A are continuous when restricted to $(\partial\backslash K)^-$ or $M\backslash K$, the natural injection of each of these spaces into X is continuous, and $1-1$ of course. In particular, $(\partial\backslash K)^-$ is imbedded homeomorphically in X. But the same is true of $M\backslash K$ since the map $\rho: X \to M$ dual to $A \to B$ clearly provides inverses for our injections. (A priori, ρ maps X into M_A; but of course it takes X into the closure of $\partial \cup (M\backslash K)$ in M_A and that closure lies in our boundary M. Note that $\hat{a}(x) = a(\rho(x))$ for $a \in A$, $x \in X$.)

Now $M\backslash K$ is in fact imbedded as an open subset of X. For each $\phi_0 \in M\backslash K$ has a compact neighborhood U in M disjoint from K of the form

$$U = \{\phi \in M: | a_i(\phi) - a_i(\phi_0) | \leqq \varepsilon, \ i \leqq n\} ,$$

and since $X = M_0^- = U^- \cup (M\backslash U)^- = U \cup (M\backslash U)^-$, $x \in X\backslash U$ implies $x \in (M\backslash U)^-$ so $| \hat{a}_i(x) - a_i(\phi_0) | \geqq \varepsilon$ for some i, whence

$$\begin{aligned} W_{\phi_0} &= \{\phi \in M: | a_i(\phi) - a_i(\phi_0) | < \varepsilon/2, \ i \leqq n\} \\ &= \{x \in X: | \hat{a}_i(x) - a_i(\phi_0) | < \varepsilon/2, \ i \leqq n\} \end{aligned}$$

is a neighborhood of ϕ_0 in X lying in $M\backslash K$, so $M\backslash K$ is open in X as asserted. Moreover, this also shows ρ is $1-1$ over $M\backslash K$ since $\hat{a}_i(x) = a_i(\rho(x))$ so $\rho(x) \in M\backslash K$ implies $x \in W_{\rho(x)} \subset M\backslash K$, and so $x = \rho(x)$.

The fact that $M\backslash K$ is open in X, so that $(M\backslash K)\backslash\partial = M\backslash(K \cup \partial)$ is also, along with local maximum modulus, shows ∂_B does not meet $M\backslash(K \cup \partial)$: for each element of B is A-holomorphic on this set (i.e., locally uniformly approximable), and precisely the trivial argument of [1, 3.2] applies. Now since ρ is $1-1$ over $M\backslash K$,

$$\rho(\partial_B) \cap (M\backslash(K \cup \partial)) = \varnothing$$

so $\rho(\partial_B) \subset K \cup \partial$. Thus for $x \in \partial_B$, if $\rho(x) \notin K$ we have $\rho(x) = x$, again since ρ is $1-1$ over $M\backslash K$, and $\rho(x) \in K \cup \partial$, so $x = \rho(x) \in \partial\backslash K$. So $\rho(\partial_B\backslash(\partial\backslash K)) \subset K$.

We can now conclude that $\partial_B \subset (\partial\backslash K)^-$. For $\partial_B\backslash(\partial\backslash K)^-$ lies in the zero set $\rho^{-1}(K)$ of the element $1 - \hat{g} = 1 - g \circ \rho$ of B (where again $g \in A$ peaks on K), and so in the topological boundary of that set since it is also in the closure in X of $M\backslash K$ (where $1-\hat{g}$ never vanishes): indeed this follows from the fact that $\partial\backslash K$ is dense in the subspace $(\partial\backslash K)^-$ of M, hence in the subspace $(\partial\backslash K)^-$ of X, so that the dense subset $(\partial\backslash K)^- \cup (M\backslash K)$ of X lies in the closure in X of $M\backslash K$. But now by [1, 2.2], $\partial_B\backslash(\partial\backslash K)^-$ must be void and we have $\partial_B \subset (\partial\backslash K)^-$, as asserted, and also the fact that $M\backslash K$ is dense in X.

Now if $\rho^{-1}(\phi)$ is a singleton $\{\psi\}$ for $\phi \in K$ then any net $\{\phi_\delta\}$ in $M\backslash K$ converging to ϕ in M can only have ψ as a cluster point in the compact space X, so converges to ψ in X as well. Since $\hat{f}$ is continuous on $X \subset M_B$, $\hat{f}(\psi) = \lim f(\phi_\delta)$ for any such net, and thus f extends continuously at such a ϕ. Moreover if $\rho^{-1}(\phi)$ is a singleton for each $\phi \in K$ then ρ is $1-1$, so X and M are homeomorphic, and $\hat{f}$ provides a continuous extension of f to M.

So it only remains to see that if the subset of M^ϕ accessible from $M\backslash K$ is convex for $\phi \in K$ then $\rho^{-1}(\phi)$ is a singleton. First, each accessible λ in M^ϕ is also multiplicative on B because our function f_ε defined in (2) lies in A: for we have $\lambda_\delta \rightarrow \lambda$ w^*, with λ_δ representing a point in $M\backslash K$ and supported by $\partial\backslash K$, so that λ is supported by $(\partial\backslash K)^-$ and

$$\lambda_\delta(f_\varepsilon^n a) = \lambda_\delta(f_\varepsilon)^n \lambda_\delta(a)\ , \quad a \in A\ , \quad n \geqq 0\ ,$$

and since $f_\varepsilon \rightarrow f$ on $M\backslash K$ and $|f_\varepsilon| \leqq |f|$, by dominated convergence

$$\lambda_\delta(f^n a) = \lambda_\delta(f)^n \lambda_\delta(a)\ .$$

But $\lambda_\delta \rightarrow \lambda$ w^* as measures on $(\partial\backslash K)^-$ while $f \mid (\partial\backslash K^-) \in C((\partial\backslash K)^-)$ so $\lambda(f^n a) = \lambda(f)^n \lambda(a)$, and λ is multiplicative on B, as asserted.

Since we know $M\backslash K$ itself is dense in X, if $\psi \in \rho^{-1}(\phi)$ then we have ϕ_δ in $M\backslash K$, $\phi_\delta \rightarrow \psi$ in X, and if λ_δ is a probability measure on ∂_B representing ϕ_δ on B then $\{\lambda_\delta\}$ has a cluster point λ carried by $(\partial\backslash K)^-$ and representing ψ on B. Trivially λ represents ϕ on A and is accessible from $M\backslash K$, so each element of $\rho^{-1}(\phi)$ is represented by one of our convex set of accessible elements of M^ϕ, all of which are multiplicative on B. But a convex set of multiplicative measures on an algebra B must all represent the same point of the spectrum[3], so $\rho^{-1}(\phi)$ is a singleton, completing our proof of Theorem 1.

(Note that if we take our λ_δ's to be Jensen measures then λ is also Jensen, hence accessible in the sense of the remark following Theorem 1, which now follows immediately. In particular, if the elements of K all have unique Jensen measures (necessarily accessible)

[3] If λ_i represents ϕ_i, $i = 1, 2$, $\phi_1 \neq \phi_2$, $(\lambda_1 + \lambda_2)/2$ is multiplicative and $b(\phi_1) = 0$, $b(\phi_2) = 1$ then $1/2 = (\lambda_1 + \lambda_2)/2\,(b^2) = ((\lambda_1 + \lambda_2)/2\,(b))^2 = 1/4$. Note that we could use any interior point of the segment joining λ_1 and λ_2 to obtain this contradiction; thus for the modification indicated in footnote 2, we need only take our measures λ_δ to be Jensen, so λ is, and all such λ agree on B by an obvious argument.

then the conclusion of Theorem 1 applies.

2. Some remarks are in order. First, (1) is not the most general hypothesis we could use (cf. [2, 3.2, and correction]), but seems nearly so. And K could equally well be a "generalized" peak set-an intersection of peak sets. To see this note that we have used the fact that our K was a true peak set mainly to construct f_ε; in all our arguments we then proceed to the fact that $\nu(f_\varepsilon) \to \nu(f)$ for some measure ν supported by $M \backslash K$. In the more general case one has to select, given ν and $\eta > 0$, a true peak set $K_1 \supset K$ so $|\nu|(K_1 \backslash K) < \eta$ and then construct f_ε using the peaking function for K_1, so that $f_\varepsilon \to f$ on $M \backslash K_1$ (and $|f_\varepsilon| \leqq |f|$ of course) whence $\overline{\lim}\,|\nu(f_\varepsilon - f)| \leqq \|f\| \cdot |\nu|(K_1 \backslash K) \leqq \|f\| \cdot \eta$, which of course suffices.

Next, using accessibility from $F \subset M \backslash K$ in Theorem 1 yields an analogous assertion about a limiting value as we approach ϕ along F. Slightly greater generality can also be obtained by noting that instead of assuming (1) for A we can assume it for any algebra between $\boldsymbol{C} + \{f \in A : f(K) = 0\}$ and A since f_ε then lies in that algebra, and so in A.

Finally even when $f \mid (\partial \backslash K)$ has no continuous extension to ∂ a bit of our argument survives to give some information on cluster values.

THEOREM 4. *Suppose A satisfies* (1) *and $K \subset M$ is a nowhere dense peak set. If $f \in C(M \backslash K)$ is A-holomorphic on $M \backslash (\partial \cup K \cup f^{-1}(0))$ then* $\operatorname{cl}(f, K)$, *the set of cluster values of f at points of K, is contained in* $\mathscr{C}(\operatorname{cl}(f \mid (\partial \backslash K), \partial \cap K))$, *the closed convex hull of the set of cluster values of $f \mid (\partial \backslash K)$.*

Indeed suppose $c \in \operatorname{cl}(f, K)$, so $c = \lim f(\phi_\delta)$, where $\phi_\delta \in M \backslash K$ and $\phi_\delta \to \phi_0 \in K$. If $\lambda_\delta \in M^{\phi_\delta}$ then $\lambda_\delta(K) = 0$, and since $f_\varepsilon \in A$, $\lambda_\delta(f_\varepsilon) = f_\varepsilon(\phi_\delta)$, we again conclude by dominated convergence that $\lambda_\delta(f) = f(\phi_\delta)$. Trivially all but ε of the mass of λ_δ is carried by a given neighborhood V of $\partial \cap K$ in ∂ for $\delta \geqq \delta_{\varepsilon, V}$, hence by $V \backslash K$, so

$$f(\phi_\delta) = \lambda_\delta(f) \in (1 - \varepsilon)\mathscr{C}(f(V \backslash K)) + \varepsilon D ,$$

where D is the closed unit disc in $\boldsymbol{C}$, and thus $c \in \mathscr{C}(f(V \backslash K))$ for all V, whence our assertion.

3. We conclude with some examples. For M compact in $\boldsymbol{C}^N$ with interior M^0 let

$$A = A(M) = \{f \in C(M) : f \text{ is analytic on } M^0\} ,$$
$$A_i = \{f \in C(M) : f \text{ is analytic in } z_i \text{ on } M^0\} .$$

and $\partial = \partial M$, not necessarily the Šilov boundary. Then $A = \bigcap A_i$, and (1) holds. For A_i satisfies local maximum modulus relative to ∂, and suppose B is a larger subalgebra of $C(M)$ with this property. Then, since each slice

$$M_0 = M \cap \{z \in C^N : z_j = z_j^0,\ j \neq i\}$$

of M is a peak set for A_i, $B \mid M_0$ is closed, and for the same reason satisfies local maximum modulus relative to $M_0 \backslash (M^0 \cap M_0)$: if U is a neighborhood in M^0 of $p \in M^0 \cap M_0$ then $|b(p)| \leqq \sup |b(\partial U)|$ for all $b \in B$ implies

$$|b(p)| = |ba^n(p)| \leqq \sup |ba^n(\partial U)| \longrightarrow \sup |b(M_0 \cap \partial U)|$$

as $n \to \infty$, where $a \in A_i$ peaks on M_0; this of course yields our assertion. But now taking U a ball in M^0 about $p \in M_0$, so $D = M_0 \cap U$ is a disc, we see from Wermer's maximality theorem [3] that $b \in B$ is analytic in z_i at p, whence $B = A_i$ as desired. (It may be worth noting that even for $N = 1$ those M for which $A(M)$ is relatively maximal have not been identified as yet, so that property (1) seems more easily applicable. A simple condition insuring the convexity of accessible measures frequently holds in this setting, but often when the conclusion can be obtained rather trivially from analytic structure. It is simply that we have a sequence $\sigma_n : K \to M\backslash K$ of continuous maps tending pointwise to the indentity, with $A \circ \sigma_n \subset A \mid K$ and $\sigma_n((\partial\backslash K)^- \cap K) \subset \partial\backslash K$. For then any λ on $(\partial\backslash K)^-$ representing $\phi \in K$ has $\sigma_n^*\lambda$ carried by $\partial\backslash K$ while $\sigma_n^*\lambda(a) = \lambda(a \circ \sigma_n) = a \circ \sigma_n(\phi) = a(\sigma_n(\phi))$, $a \in A$, so $\sigma_n^*\lambda$ represents $\sigma_n(\phi) \in M\backslash K$ and $\sigma_n^*\lambda \to \lambda$ w^* by dominated convergence. Thus all representing measures for ϕ on $(\partial\backslash K)^-$ are accessible.)

As an application of Theorem 1, let $E = \{z \in C: 1/2 \leqq |z| \leqq 1\}$ and $\{D_n\}$ be a sequence of disjoint open discs in E^0 which accumulate only on ∂E. Set $E_{2^{-n}} = E\backslash \bigcup_{j \geqq n} D_j$, $n \geqq 0$, $E_0 = E$, and for $2^{-n-1} < x < 2^{-n}$ $E_x = E\backslash \bigcup_{j \geqq n+1} D_j \cup 2^{n+1}(x - 2^{-n-1})D_n$ (so that we continuously and successively fill the holes in E_{2^0}). Now let $M_x = \{x\} \times E_x$ and $M = \bigcup_{0 \leqq x \leqq 1} M_x$, the corresponding compact set in C^2; we take $A \subset C(M)$ to consist of those f for which

$$z \longrightarrow f(x, z) \text{ is analytic on } E_x^0\,, \qquad 0 \leqq x \leqq 1\,.$$

Trivially $\partial = \bigcup_{0 \leqq x \leqq 1} \{x\} \times \partial E_x$ is compact and the Šilov boundary for A, and A satisfies local maximum modulus relative to ∂; moreover A is maximal with respect to this property, again by Wermer's maximality theorem.

Now $K = M_0$ is our peak set, and we next want to observe that for $\phi = (0, z) \in \{0\} \times E^0$, the measures in M^ϕ accessible from $M\backslash K$

form a convex set. Since $A \mid M_0$ can be identified with $A(E)$, which has a one-dimensional set of real orthogonal measures on ∂E, M^ϕ is a segment, and its set F of accessible elements fails to be convex only if F lies in two disjoint subsegments, both meeting F. Let N_1 and N_2 be disjoint w^* compact neighborhoods of these subsegments in the space of these subsegements of measures on ∂. Then for a sufficiently small closed ball N in C^2 about our ϕ each $\psi \varepsilon (M \backslash M_0) \cap N = N_0$ is represented by a convex set M^ψ of measures on ∂ lying wholly in N_1, or in N_2; thus since N_0 is connected if N is sufficiently small (because of the accumulation of the discs D_j only on ∂E) while

$$\{\psi \in N_0 \colon M^\psi \subset N_i\}$$

is closed in N_0 for $i = 1, 2$, we must have one empty, and F cannot meet both N_1 and N_2.

Note that both Theorem 2 and Corollary 3 now apply to the example. (Indeed an alternative approach is to see that for $0 < x < \varepsilon$ the functional on $C(\partial E_x)$ given by

$$f \longrightarrow \int_{|z|=3/4} \frac{\partial f^*}{\partial n} \mid dz \mid$$

(where f^* is the harmonic extension of f to E_x, and the integral gives the period of the conjugate harmonic function) is represented by a measure $\nu_x \geqq 0$ on the part E_x^1 of ∂E_x outside $|z| = 3/4$ and $\leqq 0$ on the part E_x^2 within. The restrictions ν_x^1, ν_x^2 to these parts have equal mass $\geqq c > 0$ (consider $f = 1$ and $f = \log|z|$), and thus for some $x_\delta \longrightarrow 0$, $\nu_{x_\delta}^1 \longrightarrow \lambda^1 \neq 0$, $\nu_{x_\delta}^2 \longrightarrow \lambda^2$ and $\lambda^1 - \lambda^2$ is a nonzero real measure on $\{0\} \times \partial E$ orthogonal to $A \mid M_0$, hence spans the space of such measures, and (c) of Theorem 2 follows. Note that this approach would work if E had $n > 2$ complementary components.)

(That (a) implies (b) in Theorem 2 can fail if $\partial \neq \partial_A$, as can Corollary 3, can be seen by taking $\partial = \partial_A \cup \{\phi\}$ for some ϕ in $K \backslash \partial_A$ in this example: for $\delta_\phi - \lambda$ is not in the w^*closure of $A_{K'}^\perp$ for any λ on ∂_A representing ϕ since the w^*closure is carried by $(\partial \cap (M \backslash K))^- = \partial_A$.)

We have complicated the preceding example (by the insertion of our shrinking holes) so as to avoid having our continuity obtainable in a trivial and direct way: Without the holes,

$$f(0, re^{i\theta}) = \lim f\left(\frac{1}{n}, re^{i\theta}\right)$$

uniformly for $r = 1/2$ and 1 says $f(0, \cdot)$ is the boundary value function of an element of $A(E)$. Even with the conclusion non-obvious, the example is not satisfying as an application of Theorem 1 since it really follows from the simple part, (c) implies (a), of Theorem 2.

However we can modify it by a hair and make Theorem 2 inapplicable. Indeed for $z_0 \in E^0$, now let

$$M = \bigcup_{0 \leqq x \leqq 1} M_x \cup ([-1, 0] \times \{z_0\})$$

so we have added a segment to our old M which meets that set in $\{(0, z_0)\}$. For A we take all continuous extensions of our old algebra, so our Šilov boundary is the old one plus our closed segment, and we take $K = M_0$. Again A satisfies (1) as before (there is a removable singularity at $(0, z_0)$ in $\{0\} \times E^0$ of course to be mentioned after applying Wermer), and it is trivial to give an f satisfying (3) which does not extend: $f(x, z) = z$ for $x > 0$, $f(x, z_0) = -z_0$ for $x \leqq 0$. On the other hand, every f satisfying (3) does extend to $\phi = (0, z) \neq (0, z_0)$ since N_0 is connected and our accessible measures, which must be carried by our old boundary, again lie on a segment. Our argument fails at $\phi = (0, z_0)$ since M^ϕ is a triangle; this could be changed to a segment simply by replacing E by the unit disc, but even then it fails since N_0 is not connected.)

REFERENCES

1. I. Glicksberg, *Maximal algebras and a theorem of Radó*, Pacific, J. Math., **14** (1964), 919–941; correction **19** (1966), 587.

2. ———, *A Phragmén-Lindelöf theorem for function algebras*, Pacific, J. Math., **22** (1967), 401–406.

3. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, England Cliffs, N.J., 1962.

Received November 7, 1974. Work supported in part by the N.S.F.

OXFORD UNIVERSITY
AND
UNIVERSITY OF WASHINGTON