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THE WORD PROBLEM AND POWER PROBLEM IN 1-RELATOR GROUPS ARE PRIMITIVE RECURSIVE

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The purpose of this paper is to show the solution to the word problem in a 1-relator group can be computed with respect to an effective indexing of the group by an algorithm at level at most $2+\sigma(R)$ of the Grzegorczyk hierarchy, where $\sigma(R)$ is the length of the relator, and by a primitive recursive function, always. As a consequence, it is shown that the power problem in a 1-relator group can be solved similarly. An example is given in which the Magnus algorithm for the extended word problem is at level 4 but not 3 of the Grzegorczyk hierarchy even though the word problem is solvable at level 3.

Our theorem provides a negative solution to problem 20 of [2], p. 643, which reads "is there a one-relator group whose word problem is not solvable by a primitive recursive function?" These results do not depend on a particular presentation of the group, rather they are algebraic invariants in the sense of [3]. (The algebraic invariance of the order and power problem can be shown in a manner similar to the proof of the invariance of the word problem in [3].) Sinch each finitely generated subgroup of a group with primitive recursive word problem has a primitive recursive word problem ([4] Corollary 3.7, p. 380) and there are finitely generated and finitely presented groups with recursive but not primitive recursive word problem ([5], [6] and [1] we see that a finitely generated group with solvable word problem cannot necessarily be embedded in 1-relator group. (This result can be obtained in the stronger form: a finitely presented group with primitive recursive word problem is not necessarily embeddable in a 1-relator group, from B. B. Newman's work [11] which shows a free abelian group of rank 3 cannot be a subgroup of a 1-relator group and the easily shown primitive recursive solution to the word problem in such groups.)

The proof of the main theorem follows by analyzing the proof of Magnus' theorem of the solvability of the word problem in 1-relator groups as given in [9], with respect to the Grzegorczyk hierarchy of primitive recursive functions [8]. (See also [12].) We assume a knowledge of [4], [5] and [9]. We prove that the Magnus algorithm for solving the "extended word problem" and hence the word problem in a one relator group is $\mathscr{E}^{2+\sigma(R)}$ computable in the Grzegorczyk hierarchy where $\sigma(R)$ is the length of the relator. In section 4 we verify that the order and power problem in one-relator groups are also $\mathscr{E}^{2+\sigma(R)}$ computable.

Finally we note that the Magnus algorithm is not always the "easiest" algorithm in the sense of the Grzegorczyk hierarchy, for solving the word problem in a one relator group. In § 5 we give an example of a one relator group in which the Magnus algorithm is $\mathscr{E}^4$ but not $\mathscr{E}^3$ computable and yet for which the word problem is $\mathscr{E}^3$ computable.

2. Notation and definitions.

We abbreviate finitely generated, finitely presented, word problem, primitive recursive by "f.g.", "f.p.", "w.p." and "p.r." respectively. The levels of the Grzegorczyk hierarchy are denoted by $\mathscr{E}^\alpha$, $\alpha \geqq 0$. If R is a word on $a, b, c, \cdots$ then $\sigma(R)$ denotes the length of R and $\sigma_a(R)$ denotes the exponent sum of a in R.

The following definition is basic to this paper. It is discussed in detail in [4] and [5].

A countable group G is $\mathscr{E}^\alpha$ computable, or briefly an $\mathscr{E}^\alpha$ group if the following three conditions are met:

(1) There is an injection $i\colon G \to \omega$ such that $i(G)$ is an $\mathscr{E}^\alpha$-decidable subset of ω. (If $r \in i(G)$, $i^{-1}(r)$ will be denoted by g_r.)

(2) The function $m\colon i(G) \times i(G) \to i(G)$ defined by $m(r, s) = i(g_r \cdot g_s)$, $r, s \in i(G)$ is an $\mathscr{E}^\alpha$ function. (More precisely the restriction of an $\mathscr{E}^\alpha$ function to $i(G) \times i(G)$.)

(3) The function In: $i(G) \to i(G)$ defined by

$$\mathrm{In}\,(r) = i(g_r^{-1}),\ r \in i(G)$$

is an $\mathscr{E}^\alpha$ function. (This condition is redundant for $\mathscr{E}^\alpha =$ the recursive functions.)

The function i of (1) is called an $\mathscr{E}^\alpha$ *index* of G. Of particular importance is the notion of a *standard index* which is obtained as follows. Let G have a presentation $G = gp(a_1, a_2, a_3, \cdots; R, S, T, \cdots)$ where $R, S, T, \cdots$ are words on $a_1, a_2, a_3, \cdots$. We fix a Gödel numbering of the free group F on $a_1, a_2, a_3, \cdots$ and assign to each $g \in G$ the index $i(g)$ equal to the smallest index in $\rho^{-1}(g)$ where ρ is the canonical epimorphism $F \twoheadrightarrow G$. If G is $\mathscr{E}^\alpha$ computable with respect to a f.g. standard index we say G is $\mathscr{E}^\alpha$ *standard*. It is known, in this case, that G is then $\mathscr{E}^\alpha$ standard with respect to all f.g. standard indices for this is another way of expressing the algebraic invariance of the w.p. for f.g. groups with respect to the Grzegorczyk hierarchy. See [3] and [6] for the details on the equivalence of $\mathscr{E}^\alpha$ w.p. and having an $\mathscr{E}^\alpha$ standard index for f.g. groups.

3. The main result.

We now state and prove the

THEOREM. *The word problem in a one relator group*

$$G = gp(a, b, \cdots; R(a, b, \cdots))$$

is at most $\mathscr{E}^{2+\sigma(R)}$ *solvable. In particular it is* p.r.

Proof. We prove the "extended word problem" (henceforth "e.w.p.") for G is $\mathscr{E}^{2+\sigma(R)}$ where we mean the problem of determining if a freely reduced word on the generators of G represents an element of the subgroup $H < G$ generated by a given proper subset of the generators and, if so, to rewrite it in terms of the generators of H. The proof will involve an induction on $\sigma(R)$ which raises the degree of computational complexity by at most 1 at each stage.

It should be observed that we may assume R involves all of the generators of G, and in particular that G is f.g., for otherwise we can write G as the free product of a free group with a 1-relator group in which the relator involves all of the generators. Such a free product has an $\mathscr{E}^n$ e.w.p. if the 1-relator factor does, by virtue of the normal form theorem for free products and the obvious $\mathscr{E}^3$ e.w.p. for free groups. Also, observe that by the Freiheitssatz the subgroup of G generated by deleting one relator is free. Thus, if $H < G$ is a subgroup generated by a proper subset of the generators, say $a \notin H$, and W is an arbitrary word, the e.w.p. for W reduces to the question of W representing a word in $K = gp(b, c, \cdots;) < G$ and of rewriting W in terms of $b, c, \cdots$, because the e.w.p. in K (with respect to generators $b, c, \cdots$) is $\mathscr{E}^3$. This last remark holds in particular if $H = \{1\}$, so an $\mathscr{E}^n$ solution to the e.w.p. implies an $\mathscr{E}^n$ solution to the w.p.

Now if G has one generator, G is either trivial or finite cyclic and the e.w.p. is identical to the w.p. and is $\mathscr{E}^3$. Thus, we assume G has at least two generators, say a and b, and hence that R has length at least two. The proof proceeds by recursion on the length of the relator.

Given a subgroup $H < G$ generated by all but one of the generators of G and a word W representing an element of G the recursion reduces the question of W representing an element of H in G (we write $W \in H < G$) to the question $W' \in H' < G'$, where G' is a 1-relator group with relator of length less than that of R and H' is generated by a proper subset of the generators of G'. We show at each stage of this recursion the encoding of W' is $\mathscr{E}^3$ computable from the result of the previous stage and that at each stage the level of computability is raised by at most 1, involving an (inner) recursion. The result then follows since the number of steps in the recursion is bounded by $\sigma(R)$ independent of $\sigma(W)$.

First, assume $\sigma_a(R) = 0$. Also assume $H < G$, is generated by all generators of G excluding a. Let $N < G$ be the normalizer of the set of generators of G excluding a. Then since $\sigma_a(R) = 0$, the map $G \to gp(a;)$ by $a \mapsto a$ and all other generators $\mapsto 1$ extends to an epimorphism $G \twoheadrightarrow gp(a;)$ given by $W(a, b, \cdots) \mapsto a^{\sigma_a(W(a,b,\cdots))}$. Thus, $W \in N$ iff $\sigma_a(W) = 0$ (an $\mathscr{E}^3$ decision) and since $H < N$, $W \notin H$ in G otherwise. By a Reidemeister-Schreier rewriting process (using coset representatives a^k) N has a presentation

$$N = gp(\cdots, b_{-1}, b_0, b_1, \cdots, c_{-1}, c_0, c_1, \cdots; \cdots, P_{-1}, P_0, P_1, \cdots) \text{ where}$$

$b_k = a^k b a^{-k}$, $c_k = a^k c a^{-k}$, etc. and P_k is $a^k R a^{-k} \in N$ rewritten in terms of the b_k, c_k, etc. For $W \in N$, W is rewritten by replacing b^j (or c^j etc.) by b_k^j for k equal to the exponent sum on a of the symbols preceding b_k^j. (For example $a^2b^4ac^{-1}a^{-1}$ is rewritten $b_2^4c_3^{-1}$.) In particular the rewriting lowers the length of W. Notice also that P_k may be obtained from P_0 by raising the subscripts by k.

Now for $W \in N$, $W \in H$ iff its rewrite $W' \in H'$ for $H' = gp(b_0, c_0, \cdots) < N$. Note $\sigma(W') \leqq \sigma(W)$. For simplicity of notation, assume R begins with the symbol b (conjugate R if necessary) and let n be the minimum subscript of b in P_0 ($n \leqq 0$ by the simplification) and m the maximum subscript of b in P_0 ($m \geqq 0$). Set

$$N_0 = gp(b_n, \cdots, b_m, \cdots, c_{-1}, c_0, c_1, \cdots; P_0) .$$

Note $H' < N_0$ since $n \leqq 0 \leqq m$. Recursively form

$$N_k = gp(b_{n+k}, \cdots, b_{m+k}, \cdots, c_{-1}, c_0, c_1, \cdots; P_k) .$$

It should be observed that the N_k are all isomorphic to N_0 the isomorphism given by lowering the subscripts of the b generators by k. Also note the group N_0 (and subsequent groups in the recursion playing the role of N_0) are determined by R independent of W. Thus (formal) questions and rewritings of words in N_k may be treated as questions and rewritings in N_0 by an $\mathscr{E}^3$ modification of subscripts. The significance of the above is that in the following the groups $N_{-\sigma(W)}, N_{-\sigma(W)+1}, \cdots, N_0, \cdots, N_{\sigma(W)}$ must be considered, the number dependent on W. Nevertheless, the algorithms involved depend only on R.

Form $N_{0,1} = N_0 *_\phi N_1$ where ϕ amalgamates the free subgroup generated by $b_{n+1}, \cdots, b_m, \cdots, c_{-1}, c_0, c_1, \cdots$ (all but b_n) in N_0 with the free subgroup having the same generators in N_1. Next, form $N_{-1,1} = N_0 *_\phi N_{-1}$ for ϕ amalgamating the free subgroups generated by $b_n, \cdots, b_{m-1}, \cdots, c_{-1}, c_0, c_1, \cdots$ and proceed to form $N_0 < N_{0,1} < N_{-1,1} < N_{-1,2} < N_{-2,2} < \cdots$. Now N is the union of this chain so if $W \in N$, W is eventually in one of these groups (at least by stage $N_{-\sigma(W),\sigma(W)}$). Note $H' < N_0$. Also observe that up to isomorphism by change of subscripts of the b symbols, there are two types of amalgams namely one generated by $b_{n+k}, \cdots, b_{m+k-1}, \cdots, c_{-1}, c_0, c_1, \cdots$, and one generated by $b_{n+k+1}, \cdots, b_{m+k}, \cdots, c_{-1}, c_0, c_1, \cdots$. Thus questions and rewrites in the amalgams do not depend on k and hence not on W up to an $\mathscr{E}^3$ process on the subscripts.

We must show both the question $W' \in H'$ and, if true, the process of rewriting W' in terms of $b_0, c_0, \cdots$ are $\mathscr{E}^{2+\sigma(R)}$. Then the rewriting in terms of $b, c, \cdots$ (i.e. renumbering to delete the subscripts) is $\mathscr{E}^3$ and so the e.w.p. is $\mathscr{E}^{2+\sigma(R)}$ in this case.

By an $\mathscr{E}^3$ process, $W' \equiv W_1 \cdots W_p$ in syllables, each W_i in a different N_k. We describe a recursion involving the e.w.p. in N_k (or in N_0). Inductively (recursively) the e.w.p. in N_k is $\mathscr{E}^{2+\sigma(P_0)}$ and $\sigma(P_0) < \sigma(R)$ so the computability of the (inner) recursion is at most $\mathscr{E}^{2+\sigma(P_0)+1} \subseteq \mathscr{E}^{2+\sigma(R)}$. We consider each pair of syllables $W_i W_{i+1}$. Let $W_i \in N_k$ and $W_{i+1} \in N_l$ and assume $k < l$ (for simplicity; the case $k > l$ is similar). Then we ask: can W_i be rewritten without the b_{n+k} symbol? and if so rewrite it (an $\mathscr{E}^{2+\sigma(P_0)}$ process). Next, we ask: can W_{i+1} be rewritten without the b_{m+k-1} symbol? and if so rewrite it. The rewritings are in N_{k+1} and N_{l-1} if possible. We continue until neither type of rewrite is possible or W_i has been rewritten $W_i' \in N_j$ and W_{i+1} rewritten $W_{i+1}' \in N_j$ for $k \leqq j \leqq l$. In the former event, $W_i W_{i+1}$ is left as in the original; in the latter $W_i W_{i+1}$ is replaced by the single syllable $(W_i' W_{i+1}')$ reducing the syllable length. The process continues until W' has been rewritten W'' with minimal syllable length. Now the possibilities for W'' are:

(i) $W'' = \Lambda$ (the empty word) whence $W \in H$ with rewrite Λ,

(ii) the syllable length of W'' is greater than 1 whence $W \notin H$,

(iii) the syllable length of W'' is 1 with $W'' \in N_k$ in which case we ask if W'' can be rewritten W''' involving b_0 as the only b symbol if $n + k \leqq 0 \leqq m + k$ or no b symbol otherwise (an $\mathscr{E}^{2+\sigma(P_0)}$ decision and process) so either

(iii) (a) W''' can not be obtained hence $W \notin H$

(iii) (b) $W''' \in N_0$ can be obtained but $W''' \notin H'$ (an $\mathscr{E}^{2+\sigma(P_0)}$ decision) so $W \notin H$ or

(iii) (c) $W''' \in N_0$ can be obtained and $W''' \in H'$ so $W \in H$ and W is rewritten in terms of $b, c, \cdots$ by rewriting W''' in terms of $b_0, c_0, \cdots$ and deleting the subscripts.

Recall that we had assumed above that $\sigma_a(R) = 0$ and $a \notin H$. Now assume $\sigma_a(R) = 0$ but $b \notin H$ (so $a \in H$). In this case $W \in H$

does not imply $W \in N$ but for $\alpha = \sigma_a(W)$, $W \in H$ iff $Wa^{-\alpha} \in H$ and $Wa^{-\alpha} \in N$. Thus the above argument is carried out replacing W by $Wa^{-\alpha}$ and using H', the subgroup generated by all generators of N excluding the b_k for all k. It should be observed that $H' < N_0$ and also that the rewrite W' of $W_a{}^{-\alpha}$ satisfies $\sigma(W') \leqq \sigma(W)$. Step (iii) c) must be replaced by $W''' \in H$ generated by $\cdots c_{-1}, c_0, c_1, \cdots$, $d_{-1}, d_0, \cdots$ and W is rewritten in terms of $a, c, \cdots$ by rewriting W''' replacing c_i (respectively d_i etc.) by $a^i c a^{-i}$ ($d^i c d^{-i}$ respectively), concatenating a^α and freely reducing (all $\mathscr{E}^3$ processes).

Next, assume none of the expenent sums of R is zero and the problem is to determine if $W(a, b, c, \cdots)$ is equal in G to a word not involving b and if so rewrite W without b. Let $\sigma_a(R) = \alpha \neq 0$ and $\sigma_b(R) = \beta \neq 0$. Consider $E = gp(\eta, b, c, \cdots; R(\eta^\beta, b, \cdots)) = G *_\phi gp(\eta;)$ where ϕ amalgamates the infinite cyclic groups generated by a and η^β. By an $\mathscr{E}^3$ process $W(a, b, \cdots) \in G$ can be rewritten $W(\eta^\beta, b, \cdots) \in E$. By a Tietze transformation

$$\begin{aligned} E &= gp(\eta, \zeta, b, c, \cdots; R(\eta^\beta, b, \cdots), b = \zeta\eta^{-\alpha}) \\ &= gp(\eta, \zeta, c, \cdots; R(\eta^\beta, \zeta\eta^{-\alpha}, \cdots)) \end{aligned}$$

where $\sigma_\eta(R(\eta^\beta, \zeta\eta^{-\alpha}, \cdots)) = 0$. Now $W(a, b, c, \cdots)$ is equal in G to a word involving only $a, c, \cdots$ iff $W(\eta^\beta, \zeta\eta^{-\alpha}, \cdots)$ is equal in E to a word involving only $\eta^\beta, c, \cdots$. Note that expressing $W(\eta^\beta, \zeta\eta^{-\alpha}, \cdots)$ in terms of $W(a, b, c, \cdots)$ is an $\mathscr{E}^3$ process. We proceed with the previous analysis letting N be the normalizer of $\{\eta, c, \cdots\}$ (excluding ζ) in E. Thus, by the recursion we can determine if $W(\eta^\beta, \zeta\eta^{-\alpha}, c, \cdots)$ is equal in E to a word say $V(\eta, c, \cdots)$ (excluding ζ symbols) and if so compute V by an $\mathscr{E}^{2+\sigma(R)}$ process. Now since the subgroup of E generated by $\eta, c, \cdots$ (excluding ζ) is free, it is an $\mathscr{E}^3$ process to determine if all powers of η in V are multiples of β. If so, replacing η^β by a in V is also an $\mathscr{E}^3$ process. Thus, this situation has been reduced to the previous one. This completes the proof.

4. The power problem. We now turn to the power problem in 1-relator groups. The reader will recall that the generalized word problem (g.w.p.) for a subgroup $H < G = gp(S; D)$ is the algorithmic problem of deciding whether or not an arbitrary word $W \in G$ defines an element of H. If the g.w.p. is solvable for every cyclic subgroup of G then G is said to have solvable power problem. The object of this section is to prove that the power problem is $\mathscr{E}^{2+\sigma(R)}$-decidable in 1-relator groups. We followed the discussion in McCool [10], indicating where the argument McCool must be modified to obtain the information needed to locate the decidability level with respect to the hierarchy $\{\mathscr{E}^\alpha\}$.

The first step after noting that McCool's lemma [10, p. 428] is identical to parts of our theorem above, is to show the order problem is $\mathscr{E}^{2+\sigma(R)}$-decidable. That is, the algorithmic problem of deciding the order of the element defined by a word W in a 1-relator group is $\mathscr{E}^{2+\sigma(R)}$. We follow the discussion in [9] Theorem 4.13, page 269. Now a 1-relator group has an element of finite order if and only if the relator R is a k^{th} power, $k > 1$, of some non-empty word V in the free group on the generators of the 1-relator group). Thus we assume $G = gp(a_1, \cdots, a_n; V^k(a_1, \cdots, a_n))$, $k > 1$. Furthermore, we may assume V is cyclically reduced since we assume V^k is. (The reduction of a word W to a cyclically reduced word can be accomplished by an $\mathscr{E}^3$ process because a word is cyclically reduced if and only if each of its cyclic permutations is freely reduced. Clearly all cyclic permutations and their subsequent free reductions are obtainable from the index of W by an $\mathscr{E}^3$ process.) Thus given V^k, since $k \leqq \sigma(V^k)$ and $\sigma(V) \leqq \sigma(V^k)$ we can use bounded minimalization, which preserves level $\mathscr{E}^3$, to obtain V and k by an $\mathscr{E}^3$ process. Then $W \in G$ has finite order ($\leqq k$) if and only if W is conjugate to a power of V ([9] Theorem 4.13, p. 269). Therefore W has finite order if and only if $W^k = 1$, and $\mathscr{E}^{2+\sigma(R)}$ decision, since the word problem in G is $\mathscr{E}^{2+\sigma(R)}$. If the order of W is finite, it is the minimal $m \leqq k$ such that $W^m = 1$ and so is $\mathscr{E}^{2+\sigma(R)}$ computable by a bounded minimalization on the decision procedure for the word problem. The remainder of Case 1 of McCool [10], page 428 can be decided at level $\mathscr{E}^{2+\sigma(R)}$, since the induction used is identical to the recursion given above for the $\mathscr{E}^{2+\sigma(R)}$-decidability of the e.w.p. As above, Case 2 reduces to Case 1. Thus we have the

THEOREM. *The order and power problems in a* 1-*relator group are* $\mathscr{E}^{2+\sigma(R)}$-*decidable. In particular, they are* p.r.

5. Remarks, example and questions. We remark again that the groups considered in the Magnus process are determined, up to isomorphism by relabelling subscripts, only by the relator and not by the word under consideration. The induction on the length of the relator introduces at each stage two new groups, the amalgams. Thus the number of groups which must be considered is $\mathscr{E}^{\alpha(R)+1} - 1$ where $\alpha(R)$ is the number of stages in the induction and depends only on the relator R. It is also clear that a better bound on the computability level is $\mathscr{E}^{3+\alpha(R)}$ where $\alpha(R) \leqq \sigma(R)$-1.

In fact each stage of the recursion need not raise the computability level since in many cases the length of the word currently under consideration is reduced permitting a bound on the resursion. However every application of the case type $\sigma_{\text{generator}}$ (Relator) $\neq 0$ in the course of the recursion can increase the length of the rewriting. This being the only instance in which the length of the rewrite of W can increase, a better bound than $\mathscr{E}^{3+\alpha(R)}$ can be obtained for particular groups.

In view of the above, the authors had long erroneously conjectured that the computability bound $\mathscr{E}^3$ could be obtained for the Magnus process on any one relator group. The following example which shows the conjecture false has been brought to their attention. Let $G = gp(a, b; a^{-1}b^{-1}aba^{-1}bab^{-2})$, $X_0 = b$, $X_{k+1} = a^{-1}X_k^{-1}aba^{-1}X_k a$. Then $\sigma(X_k) = 2^k + 5(2^k - 1)$, an $\mathscr{E}^3$ function of k and $X \underset{KG}{=} b^{f_4(2,k)}$ for f_4 as defined in [4], an $\mathscr{E}^4 - \mathscr{E}^3$ function of k. This shows the Magnus process for G has a lower computability bound of $\mathscr{E}^4$. In fact a careful consideration of the recursion in the Magnus process for G shows one occurrence of the $\sigma_{\text{generator}}$ (Relator) $\neq 0$ case and exactly the $\mathscr{E}^4 - \mathscr{E}^3$ computability of the Magnus algorithm.

It is tempting to conjecture that the group G in the above example has $\mathscr{E}^4 - \mathscr{E}^3$ computable w.p. To see however that G has an $\mathscr{E}^3$ computable w.p. first observe $G \cong gp(a, b, c; c^{-1}bc = b^2, a^{-1}ba = c)$ by a Tietze transformation. Let $F = gp(b;)$ and $\phi_1: F \to F$ by $b \mapsto b^2$ (an $\mathscr{E}^3$ isomorphism in F an defined in [7]). Then by Tietze transformations the strong Britton extension $F_{\phi_1} = gp(b, c; c^{-1}bc = b^2) < gp(b, c;) *_{\psi_1} gp(\hat{b}, s;) = M$ where ψ_1 amalgamates H, the subgroup of $gp(\hat{b}, c;)$ generated (freely) by b and $c^{-1}bc$ with the subgroup $\hat{H}$ of $gp(\hat{b}, s;)$ generated (freely) by $\hat{b}$ and $s^{-1}\hat{b}^2s$ (see [7] proof of Theorem 3.1). We show M has $\mathscr{E}^3$ computable w.p. and hence so does F_{ϕ_1}. To see the w.p. in M is $\mathscr{E}^3$ let W be a word on $b, c, \hat{b}, s$. Then following the proof of Theorem 4.10 of [4], $W = W_1W_2 \cdots W_p$ in syllables and the w.p. for M is solved by rewriting the syllables to obtain a word of shorter syllable length. Since the rewriting are $\hat{b} \leftrightarrow b$ and $c^{-1}bc \leftrightarrow s^{-1}\hat{b}^2s$, the length of a rewrite is at most twice the length of the original and since each rewrite lowers the syllable length, the length of any rewrite is bounded by $2^p\sigma(W) \leqq 2^{\sigma(W)}\sigma(W)$, an $\mathscr{E}^3$ function of W. Thus the recursion has an $\mathscr{E}^3$ bound and the w.p. in M is $\mathscr{E}^3$. Similarly, ϕ_2 given by $\phi_2: b \mapsto c$ is an $\mathscr{E}^3$ isomorphism in F_{ϕ_1} and hence in M so $F_{\phi_1,\phi_2} < M_{\phi_2}$ a subgroup of a free product with amalgamation where in this case the rewrites do not increase the length of the word under consideration. Thus M_{ϕ_2} hence $F_{\phi_1,\phi_2} = gp(a, b, c; c^{-1}bc = b^2, a^{-1}ba = c) \cong G$ has $\mathscr{E}^3$ w.p.

Thus the question of whether or not the w.p. for one relator groups is $\mathscr{E}^3$ remains open. One might also ask if the example given can be generalized to obtain one relator groups with strictly $\mathscr{E}^n$ Magnus process and $\mathscr{E}^3$ w.p.

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