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FINITELY GENERATED IDEALS IN REGULAR F-ALGEBRAS

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Let A be a regular, semisimple, commutative F-algebra with identity. For each point in the spectrum of A, let $\mathscr{A}_p$ denote the local algebra of germs at p of elements of A and let $\mathscr{M}_p$ denote its maximal ideal. When $\mathscr{M}_p$ is finitely generated we show to what extent representatives of its generators are generators of the maximal ideals in the algebras of functions locally belonging to A on some neighborhood of p. We show that if $\mathscr{M}_p$ is finitely generated, then all point derivations of A at p are continuous. Using this last fact, we describe the generators of maximal ideals when the polynomials in finitely many elements of the algebra are dense in the algebra.

1. Preliminaries. Throughout we assume that all algebras are commutative algebras with identity over the complex field C and that all homomorphisms of algebras carry the identity of one to the identity of the other. For general references in topological algebras we refer the reader to [1] and [6].

For X a Hausdorff topological space, we denote by $C(X)$ the algebra of all complex-valued, continuous functions on X where $C(X)$ has the pointwise operations and the topology of compact convergence. The seminorms of this topology will be denoted $\|\cdot\|_K$ where K is a compact subset of X and for $f \in C(X)$, $\|f\|_K = \sup\{|f(x)|: x \in K\}$.

A *locally m-convex* (LMC) *algebra* is a locally convex (Hausdorff) topological algebra A with a topology given by a family $\{\|\cdot\|_n: n \in (D, \leqq)\}$ of submultiplicative seminorms. An F-algebra is a complete LMC algebra with a topology given by a countable family of seminorms. It can always be assumed that these families of seminorms are directed (i.e., if $n \leqq m$ in D, then $\|a\|_n \leqq \|a\|_m$ for all $a \in A$). If A is an LMC algebra and if $\{\|\cdot\|_n: n \in D\}$ is a directed family of seminorms for A, then for each $n \in D$, the set $\{x: \|x\|_n = 0\}$ is a closed ideal in A and $A/\{x: \|x\|_n = 0\}$ is a normed algebra with norm $\|\pi_n x\| = \|x\|_n$, where π_n is the natural map. Let A_n denote the completion of this algebra. If $n \leqq k$, then the maps π_n and π_k induce a norm-decreasing homomorphism π_{nk} of A_k onto a dense subalgebra of A_n, and $\{A_n, \pi_{nk}, D\}$ forms a dense inverse limit system. Moreover, $\lim\operatorname{inv} A_n$ is topologically and algebraically the completion of A, where A is imbedded via $\pi(x) = \{\pi_n x\}$. If A

is complete, then π is surjective and we identify A and $\lim\operatorname{inv} A_n$. Of interest to us later are the facts that an inverse limit of F-algebras is a complete LMC algebra and that the inverse limit of a countable family of F-algebras is an F-algebra.

The *spectrum* of A, denoted $\mathrm{Sp}(A)$, is the space of all nonzero, continuous, multiplicative, linear functionals on A with the Gelfand (relative weak*) topology. If A is a commutative F-algebra with identity, $\{\|\cdot\|_n\}$ is an increasing sequence of seminorms for A, and A_n and π_n are as above, then π_n induces a topological map of $\mathrm{Sp}(A_n)$ onto a compact set S_n of $\mathrm{Sp}(A)$ such that $S_1 \subset S_2 \subset \cdots$ and $\mathrm{Sp}(A) = \bigcup S_n$. Moreover, every compact subset of $\mathrm{Sp}(A)$ is contained in some S_n; hence, $\mathrm{Sp}(A)$ is hemicompact. This implies that $\mathrm{Sp}(A)$ is Lindelöf and $\mathrm{Sp}(A)$ is also completely regular and Hausdorff. Since $\mathrm{Sp}(A)$ is both Lindelöf and regular, it is paracompact and normal.

For each $f \in A$, define the mapping $\hat{f}: \mathrm{Sp}(A) \to C$ by $\hat{f}(x) = x(f)$, $x \in \mathrm{Sp}(A)$. $\hat{f}$ is called the Gelfand transform of f, and the mapping $f \to \hat{f}$ is a homomorphism of A onto a separating subalgebra $\hat{A}$ of $C(\mathrm{Sp}(A))$. A is called *semi-simple* if $a \in A$ and $\hat{a} \equiv 0$ on $\mathrm{Sp}(A)$ implies that $a = 0$. If A is semi-simple, then the Gelfand mapping $a \to \hat{a}$ is an algebraic isomorphism and we can regard A as an algebra of complex-valued functions on $\mathrm{Sp}(A)$ with the topology transferred from A via this isomorphism. This topology is weaker than the topology of compact convergence. Throughout, whenever an algebra is semi-simple we assume that it has been identified in this way.

A commutative LMC algebra A is said to be *regular* if for each closed set F in $\mathrm{Sp}(A)$ and each point $x \in \mathrm{Sp}(A)\backslash F$, there is an element a in A such that $\hat{a} = 0$ on F and $\hat{a}(x) = 1$. The algebra A is *normal* if for each pair F_1 and F_2 of disjoint closed subsets of $\mathrm{Sp}(A)$, there exists an element a in A such that $\hat{a} = 0$ on F_1 and $\hat{a} = 1$ on F_2. Regular F-algebras are normal (see [8, p. 160] or [3, p. 266]).

Let A be a regular, semi-simple, commutative F-algebra with identity. For S a subset of $\mathrm{Sp}(A)$, let $A \mid S$ denote the algebra of restrictions of functions in A to the set S. Let F be a closed subset of $\mathrm{Sp}(A)$. It is easy to see that the mapping $f \mid F \to f + k(F)$ gives an algebraic isomorphism of $A \mid F$ onto $A/k(F)$ where $k(F) = \{f \in A: f = 0 \text{ on } F\}$. But $A/k(F)$ with the quotient topology is a regular, semi-simple, commutative F-algebra with identity (see for instance [2, p. 264]). We transfer the topology of $A/k(F)$ to $A \mid F$ via the isomorphism described above.

If V is an open subset of $\mathrm{Sp}(A)$ and $f \in C(V)$, then we say that f *locally belongs* to A on V if for each point $x \in V$, there exists an open neighborhood U of x in V and an element $a \in A$ such that $a \mid U = f \mid U$. Let $A(V)$ denote the collection of all such function. It is shown in [3, p. 271] that $A(\mathrm{Sp}(A)) = A$. If $\mathrm{Sp}(A)$ is locally compact and V is an open subset of $\mathrm{Sp}(A)$, then we shall give $A(V)$ a locally m-convex topology by realizing it as a dense inverse limit of F-algebras of the form $A \mid K$ where K is a compact subset of V.

Let A be a regular, semi-simple, commutative F-algebra with identity such that $\mathrm{Sp}(A)$ is locally compact and let V be an open subset of $\mathrm{Sp}(A)$. Let $\{K_\lambda : \lambda \in \Lambda\}$ be the collection of all compact subsets of V where $\Lambda = \{K : K \text{ compact}, K \subset V\}$ is partially ordered by $\lambda \leqq \mu$ if and only if $K_\lambda \subset K_\mu$. For each $\lambda \in \Lambda$ let $A_\lambda = A \mid K_\lambda$ with its F-algebra topology defined above (K_λ is closed in V). Hence A_λ is a regular, semi-simple F-algebra with identity. If $\lambda \leqq \mu$, then $K_\lambda \subset K_\mu$ and the restriction mapping $r_\lambda^\mu : A_\mu \to A_\lambda$ is defined, continuous, and surjective. Hence $\{A_\lambda, r_\lambda^\mu, \Lambda\}$ is a dense inverse limit system of F-algebras. Let $A'(V) = \lim \mathrm{inv}\, A_\lambda$. Since each A_λ is an F-algebra, we have that $A'(V)$ is a complete, locally m-convex algebra. We next show that $A'(V)$ is algebraically isomorphic to $A(V)$. If $f \in A'(V)$, then we may represent $f = \{f_\lambda\}_{\lambda \in \Lambda}$ where $f_\lambda \in A_\lambda$ and $r_\lambda f_\mu = f_\lambda (\lambda \leqq \mu)$. For each $f \in A'(V)$, define $\tilde{f} : V \to C$ by $\tilde{f}(x) = f_\lambda(x)$ if $x \in K_\lambda$. Suppose $x \in K_\lambda \cap K_\mu$ and let $\nu \geqq \lambda, \mu$. Then $f_\lambda(x) = (r_\lambda^\nu f_\nu)(x) = f_\nu(x) = (r_\mu^\nu f_\nu)(x) = f_\mu(x)$. Hence $\tilde{f}$ is well-defined and $\tilde{f}$ is that unique function on V such that $\tilde{f} \mid K_\lambda = f_\lambda (\lambda \in \Lambda)$. Since V is locally compact, each $\tilde{f}$ is continuous on V. If $\tilde{f} \equiv 0$, then $f_\lambda = 0$ in A_λ for each λ since A_λ is semi-simple and thus $f = 0$ in $A'(V)$. Therefore $f \to \tilde{f}$ is a monomorphism of $A'(V)$ into $C(V)$. Furthermore, it is clear that the image is $\{f \in C(V) : f \mid K_\lambda \in A_\lambda (\lambda \in \Lambda)\}$ which is just $A(V) = \{f \in C(V) : F \text{ locally belongs to } A \text{ on } V\}$. To verify this statement, it is clear that $\{f \in C(V) : f \mid K_\lambda \in A_\lambda(\lambda \in \Lambda)\} \subseteqq A(V)$ because V is locally compact. To show the opposite inclusion, let $f \in A(V)$ and let K be a compact subset of V. Let W be an open set such that $K \subset W \subset \bar{W} \subset V$. Since K and $\mathrm{Sp}(A) \backslash W$ are closed subsets of $\mathrm{Sp}(A)$ and since A is normal, there exists g in A such that $g \mid K = 1$ and $g \mid \mathrm{Sp}(A) \backslash W = 0$. Now, $f \cdot g$ locally belongs to A on $\mathrm{Sp}(A)$; consequently, $f \cdot g \in A$. But $f \cdot g = f$ on K. Hence, $f \in A_K$. From this, the set inclusion is proven. Thus we may identify $A(V)$ via this isomorphism with $A'(V)$ and transfer the topology of $A'(V)$ to $A(V)$. Call that topology τ_V.

Since $A'(V) = \lim \mathrm{inv}\, A_\lambda$ and since $\bigcup_{\lambda \in \Lambda} \mathrm{Sp}(A_\lambda) = \bigcup_{\lambda \in \Lambda} K_\lambda = V$, it is clear that we may identify $\mathrm{Sp}(A'(V))$ with V. That identification we will call h and it is given of course by: if $\varphi \in \mathrm{Sp}(A'(V))$, then $h(\varphi)$ is that unique point in V such that $\varphi(f) = \tilde{f}(h(\varphi))$ for every $f \in A$. We need still show that the topologies are the same.

Let $\varphi_\alpha \rightarrow \varphi$ in $\mathrm{Sp}(A'(V))$. Then $f(\varphi_\alpha) \rightarrow f(\varphi)$ for every $f \in A'(V)$. If $g \in A$, then $g' = \{g \mid K_\lambda\} \in A'(V)$. Hence $g(h(\varphi_\alpha)) = g'(\varphi_\alpha) \rightarrow g'(\varphi) = g(h(\varphi))$. Hence, $h(\varphi_\alpha) \rightarrow h(\varphi)$ in V the relative topology from $\mathrm{Sp}(A)$. Therefore $h(\varphi_\alpha) \rightarrow h(\varphi)$ in V. It is clear that h^{-1} is continuous since the functions $\tilde{f}$ are in $C(V)$. Thus $\mathrm{Sp}(A'(V)) = V$.

Thus, if $\mathrm{Sp}(A)$ is locally compact and V is open, then $(A(V), \tau_V)$ is a semi-simple, commutative, complete, LMC algebra with identity such that $\mathrm{Sp}(A(V)) = V$. Furthermore, since $A \mid V$ is contained in $A(V)$, we have that $A(V)$ is regular.

If $\mathrm{Sp}(A)$ is second countable, then, since $\mathrm{Sp}(A)$ is also hemicompact, we have that $\mathrm{Sp}(A)$ is locally compact. Furthermore, if $\mathrm{Sp}(A)$ is second countable, we can choose a sequence $\{K_n\}$ of compact subsets of V covering V such that $A'(V) = \lim\mathrm{inv}\, A \mid K_n$. Consequently $A(V)$ is an F-algebra if $\mathrm{Sp}(A)$ is second countable. This topology will be used in Corollary 2.6 of the next section.

Notice that the topologies which have been given for $A \mid F$ and $A(V)$ are natural generalizations of the relationship between the topologies found in familiar examples: for instance, $C(R)$, $C((0, 1))$, and $C([0, 1])$.

2. Local maximal ideal structure.

Throughout this section, A is assumed to be a regular, semi-simple, commutative F-algebra with identity. At each point $p \in \mathrm{Sp}(A)$, we define the local algebra $\mathscr{A}_p$ of germs at p of functions in A. In this section information is obtained concerning the algebra A when the maximal ideal of $\mathscr{A}_p$ is finitely generated. Specific information is obtained about representatives of generators of the maximal ideal, about the number of generators of the maximal ideal, and about continuity of point derivations.

For V an open subset of $\mathrm{Sp}(A)$, $A(V)$ denotes the algebra of all continuous, complex-valued functions on V which locally belong to A on V. If F is a closed subset of $\mathrm{Sp}(A)$, then $A \mid F$ denotes the algebra of restrictions of elements of A to the set F; for a description of the topology of $A \mid F$ and a topology for $A(V)$ when V is locally compact see Section 1. For $p \in \mathrm{Sp}(A)$ let M_p, $M_p(V)$, and $M_p \mid F$ denote the maximal ideal of all elements of A, $A(V)$, and $A \mid F$, respectively, which vanish at p. Let J_p denote the ideal of all elements of A vanishing in neighborhoods of p, let $\mathscr{A}_p$ denote the factor algebra A/J_p with γ_p the natural projection of A onto $\mathscr{A}_p$, and let $\mathscr{M}_p = \gamma_p(M_p)$. Thus $\mathscr{A}_p$ is the algebra of germs at p of elements of A. It is easy to see that $\mathscr{A}_p$ is a *local algebra* (that is, $\mathscr{A}_p$ is a complex algebra with a unique maximal ideal) and that $\mathscr{M}_p$ is its unique maximal ideal.

LEMMA 2.1. *If $\{p_n\}$ is a sequence of distinct points such that $p_n \to p$ in* Sp(A), *then there exists $G \in \bar{J}_p$ such that $G(p_n) \neq 0$ for each n.*

Proof. Let $\{\|\cdot\|_k\}_{k=1}^{\infty}$ be an increasing sequence of semi-norms determining the topology of A. Since A is regular, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ contained in J_p such that $g_n(p_k) = 0$ if $k \neq n$, $g_n(p_n) \neq 0$, and $\|g_n\|_n < 1/2^n$. Let $G_n = \sum_{k=1}^{n} g_k$. Then $\{G_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A and consequently converges to some $G \in \bar{J}_p$. Since for each n, $G(p_n) = g_n(p_n)$, the proof if complete.

If p is isolated in Sp(A), then $J_p = M_p$ and $\mathscr{A}_p \cong C$. Throughout the rest of this section, we assume that p is not isolated in Sp(A) and also that Sp(A) has a countable neighborhood base at p. As an immediate consequence of these assumptions, Lemma 2.1 implies that J_p is not closed; hence $J_p \neq M_p$ and $\mathscr{A}_p$ is nontrivial.

We now obtain information about representatives of generators of $\mathscr{M}_p$ when $\mathscr{M}_p$ is finitely generated. An ideal is *n-generated* if it contains elements $a_1, \cdots, a_n$ such that each element of the ideal is of the form $\sum_{i=1}^{n} a_i b_i$. From the definition of $\mathscr{A}_p$, we see that $\mathscr{M}_p$ is finitely generated if and only if there exist finitely many functions $f_1, \cdots, f_n \in M_p$ such that to each $g \in A$ correspond an open neighborhood V of p, functions $g_1, \cdots, g_n \in A$, and $G \in k(V)$ such that $g - g(p) = \sum_{i=1}^{n} g_i f_i + G$. Notice that in general the neighborhood V may depend on the function g. The next theorem states that it is possible to choose the neighborhood independently of the particular function. We first need two lemmas. For functions $f_1, \cdots, f_n \in A$, let $Z(f_1, \cdots, f_n) = \{x \in \mathrm{Sp}(A): f_1(x) = \cdots = f_n(x) = 0\}$.

LEMMA 2.2. *If $f_1, \cdots, f_n \in M_p$ and $\gamma_p(f_1), \cdots, \gamma_p(f_n)$ generate $\mathscr{M}_p$, then there is an open neighborhood V of p in* Sp(A) *such that $Z(f_1, \cdots, f_n) \cap V = \{p\}$. Consequently* $[\mathrm{Sp}(A) \backslash Z(f_1, \cdots, f_n)] \cup \{p\}$ *is open in* Sp(A).

Proof. If not, since there is a countable neighborhood base at p, there exists a sequence of points $\{p_k\}_{k=1}^{\infty}$ converging to p which are contained in $Z(f_1, \cdots, f_n)$. By Lemma 2.1, there is an element g of A such that $g(p) = 0$ but $g(p_k) \neq 0$ for each k. By earlier comments, there exist a neighborhood U of p, functions $g_1, \cdots, g_n \in A$ and $G \in k(U)$ such that $g = \sum_{i=1}^{n} g_i f_i + G$. Consequently $g(p_k) = 0$ for k sufficiently large which gives a contradiction.

LEMMA 2.3. *Let V be an open subset of* Sp(A) *and let $f_1, \cdots, f_n$ be elements of A such that $Z(f, \cdots, f_n) \subset V$. If $g \in k(V)$, then* *there exist* $g_1, \cdots, g_n \in A$ *such that* $g = \sum_{i=1}^n g_i f_i$.

Proof. Let $F = \mathrm{Sp}(A) \backslash V$. Since $f_1 | F, \cdots, f_n | F$ have no common zero on F, the spectrum of $A | F$, there exist $h_1, \cdots, h_n \in A$ such that $(\sum_{i=1}^n h_i f_i) | F = 1$. Letting $g_i = gh_i$, $1 \leqq i \leqq n$, we have that $g = \sum_{i=1}^n g_i f_i$.

THEOREM 2.4. *Let* $f_1, \cdots, f_n$ *be representatives of generators of* $\mathscr{M}_p$, *let* $W = \mathrm{Sp}[(A) \backslash Z(f_1, \cdots, f_n)] \cup \{p\}$, *and let* V *be an open neighborhood of* p *such that* $\bar{V} \subset W$. *Then for each* $g \in A$, *there exist* $g_1, \cdots, g_n \in A$ *and* $G \in k(V)$ *such that* $g - g(p) = \sum_{i=1}^n g_i f_i + G$. *Furthermore, if* $Z(f_1, \cdots, f_n) = \{p\}$, *then* $f_1, \cdots, f_n$ *generate* M_p.

Proof. In the case that $Z(f_1, \cdots, f_n) = \{p\}$, then $W = \mathrm{Sp}(A)$, $k(W) = \{0\}$ since A is semi-simple, and we may choose $V = W$. Let $g \in A$. By an earlier remark, there exist an open neighborhood U of p (which is contained in V) and functions $g'_1, \cdots, g'_n \in A$ and $G' \in k(U)$ such that $g - g(p) = \sum_{i=1}^n g'_i f_i + G'$. Applying Lemma 2.3 to the algebra $A | \bar{V}$ where $Z(f_1 | \bar{V}, \cdots, f_n | \bar{V}) = \{p\} \subset U \subset \mathrm{Sp}(A | \bar{V})$ and $G' | \bar{V}$ vanishes on U, we have that there exist $h_1, \cdots, h_n \in A$ such that $G' | \bar{V} = (\sum_{i=1}^n h_i f_i) | \bar{V}$. Let $g_i = g'_i + h_i$, $1 \leqq i \leqq n$, and $G = G' - \sum_{i=1}^n h_i f_i$. Then $G \in k(V)$ and $g - g(p) = \sum_{i=1}^n g_i f_i + G$.

Let $N = N(p)$ and $n = n(p)$ denote the minimal number of generators of M_p and $\mathscr{M}_p$ respectively. We have not been able to show that $N = n$ except in special cases (for instance, if A is closed under complex conjugation), but we do get the following:

COROLLARY 2.5. M_p *is finitely generated if and only if* $\mathscr{M}_p$ *is finitely generated. In fact,* $n \leqq N \leqq n + 1$.

Proof. Assume that $n < \infty$ and let $f_1, \cdots, f_n$ be representatives of generators of $\mathscr{M}_p$. Let U and V be open neighborhoods of p such that $\bar{U} \subset V$ and $Z(f_1, \cdots, f_n) \cap V = \{p\}$. Since A is regular, there exists a function $f \in A$ such that $f(p) = 0$ and $f = 1$ on $\mathrm{Sp}(A) \backslash U$. But since $\gamma_p(f_1), \cdots, \gamma_p(f_n), \gamma_p(f)$ generate $\mathscr{M}_p$ and $Z(f_1, \cdots, f_n, f) = \{p\}$, Theorem 2.4 guarantees that $f_1, \cdots, f_n, f$ generate M_p and thus $N \leqq n + 1$. The rest of the proof of this corollary is clear.

If $f_1, \cdots, f_m$ generate the maximal ideal M_p, then p must be their only common zero. If $f_1, \cdots, f_m$ are representatives of generators of $\mathscr{M}_p$, then p might not be their only common zero. To what extent they can generate a maximal ideal is given in Theorem 2.4

and the following corollary.

COROLLARY 2.6. *If $\gamma_p(f_1), \cdots, \gamma_p(f_n)$ generate $\mathscr{M}_p$, $f_1, \cdots, f_n$ are representatives of these generators, and* $W = [\mathrm{Sp}(A)\backslash Z(f_1, \cdots, f_n)] \cup \{p\}$, *then* (i) *if F is a closed subset of W containing p, then $f_1 | F, \cdots, f_n | F$ generate $M_p | F$ and* (ii) *if U is a second countable open subset of W containing p, then $f_1 | U, \cdots, f_n | U$ generate $M_p(U)$.*

Proof. Let V be an open set such that $F \subset V \subset \bar{V} \subset W$. If $g \in A$, there exist $g_1, \cdots, g_n \in A$ and $G \in k(V)$ such that $g - g(p) = \sum_{i=1}^n g_i f_i + G$. By restricting to F, we see that (i) is established. If U is second countable, then we give $A(U)$ an F-algebra topology such that $A(U)$ is regular and $\mathrm{Sp}(A(U)) = U$. (See Section 1 for details.) But $Z(f_1 | U, \cdots, f_n | U) = \{p\}$, and clearly the germs of $f_1 | U, \cdots, f_n | U$ in the algebra of germs of $A(U)$ functions at p generate the maximal ideal; hence Theorem 2.4 applies to the algebra $A(U)$ and (ii) follows.

In order to obtain more information about generators of $\mathscr{M}_p$, it is convenient to study point derivations and tangent vectors on A. For $p \in \mathrm{Sp}(A)$, there is a natural notion of the value of a germ α at p since representatives of α must agree in value at p. Define $\alpha(p) = f(p)$ where $f \in A$ and $\gamma_p(f) = \alpha$. A *tangent vector* of A at p is a linear functional v on $\mathscr{A}_p$ satisfying $v(\alpha\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha)$ for all $\alpha, \beta \in \mathscr{A}_p$. $T(\mathscr{A}_p)$ will denote the collection of all tangent vectors of A at p. A *point derivation* of A at p is a linear functional D on A satisfying $D(fg) = f(p)D(g) + g(p)D(f)$ for all $f, g \in A$. Let $T_p(A)$ denote the collection of all point derivations of A at p.

$T(\mathscr{A}_p)$ and $T_p(A)$ with the natural operations of addition and scalar multiplication are vector spaces over C. Let $[\mathscr{M}_p/\mathscr{M}_p^2]^*$ and $[M_p/M_p^2]^*$ denote the algebraic duals of the vector spaces $\mathscr{M}_p/\mathscr{M}_p^2$ and M_p/M_p^2 respectively (vector spaces with the quotient operations).

LEMMA 2.7. $[\mathscr{M}_p/\mathscr{M}_p^2]^* \cong T(\mathscr{A}_p) \cong T_p(A) \cong [M_p/M_p^2]^*$. *If $\mathscr{M}_p$ is finitely generated, then $\mathscr{M}_p/\mathscr{M}_p^2 \cong T(\mathscr{A}_p) \cong T_p(A) \cong M_p/M_p^2$ and each of these vector spaces is finite dimensional.*

Proof. In the first statement, the outside isomorphisms follow since $T(\mathscr{A}_p)$ consists precisely of those linear functionals on $\mathscr{A}_p$ which vanish on $\mathscr{M}_p^2 + C$ and that $T_p(A)$ consists precisely of those linear functionals on A which vanish $M_p^2 + C$ (see [10, p. 263]). Define φ: $T(\mathscr{A}_p) \to T_p(A)$ by $\varphi(v) = v \circ \gamma_p$ for $v \in T(\mathscr{A}_p)$. It is clear that φ is linear and injective. Since every $D \in T_p(A)$ vanishes on J_p, φ is surjective, and the first statement has been proved.

If $\mathscr{M}_p$ is finitely generated, so is M_p. To establish this lemma, we shall only show that M_p/M_p^2 is finite dimensional since the argument that $\mathscr{M}_p/\mathscr{M}_p^2$ is finite dimensional is similar. The isomorphisms in the second statement follow from this finite dimensionality and the isomorphisms in the first part. Let $f_1, \cdots, f_m$ generate M_p, and let $g \in M_p$. Then there exist $g_1, \cdots, g_m \in A$ such that $g = \sum_{i=1}^m g_i f_i = \sum_{i=1}^m g_i(p) f_i + \sum_{i=1}^m (g_i - g_i(p)) f_i$. Since the latter sum is in M_p^2, we see that $\{f_i + M_p^2\}_{i=1}^m$ spans M_p/M_p^2 and the proof if complete.

To describe a basis for $T(\mathscr{A}_p)$ when $\mathscr{M}_p$ is finitely generated, we suppose that $n = n(p)$ is finite and let $\alpha_1, \cdots, \alpha_n$ be generators of $\mathscr{M}_p$. Define $\theta_1, \cdots, \theta_n \in T(\mathscr{A}_p)$ such that $\theta_k(\alpha_j) = \delta_{kj}$ (the Kronecker delta) by $\theta_k(\beta) = \beta_k(p)$ where $\beta_1, \cdots, \beta_n \in \mathscr{A}_p$ satisfy $\beta - \beta(p) = \sum_{j=1}^n \beta_j \alpha_j$. Since $\mathscr{A}_p$ is a local algebra and since n is the minimum number of generators of $\mathscr{M}_p$, we have that each θ_k is well-defined. It is straightforward to verify that $\theta_1, \cdots, \theta_n \in T(\mathscr{A}_p)$. The proof of the following lemma is omitted. (The proof is similar to a proof in [7, p. 57].)

LEMMA 2.8. *If $\alpha_1, \cdots, \alpha_n$ generate $\mathscr{M}_p$, then the tangent vectors $\theta_1, \cdots, \theta_n$ defined above form a basis for $T(\mathscr{A}_p)$. If $D_k = \theta_k \circ \gamma_p$, $1 \leqq k \leqq n$, then $D_1, \cdots, D_n$ form a basis for $T_p(A)$.*

THEOREM 2.9. *If $\mathscr{M}_p$ is finitely generated, then every point derivation of A at p is continuous.*

Proof. Because $\mathscr{M}_p$ is finitely generated, so also is M_p finitely generated, and M_p^2 has finite codimension in M_p. We now prove that M_p^2 is closed in M_p as follows. Suppose that $f_1, \cdots, f_n$ generate M_p. Let A_n be the direct product of n copies of M_p. Now, M_p is a Fréchet space; consequently, A_n with the product topology is also a Fréchet space. Let Φ be the mapping of A_n into M_p defined by $\Phi(g_1, \cdots, g_n) = f_1 g_1 + \cdots + f_n g_n$. Then, Φ is a continuous linear map of A_n into M_p whose range $\Phi(A_n)$ is M_p^2. Thus its range has finite codimension. Using the Open-mapping Theorem as in the proof of the corresponding theorem for Banach Spaces (see [5, p. 186]), we conclude that M_p^2 is closed.

To complete the proof, every element $D \in T_p(A)$ factors as $D = D^* \circ \pi \circ T$ where $D^* \in (M_p/M_p^2)^*$, π is the natural projection of M_p onto M_p/M_p^2, and $T: A \to M_p$ is defined by $T(f) = f - f(p)$. Because M_p^2 is closed, M_p/M_p^2 with the quotient topology is a Hausdorff, finite-dimensional vector space. Hence D^* is continuous and it is clear that π and T are continuous; therefore D is continuous.

We will use the information that we have derived about point derivations of A at p to obtain more information about generators of $\mathscr{M}_p$. As before we let $n = n(p)$ denote the minimal number of generators of $\mathscr{M}_p$.

LEMMA 2.10. *Suppose that* $\alpha_1, \cdots, \alpha_n$ *generate* $\mathscr{M}_p$ *and define tangent vectors* $\theta_1, \cdots, \theta_n$ *with respect to these generators. If* $\beta \in \mathscr{M}_p$ *and* $\theta_1(\beta) \neq 0$, *then* $\beta, \alpha_2, \cdots, \alpha_n$ *generate* $\mathscr{M}_p$.

Proof. Let $\beta_1, \cdots, \beta_n \in \mathscr{A}_p$ satisfy $\beta = \sum_{i=1}^n \beta_i \alpha_i$. β_1 is invertible in $\mathscr{A}_p$ since $\beta_1(p) = \theta_1(\beta) \neq 0$. Hence α_1 is in the span of $\beta, \alpha_2, \cdots, \alpha_n$ and the conclusion follows.

The next theorem describes the generators of a finitely generated maximal ideal when the polynomials in finitely many elements are dense in the algebra. (This extends to regular F-algebras a theorem of Banach algebras [4, Theorem 2.2]. Also, compare this theorem to [9, Proposition 8.3] since $n(p)$ is the dimension of $T_p(A)$ when $\mathscr{M}_p$ is finitely generated.)

THEOREM 2.11. *Suppose that the polynomials in* $u_1, \cdots, u_m$ *are dense in* A *and that* $\mathscr{M}_p$ *is finitely generated. Then* M_p *is generated by* $u_1 - u_1(p), \cdots, u_m - u_m(p)$, $\mathscr{M}_p$ *is generated by* $n = n(p)$ *of* $\gamma_p(u_1 - u_1(p)), \cdots, \gamma_p(u_m - u_m(p))$, *and* $N(p) \leqq m$.

Proof. Let $\beta_i = \gamma_p(u_i - u_i(p))$, $1 \leqq i \leqq m$. It suffices to show that n of $\beta_1, \cdots, \beta_m$ generate $\mathscr{M}_p$ since $Z(u_1 - u_1(p), \cdots, u_m - u_m(p)) = \{p\}$. This proof consists in inductively applying Lemma 2.10 to specific sets of generators of $\mathscr{M}_p$. Let $\alpha_1, \cdots, \alpha_n$ generate $\mathscr{M}_p$ and define the tangent vectors $\theta_{11}, \cdots, \theta_{1n}$ with respect to these generators and let $D_{1k} = \theta_{1k} \circ \gamma_p$ as in Lemma 2.8. Since each D_{1k} is continuous on A and nontrivial, and since the polynomials in $u_1, \cdots, u_m$ are dense in A, there exist integers j and k, $1 \leqq j \leqq m$, $1 \leqq k \leqq n$, such that $\theta_{1k}(\beta_j) = D_{1k}(u_j) \neq 0$. For definiteness, we assume that $\theta_{11}(\beta_1) \neq 0$ and by Lemma 2.10, we have that $\beta_1, \alpha_2, \cdots, \alpha_n$ generate $\mathscr{M}_p$. If $n = 1$, the proof is complete. If not, define tangent vectors $\theta_{21}, \cdots, \theta_{2n}$ and corresponding point derivations $D_{21}, \cdots, D_{2n}$ with respect to $\beta_1, \alpha_2, \cdots, \alpha_n$. As before, we can conclude that for some integers j and k, $2 \leqq j \leqq m$, $2 \leqq k \leqq n$, $\theta_{2k}(\beta_j) = D_{2k}(u_j) \neq 0$. Thus we can replace β_j and α_k in the system of generators of $\mathscr{M}_p$. Continuing this argument inductively gives the desired conclusion since it will be clear that n can be no greater than m.

We now give two examples; the first shows that there may be strict inequality in the conclusion of the last theorem. Before we consider the examples we prove a lemma which we will use.

LEMMA 2.12. *If A is closed under complex-conjugation, then* $N(p) = n(p)$.

Proof. By Corollary 2.5 we need only consider the case that $n = n(p)$ is finite. Since A is closed under conjugation, applications of Lemma 2.10 to real and imaginary parts of generators give that there are real-valued functions $f_1, \cdots, f_n$ in A such that their germs generate $\mathscr{M}_p$. Let V be a neighborhood of p such that $Z(f_1, \cdots, f_n) \cap V = \{p\}$. Since A is regular, it is also normal; hence, because $\mathrm{Sp}(A)$ is a normal topological space, there is a real-valued function f in J_p such that $f \equiv 1$ in a neighborhood of $\mathrm{Sp}(A)\backslash V$. Since $\gamma_p(f_1 + if), \gamma_p(f_2), \cdots, \gamma_p(f_n)$ generate $\mathscr{M}_p$ and $Z(f_1 + if, f_2, \cdots, f_n) = \{p\}$, we conclude that $f_1 + if, f_2, \cdots, f_n$ generate M_p and that $N(p) \leqq n(p)$. Therefore $N(p) = n(p)$.

EXAMPLE 2.13. The algebra $C^\infty(R^2)$ is a regular, semi-simple, commutative F-algebra with identity such that all of its maximal ideals are two-generated. Let $F = \{(r, |r|): r \in R\}$ and let A_1 be the restriction of $C^\infty(R^2)$ to the set F with the quotient topology (see Section 1). Let $h: R \rightarrow F$ be the homeomorphism given by $h(r) = (r, |r|)$, and let $A = \{f \circ h: f \in A_1\}$. Then A is algebraically isomorphic to A_1 via the isomorphism induced by this homeomorphism and we can transfer the topology of A_1 to A. Then A is a regular, semi-simple, commutative F-algebra with identity such that $\mathrm{Sp}(A) = R$. Since the polynomials in the coordinate functions on R^2 are dense in $C^\infty(R^2)$, we see that the polynomials in x and $|x|$ are dense in A where x denotes the coordinate function of R. Since A is closed under conjugation $N(p) = n(p) \leqq 2$ for every $p \in R$. We will now show that $n(p) = 1$ for $p \neq 0$ and $n(0) = 2$. Since the maximal ideals of $C^\infty(R^2)$ are generated by appropriate translates of the coordinate functions, it is easy to see that M_p is generated by $x - p$ and $|x| - p$ for every $p \in R$. But since $\gamma_p(x - p) = \pm\gamma_p(|x| - |p|)$ for $p \neq 0$, we see that for $p \neq 0$, $\mathscr{M}_p$ is generated by $\gamma_p(x - p)$ and $n(p) = 1$. To show that $n(0) = 2$, we assume that $n(0) = 1$ and that h is a generator of M_0. Consequently, there exist h_1 and h_2 in A such that $x = h_1 h$ and $|x| = h_2 h$. It is easy to show that $h_1(0) = h_2(0) = 0$. But since the polynomials in x and $|x|$ are dense in A and since there are nontrivial point derivations on A at 0, we have a contradiction. Therefore $n(0) = 2$.

The final example shows that in a regular F-algebra a finitely generated maximal ideal can be isolated. This cannot happen in a Banach algebra (see [4, Theorem 2.1)].

EXAMPLE 2.14. For all positive integers k and n, let $K_n = [-n, n]$, $I_n = (-1/n, 1/n)$, and $I_{n,k} = [-1/n + 1/nk, 1/n - 1/nk]$. Let A be the algebra of all continuous, complex-valued functions on R which are n-times continuously differentiable on I_n for each n. For a compact subset K of R, $\|\cdot\|_K$ will denote the supremum semi-norm, and for positive integers n and j, $\|\cdot\|_{n,j}$ will denote the semi-norm on A given by $\|f\|_{n,j} = \sum_{i=0}^{n}(1/i!)\|f^{(i)}\|_{I_{n,j}}$. Give A the topology induced by the semi-norms $\{\|\cdot\|_{K_n}, \|\cdot\|_{n,j}: n, j = 1, 2, \cdots\}$. Then A with this topology is a semi-simple, commutative F-algebra with identity. Furthermore, A contains $C^\infty(R)$, the polynomials in the coordinate function x are dense in A, $\mathrm{Sp}(A) = R$, and A is regular. It is straightforward (for example, by using L'Hospital's rule repeatedly) to show that M_0 is generated by x. For $p \neq 0$, M_p is not finitely generated, for if it were it would have to be generated by $x - p$ (Theorem 2.11). But it is an easy matter to construct functions in A which are not divisible by $x - p$ (construct such a function to have the minimum amount of differentiability required at p). Hence, M_0 is the only finitely generated maximal ideal in A. Every function in A is infinitely differentiable at 0; we will now show that not only does there not exist a fixed neighborhood of 0 such that all functions in A are infinitely differentiable in that neighborhood, but that there exist functions in A which are not infinitely differentiable in any neignborhood of 0. Let $\{\|\cdot\|_n\}$ be an increasing sequence of semi-norms determining the topology of A. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in A satisfying (1) $f_n^{(n+1)}$ does not exist at some point p_n of $(1/(n+1), 1/n)$, (2) $f_n \equiv 0$ off $(1/(n+1), 1/n)$ and (3) $\|f_n\|_n \leqq 1/2^n$. Define $g = \sum_{n=1}^\infty f_n$, which exists in A by (3). Furthermore by (1) and (2), $g^{(n+1)}(p_n)$ does not exist and hence, since $p_n \to 0$, g is not infinitely differentiable in any neighborhood of 0.

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