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THE DECOMPOSITION OF MULTIPLICATION OPERATORS ON L_p-SPACES

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A multiplication operator on an L_p-space is factored as the direct sum of cyclic parts and a singular part. The equivalence of this decomposition with Rohlin's Theorem on decomposition of measure spaces is shown.

1. Introduction. Let (X, Σ, μ) be a separable measure space and suppose f is in $L_\infty(X, \Sigma, \mu)$. The bounded operator M_f on $L_P(X, \Sigma, \mu)$ defined by $M_f(g) = f \cdot g$, for $g \in L_p(X, \Sigma, \mu)$, is called a multiplication operator.

If $p = 2$, then a multiplication operator is normal on $L_2(X, \Sigma, \mu)$. Thus it may be decomposed as the direct sum of cyclic normal operators. These operators need not themselves be multiplication operators. If $1 \leqq p < \infty$ and $p \neq 2$, then in general, it is not possible to decompose $L_p(X, \Sigma, \mu)$ into the p-direct sum of subspaces such that the restriction of a multiplication operator to each of these subspaces is cyclic. (For the definition of a p-direct summand see [7], Definition 1.1.)

With the aid of Rohlin's Theorem ([5]) in the form presented by Akcoglu ([1]), we obtain a decomposition theorem for multiplication operators on L_p-spaces. A multiplication operator on $L_p(X, \Sigma, \mu)$, $p \neq 2$, is shown to be the direct sum of a regular part and a singular part. The regular part is decomposible as a direct sum of cyclic subparts while the singular part does not possess a cyclic subpart.

We show, in turn, that this decomposition theorem implies Rohlin's theorem.

2. Preliminaries. Let (X, Σ, μ) be a separable measure space. If X is a topological space, then Σ will be the Borel σ-algebra denoted by $\mathscr{B}(X)$ (or simply $\mathscr{B}$ if no ambiguity arises). If X is the unit interval, then we will denote X by J and the usual Borel measure space will be represented as $(J, \mathscr{B}(J), \lambda)$.

For ease of notation we will abbreviate $L_p(X, \Sigma, \mu)$ by $L_p(\mu)$, for $1 \leqq p \leqq \infty$, when no confusion will arise.

Suppose $f \in L_\infty(X, \Sigma, \mu)$.

DEFINITION 2.1. The measure ϕ_f on $\{\mathbf{C}, \mathscr{B}(\mathbf{C})\}$ defined by $\phi_f(B) = \mu\{f^{-1}(B)\}$, for $B \in \mathscr{B}(\mathbf{C})$, is called *the measure associated with f.*

We shall consider the multiplication operator $M_f \in B\{L_p(\mu)\}$, $1 \leqq p < \infty$. We denote its spectrum by $\sigma(M_f)$. Then the measure associated with f may be thought of as the measure associated with the operator M_f. Since $\sigma(M_f)$ is the essential range of f, we see that the support of ϕ_f is just $\sigma(M_f)$. Thus we interchangeably think of ϕ_f as a measure on $(\mathbf{C}, \mathscr{B}(\mathbf{C}))$ or on $(\sigma(M_f), \mathscr{B}(\sigma(M_f)))$.

Associated with a multiplication operator M_f is a spectral measure $\Phi_f: \mathscr{B}(\mathbf{C}) \to B(L_p(\mu))$ defined by $\Phi_f(B) = M_{\chi\{f^{-1}(B)\}}$, and $\phi_f(B) = \int_X \Phi_f(B)\chi(X)d\mu$, an extended real number, for $B \in B(\mathbf{C})$.

Let $g \in L_p(X, \Sigma, \mu)$ where $1 \leqq p < \infty$. The measure ω_g defined on $(\mathbf{C}, \mathscr{B}(\mathbf{C}))$ by $\omega_g(B) = \int_X |\Phi_f(B)g|^p d\mu$ is clearly absolutely continuous with respect to ϕ_f.

If $A \in \Sigma$, then $M_{f|_A}$ is a multiplication operator on the space $L_p(A, \Sigma|_A, \mu|_A)$ which is identified with the subspace $M_{\chi(A)}(L_p(X, \Sigma, \mu))$ of $L_p(X, \Sigma, \mu)$. We see that $\phi_{f|_A} \ll \phi_f$.

DEFINITION 2.2. Let ϕ be any σ-finite measure on $(\mathbf{C}, \mathscr{B}(\mathbf{C}))$. Then $\mathscr{L}_\phi \equiv \{g \in L_p(X, \Sigma, \mu) | \omega_g \ll \phi\}$ is *the subspace of* $L_p(\mu)$ *generated by* ϕ.

DEFINITION 2.3. Let g be a measurable function on (X, Σ, μ). Then the support of g (written supp(g)) is $\{x \in X \mid |g(x)| > 0\}$.

Let $f \in L_\infty(X, \Sigma, \mu)$.

LEMMA 2.1. *If ϕ is any σ-finite measure on $\{\mathbf{C}, \mathscr{B}(\mathbf{C})\}$ such that $\phi \ll \phi_f$, then there exists $g \in L_p(\mu)$ such that $\omega_g \approx \phi$. Moreover, there exists $A_\phi \in \Sigma$ such that $\mathscr{L}_\phi = M_{\chi(A_\phi)}(L_p(\mu))$ and $\omega_g \approx \phi_{f|A_\phi}$.*

Proof. Without loss of generality we may assume that ϕ is a finite measure. The Radon-Nikodym derivative $d\phi/d\phi_f \equiv h$ is in $L_1\{\mathbf{C}, \mathscr{B}(\mathbf{C}), \phi_f\}$. Clearly if B and D are in $\mathscr{B}(\mathbf{C})$, then $\int_B \chi(D)d\phi_f = \int_{f^{-1}(B)} \chi(D)\circ f d\mu$. By the Monotone Convergence Theorem it follows that $\phi(B) = \int_B h d\phi_f = \int_{f^{-1}(B)} h \circ f d\mu$. Let g be $(h \circ f)^{1/p}$. Then we see that $g \in L_p(\mu)$ and $\omega_g(B) = \phi(B)$, for $B \in \mathscr{B}(\mathbf{C})$.

There is a Lebesgue decomposition of ϕ_f such that $\phi_f = \rho + \eta$ where $\rho \approx \phi$ and $\eta \perp \phi$. There exists $B_0 \in \mathscr{B}(\mathbf{C})$ such that $\eta(B_0) = \rho(\mathbf{C}\backslash B_0) = 0$. Let A_ϕ be $f^{-1}(B_0)$. Then $M_{\chi(A_\phi)}\{L_p(\mu)\} \subset \mathscr{L}_\phi$ and $\phi_{f|A_\phi} = \rho$.

Suppose there exists $g_0 \in \mathcal{L}_\phi$ such that $F \equiv \operatorname{supp}(g_0) \cap (X \backslash A_\phi)$ is not equal to the empty set a.e. μ. Then there exists $G \in \mathcal{B}(\mathbf{C})$ such that $G \cap B_0 = \varnothing$ a.e. ϕ_f and $f^{-1}(G) \supset F$. Hence $\omega_{g_0}(G) > 0$ while $\phi(G) = 0$ which is a contradiction. Thus $\mathcal{L}_\phi \subset M_{\chi(A_\phi)}(L_p(\mu))$.

DEFINITION 2.4. The set A_ϕ associated with the measure $\phi \ll \phi_f$ (as in Lemma 2.1) is called *the pre-support of* ϕ.

In the sequel, we adopt the notation $\{a_n\}_{n=1}^{L\leqq\infty}$ to mean the finite sequence $\{a_n\}_{n=1}^{L}$, if $L < \infty$, or the countably infinite sequence $\{a_n\}_{n\in\mathbf{N}}$ if $L = \infty$. We shall use similar notation in sums, unions, etc. In addition, if $L = \infty$, then the expression "$1 \leqq n \leqq L$" will mean "all $n \in \mathbf{N}$".

3. A decomposition theorem. Let f be an element of $L_\infty(X, \Sigma, \mu)$.

DEFINITION 3.1. If A is in Σ, then the multiplication operator $M_{f|_A}$ on $L_p(A, \Sigma|_A, \mu|_A)$ is called a *part* of M_f (on $L_p(\mu|_A)$).

DEFINITION 3.2. The operator M_f is *cyclic* if there exists a function $g \in L_p(\mu)$ such that the set $\{p(M_f)(g) \mid p(z)$ is a polynomial in $z\}$ is a norm-dense subset of $L_p(\mu)$. We say that M_f is *singular* if it has no cyclic parts and that M_f is *regular* if it has no nonzero singular parts.

DEFINITION 3.3. Let Y and Z be Banach spaces. A bounded operator T on Y is *isometrically equivalent* to a bounded operator U on Z if there exists a surjective isometry $K: Y \to Z$ such that $KT = UK$.

REMARK 3.1. Let (X, Σ, μ) be a separable measure space and let $\{A_i\}_{i=1}^{L\leqq\infty}$ be a sequence of pairwise disjoint sets of Σ with $\bigcup_{i=1}^{L} A_i = X$ a.e. μ and $A_i \neq \varnothing$ a.e. μ for $1 \leqq i \leqq L$. Then $L_p(X, \Sigma, \mu)$ is isometrically isomorphic to $\bigoplus_{i=1}^{L} L_p(A_i, \Sigma|_{A_i}, \mu|_{A_i})$ via the mapping $g \to \Sigma_{i=1}^{L} g|_{A_i}$ for g in $L_p(X, \Sigma, \mu)$. Under this mapping, a multiplication operator M_f on $L_p(X, \Sigma, \mu)$ is isometrically equivalent to $\bigoplus_{i=1}^{L} M_{f|_{A_i}}$. Thus we will say that $M_f = \bigoplus_{i=1}^{L} M_{f|_{A_i}}$.

DEFINITION 3.4. A multiplication operator M_f on $L_p(X, \Sigma, \mu)$, with associated measure ϕ_f, has a *cyclic decomposition* if

$$M_f = \bigoplus_{i=1}^{L\leqq\infty} M_{f|_{A_i}} \qquad \text{on} \qquad \bigoplus_{i=1}^{L} L_p(A_i, \Sigma|_{A_i}, \mu|_{A_i}),$$

where $\{A_i\}_{i=1}^{L}$ is a pairwise disjoint sequence of sets of Σ with $\bigcup_{i=1}^{L} A_i = X$ a.e. μ, such that $M_{f|_{A_i}}$ is cyclic on $L_p(\mu|_{A_i})$ and its associated measure $\phi_{f|_{A_i}}$ is equivalent to ϕ_f for $1 \leqq i \leqq L$.

REMARK 3.2. Suppose M_f on $L_p(\mu)$ has a cyclic decomposition; then the cardinality of this decomposition is unique, i.e., any two cyclic decompositions for M_f have the same cardinality (see **[4]** Theorem 10.4.7, [7] Theorem 2.5).

DEFINITION 3.5. Let M_f be a regular multiplication operator on $L_p(X, \Sigma, \mu)$. Suppose $\phi \ll \phi_f$ is a measure with pre-support $A_\phi \in \Sigma$. Then ϕ is an *invariant* for M_f if:

(i) $M_{f|A_\phi}$ on $L_p(A_\phi, \Sigma|_{A_\phi}, \mu|_{A_\phi})$ has a cyclic decomposition;

(ii) if $\tau \ll \phi_f$ is a measure with pre-support $A_\tau \in \Sigma$ such that $M_{f|A_\tau}$ on $L_p(A_\tau, \Sigma|_{A_\tau}, \mu|_{A_\tau})$ has a cyclic decomposition of the same cardinality as that for $M_{f|A_\phi}$, then τ is absolutely continuous with respect to ϕ.

The cardinality of the cyclic decomposition of $M_{f|A_\phi}$, for ϕ an invariant, is called the *multiplicity* of ϕ (written $\mathcal{M}(\phi)$).

THEOREM 3.1. *If ϕ_1 and ϕ_2 are two invariants of the operator M_f on $L_p(X, \Sigma, \mu)$, then either ϕ_1 is equivalent to ϕ_2 or else ϕ_1 is singular with respect to ϕ_2.*

Proof. Let A_{ϕ_1} and A_{ϕ_2} be the pre-supports of ϕ_1 and ϕ_2 respectively. Suppose $\bigoplus_{i=1}^{\mathcal{M}(\phi_1)} M_{f|B_i}$ and $\bigoplus_{i=1}^{\mathcal{M}(\phi_2)} M_{f|C_i}$ are cyclic decompositions for $M_{f|A_{\phi_1}}$ and $M_{f|A_{\phi_2}}$ respectively. If $\phi_1 \not\perp \phi_2$, then there is a Lebesgue decomposition for ϕ_2 such that $\phi_2 = \phi_2^1 + \phi_2^2$ where $\phi_2^1 \ll \phi_1$ and $\phi_2^2 \perp \phi_1$ with $\phi_2^1 \neq 0$. Thus we have $\mathcal{L}_{\phi_2^1} \subset \mathcal{L}_{\phi_1}$ and $\mathcal{L}_{\phi_2^1} \neq (0)$. Let $A_{\phi_2^1}$ be the pre-support of ϕ_2^1. Then we have $A_{\phi_2^1} \subset A_{\phi_1}$ a.e. μ and $M_{f|A_{\phi_2^1}}$ has a cyclic decomposition given by $\bigoplus_{i=1}^{\mathcal{M}(\phi_1)} M_{f|B_i \cap A_{\phi_2^1}}$. But $\phi_2^1 \ll \phi_2$ implies that $\mathcal{L}_{\phi_2^1} \subset \mathcal{L}_{\phi_2}$ and thus $M_{f|A_{\phi_2^1}}$ has a cyclic decomposition given by $\bigoplus_{i=1}^{\mathcal{M}(\phi_2)} M_{f|C_i \cap A_{\phi_2^1}}$. Thus we conclude that $\mathcal{M}(\phi_1) = \mathcal{M}(\phi_2)$ and hence $\phi_1 \approx \phi_2$.

LEMMA 3.1. *Let M_f be a regular multiplication operator on $L_p(X, \Sigma, \mu)$ with associated measure ϕ_f. Suppose there exists a sequence of measures $\{\phi_i\}_{i=1}^{L \leqq \infty}$ such that for $1 \leqq i \leqq L$:*

(i) *$\phi_i \ll \phi_f$ with pre-support $A_{\phi_i} \in \Sigma$;*

(ii) *$\phi_f = \Sigma_{i=1}^{L} \phi_i$;*

(iii) *$M_{f|A_{\phi_i}}$ has a cyclic decomposition of cardinality C_i;*

(iv) *$C_i \neq C_j$ if $i \neq j$.*

Then $\{\phi_i\}_{i=1}^{L}$ is a sequence of invariants for M_f.

Proof. Consider ϕ_{i_0} where i_0 is a fixed index such that $1 \leqq i_0 \leqq L$. Suppose $\tau \ll \phi_f$ is a measure with pre-support $A_\tau \neq \varnothing$ a.e. μ and such that $M_{f|A_\tau}$ has a cyclic decomposition $\bigoplus_{i=1}^{J_{i_0}} M_{f|A_i}$ of cardinality C_{i_0}. Suppose $\tau \not\ll \phi_{i_0}$. Then $\tau = \tau_1 + \tau_2$ where $\tau_1 \ll \phi_{i_0}$ and $\tau_2 \perp \phi_{i_0}$ with $\tau_2 \neq 0$. There exists an index j_0, $1 \leqq j_0 \leqq L$, with $j_0 \neq i_0$, such that

$\tau_2 \not\ll \phi_{j_0}$. Without loss of generality we may assume that $\tau_2 \ll \phi_{j_0}$. Suppose A_{τ_2} is the pre-support of τ_2. Then $\bigoplus_{i=1}^{J_i} M_{f|A_i \cap A_{\tau_2}}$ is a cyclic decomposition for $M_{f|A_{\tau_2}}$. But if $\bigoplus_{i=1}^{J_{i_0}} M_{f|B_i}$ is a cyclic decomposition for $M_{f|A_{\phi_{i_0}}}$, where $A_{\phi_{i_0}}$ is the pre-support of ϕ_{j_0}, then $\bigoplus_{i=1}^{J_i} M_{f|B_i \cap A_{\tau_2}}$ is a cyclic decomposition for $M_{f|A_{\tau_2}}$ of cardinality C_{j_0}. But then we have $C_{i_0} = C_{j_0}$. This is a contradiction. Thus ϕ_{i_0} is an invariant.

DEFINITION 3.6. A sequence of measures $\{\phi_i\}_{i=1}^L$ satisfying the conditions (i) to (iv) of Lemma 3.1 is called *a complete set of invariants for* M_f.

REMARK 3.3. It follows directly from Theorem 3.1 that two complete sets of invariants, for the same regular multiplication operator M_f, are merely permutations of each other.

LEMMA 3.2. *Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces. If $M_f \in B(L_p(\mu))$ and $M_g \in B(L_p(\nu))$ are isometrically equivalent multiplication operators, then ϕ_f is equivalent to ϕ_g.*

Proof. If $p = 2$, this result follows from the uniqueness of the resolution of the identity for a normal operator (see, e.g., **[2]** Theorem 1, p. 65).

Suppose we have $p \neq 2$. There exists a surjective isometry $K: L_p(\mu) \to L_p(\nu)$ such that $KM_f = M_g K$ and K induces a set-isomorphism $\Gamma: (X, \Sigma, \mu) \to (Y, \Phi, \nu)$ as follows. Let $A \in \Sigma$. If h is in $L_p(\mu)$ and $\text{supp}(h) = A$ a.e. μ, then $\Gamma(A) = \text{supp}\{K(h)\}$ a.e. ν independent of the choice of the function h (see **[7]** Theorem 1.2 and **[3]** Theorem 3.1).

For $A \in \Sigma$, define K_A equal to $K_{|L_p(\mu|_A)}$. Then K_A is a surjective isometry from $L_p(\mu|_A)$ to $L_p(\nu|_{\Gamma(A)})$ and $K_A M_{f|_A} = M_{g|_{\Gamma(A)}} K_A$.

Now suppose that there exists G a Borel subset of $\mathbf{C}$ such that $\phi_f(G) > 0$. Then there exists $A_G \in \Sigma$, with $\mu(A_G) > 0$, such that $\sigma(M_{f|A_G}) \subset G$. Thus we see that $\sigma(M_{g|\Gamma(A_G)}) \subset G$ since under K_{A_G}, the spectrum is preserved. Clearly $M_{g|\Gamma(A_G)} \neq 0$. It follows that $\nu\{\Gamma(A_G)\} > 0$ and that $\phi_g(G) > 0$. Thus $\phi_g \gg \phi_f$. The converse is proved similarly using Γ^{-1}.

REMARK 3.4. Let ν be a measure on $\{J, \mathcal{B}(J)\}$. Suppose M_f is a multiplication operator on $L_p(J, \mathcal{B}(J), \nu)$. Let $\{\delta_i\}_{i=1}^{\infty}$ be the measures on $(J, \mathcal{B}(J))$ defined by

$$\delta_i(B) = \begin{cases} 1, & 1 - 1/i \in B \\ 0, & 1 - 1/i \notin B \end{cases}$$

for $B \in \mathscr{B}(J)$ and $i \in \mathbf{N}$. There exists a sequence of Borel measures $\{\mu_i\}_{i=0}^{L \leqq \infty}$ on $(\sigma(M_f), \mathscr{B}(\sigma(M_f)))$ such that $\mu_i \gg \mu_{i+1}$, for $1 \leqq i \leqq L$, and a point isomorphism γ from $(J, \mathscr{B}(J), \nu)$ to the Borel measure space $(E, \mathscr{B}(E), \tau)$, where E is the set $\sigma(M_f) \times J$ and τ is $\mu_0 \times \lambda + \Sigma_{i=1}^{L} \mu_i \times \delta_i$, such that $f = \pi_1 \circ \gamma$ a.e. ν (the map π_1 is the projection of E onto $\sigma(M_f)$). This is just the formulation of Rohlin's Theorem (**[5]** § IV) presented by Akcoglu (**[1]** Theorem 5.2).

THEOREM 3.2. *Let* (X, Σ, μ) *be a separable* σ*-finite measure space. Suppose* M_f *is a multiplication operator on* $L_p(\mu)$. *Then it follows that*:

(i) *there exists* $A_r \in \Sigma$, *depending only on f, such that* $M_f = M_{f|_{A_r}} \oplus M_{f|_{A_s}}$, *where* $A_s = X \backslash A_r$, $M_{f|_{A_r}} \equiv M_{f_r}$ *is regular, and* $M_{f|_{A_s}} \equiv M_{f_s}$ *is singular*;

(ii) *if* $A \neq \varnothing$ a.e., *then* $(A_s, \Sigma|_{A_s}, \mu|_{A_s})$ *is nonatomic, and if* ϕ_s *is the measure associated with* M_{f_s}, *there exists a surjective isometry* $K: L_p(\mu|_{A_s}) \to L_p(E, \mathscr{B}(E), \phi_s \times \lambda)$, *where* $E = \sigma(M_f) \times J$, *such that* $M_{\pi_1} K = K M_{f_s}$ *for* π_1 *the projection of E onto* $\sigma(M_{f_s})$.

(iii) *if* $A_r \neq \varnothing$ a.e. μ *then* M_{f_r} *has a complete set of invariants.*

Proof. There exists a set isomorphism Γ between (X, Σ, μ) and $(J, \mathscr{B}(J), \nu)$ for some Borel measure ν (see **[6]** Theorem 2, p. 264). Thus there exists a surjective isometry $I: L_p(\mu) \to L_p(\nu)$ such that I is induced by Γ and $M_f = I^{-1} M_{f'} I$ for some multiplication operator on $M_{f'}$ on $L_p(\nu)$ (see **[7]** Theorem 1.3). Since the singularity and regularity are preserved and the associated measures of the operators M_f and $M_{f'}$ are equivalent under I, we shall assume that (X, Σ, μ) is $(J, \mathscr{B}(J), \nu)$ and that M_f is a multiplication operator on $L_p(\nu)$.

Consider the measure space $(E, \mathscr{B}(E), \tau)$ as in Remark 3.4. Let γ be the point isomorphism $(J, \mathscr{B}(J), \nu) \to \{E, \mathscr{B}(E), \tau\}$ such that $f = \pi_1 \circ \gamma$. We partition the set E into disjoint sets C and D such that $C = \bigcup_{i=1}^{L} C_i$, where $C_i = \{(x, 1 - 1/i) \mid x \in \sigma(M_f)\}$ and $D = E \backslash C$. We have $\tau|_D = \mu_0 \times \lambda$ and $\tau|_{C_i} = \mu_i \times \delta_i$, $1 \leqq i \leqq L$.

Clearly the measure space $(D, \mathscr{B}(E)|_D, \tau|_D)$ is point isomorphic to $(E, \mathscr{B}(E), \mu_0 \times \lambda)$ under the identity mapping $\tau: D \to E$.

Let A_r be $\gamma^{-1}(C)$. Then A_s is $\gamma^{-1}(D)$. Since $(E, \mathscr{B}(E), \mu_0 \times \lambda)$ is nonatomic, it follows that $(A_s, \mathscr{B}(J)|_{A_s}, \nu|_{A_s})$ is nonatomic. If A is a Borel subset of A_s with $A \neq \varnothing$ a.e. ν, then we see that $f|_A = \pi_1 \circ \gamma|_A$ is not univalent on the compliment of any subset of A of measure zero and thus $M_{f|_A}$ is not cyclic on $L_p(\nu|_A)$. Suppose $A_r \neq \varnothing$ a.e. ν and $B \neq \varnothing$ a.e. ν is a Borel subset of A_r. If B is an atom, then the operator $M_{f|_B}$ on $L_p(\nu|_B)$ is cyclic since $L_p(\nu|_B)$ is one dimensional. If B is nonatomic, then $\gamma(B) = \bigcup_{i=1}^{L} \gamma(B) \cap C_i$. If for some index i_0 we have $\gamma(B) \cap C_{i_0} \neq \varnothing$ a.e. τ, then $B_{i_0} \equiv \gamma^{-1}\{\gamma(B) \cap C_{i_0}\} \neq \varnothing$ a.e. ν and $f|_{B_{i_0}}$ is univalent. Thus $M_{f|B_{i_0}}$ is cyclic on $L_p(\nu|_{B_{i_0}})$ and $M_{f|B}$ is thus seen be be the direct sum of cyclic parts. It follows immediately that $M_{f|A_r} \equiv M_{f_r}$ is regular and that $M_{f|A_s} \equiv M_{f_s}$ is singular and that $M_f = M_{f_r} \oplus M_{f_s}$ (see **[7]** Theorem 3.3).

Suppose $A_s \neq \emptyset$ a.e. ν. Since $\phi_s(B) = \nu\{f|_{A_s}^{-1}(B)\}$ for B a Borel subset of $\sigma(M_f)$, we see that $f|_{A_s}^{-1}(B) = \gamma^{-1}\{D \cap \pi_1^{-1}(B)\}$ implies $\phi_s(B) = \mu_0(B)$. It follows that $\gamma|_{A_s}$ is a point isomorphism between $(A_s, \mathscr{B}(J)|_{A_s}, \nu|_{A_s})$ and $(E, \mathscr{B}(E), \phi_s \times \lambda)$.

By standard methods it follows that there exists a surjective isometry $K: L_p(\nu|_{A_s}) \to L_p(E, \mathscr{B}(E), \phi_s \times \lambda)$ defined for $g \in L_p(\nu|_{A_s})$ by $K(g) = h \cdot (g \circ \gamma|_D^{-1})$ for some h measurable on $(E, \mathscr{B}(E), \phi_s \times \lambda)$ such that $KM_{f|A_s} = M_{\pi_1}K$ (see, e.g., **[7]** Remark 1.1).

The sequence of measures $\{\mu_i\}_{i=1}^{L}$ has one of the following two properties:

(1) given i_0 with $1 \leqq i_0 < L$, there exists $j_0 > i_0$ with $j_0 < L$ such that $\mu_{j_0} \ll \mu_{i_0}$ but $\mu_{j_0} \not\gg \mu_{i_0}$;

(2) there exists some index i_0 such that $\mu_i \approx \mu_j$ for $1 \leqq i_0 \leqq i, j \leqq L$.

In order to establish (iii), we shall assume (1) is true since (2) is handled in a similar manner.

First note that we must conclude that $L = \infty$. Now let ψ_0 be the zero measure on the Borel sets of $\sigma(M_f) \equiv S$. Define $G_0 = \emptyset$ and choose the nonnegative integer $n_0 = 0$. Suppose that the measure ψ_j on $\{S, \mathscr{B}(S)\}$, the set $G_j \in \mathscr{B}(S)$, and the nonnegative integer n_j have been chosen for $0 \leqq j \leqq i < \infty$. We define ψ_{i+1}, G_{i+1}, and n_{i+1} as follows: let $S_i = S \setminus \bigcup_{j=0}^{i} G_j$ and compare the measure $\mu_1|S_i$ with each of the measures $\mu_k|S_i$. There exists a smallest integer $k_i > n_i$ such that $\mu_k|S_i$ is equivalent to $\mu_1|S_i$ for $1 \leqq k \leqq k_i$ while $\mu_k|S_i \not\approx \mu_1|S_i$ for $k > k_i$. Set $n_{i+1} = k_i$. Then there exists Borel measures ω_1 and ω_2 such that $\mu_1|S_i = \omega_1 + \omega_2$ where $\omega_1 \approx \mu_{k_i+1}|_{S_i}$ and $\omega_2 \perp \mu_{k_i+1}|S_i$. There exists G_{i+1}, a Borel subset of S_i such that $\omega_1(G_{i+1}) = \omega_2(S \setminus G_{i+1}) = 0$. Set $\psi_{i+1} = \Sigma_{j=1}^{n_{i+1}} \mu_j | G_{i+1}$. If we define $G_\infty = S \setminus \bigcup_{j=0}^{\infty} G_i$ then one of the following possibilities can occur:

(a) for all $k \in \mathbf{N}$, $\mu_k | G_\infty \approx \mu_1 | G_\infty \neq 0$, or

(b) $\mu_1 | G_\infty = 0$.

If (a) is true, we define $\psi_\infty = \Sigma_{i=1}^{\infty} \mu_i | G_\infty$. If (b) is true ψ_∞ is not defined. Without loss of generality, we shall assume (a) holds. The collection of measures $\{\psi_i\}_{i=1}^{\infty} \cup \{\psi_\infty\}$ has the following properties:

(1) $\psi_i \perp \psi_j$ for $j \neq i$

(2) $\Sigma_{i=1}^{\infty} \mu_i = \Sigma_{i=1}^{\infty} \psi_i + \psi_\infty = \varphi_r$, the measure associated with M_{f_r}

(3) for each $i \in \bar{\mathbf{N}}$, where $\bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ we have $\bigoplus_{j \in F} M_{f|\gamma^{-1}\{\pi_1^{-1}(G_i) \cap C_j\}}$, where $F = \{j \in \mathbf{N} | \mu_j(\pi_i^{-1}(G_i) \cap C_j) > 0\}$ is a cylic decomposition for $M_{f|\gamma^{-1}\{\pi_1^{-1}(G_i) \cap C_j\}}$ which has associated measure ψ_i.

Thus by Lemma 3.1, $\{\psi_i\}_{i \in \bar{\mathbf{N}}}$ is a complete set of invariants for M_{f_r} and $\mathscr{M}(\psi_i) = n_i$ for $i \in \mathbf{N}$, while $\mathscr{M}(\psi_\infty) = \aleph_0$.

We have thus shown that Rohlin's Theorem (Remark 3.4) implies Theorem 3.2

THEOREM 3.3. *Theorem* 3.2 *implies Remark* 3.4.

Proof. Let $f \in L_\infty(J, \mathcal{B}(J), \nu)$. Then M_f on $L_p(\nu)$ has a regular part M_{f_r} and a singular part M_{f_s} with $M_f = M_{f_r} \oplus M_{f_s}$. In order to consider the most general situation, we assume that neither M_{f_r} nor M_{f_s} is zero. We let ϕ_r and ϕ_s be the measures associated with M_{f_r} and M_{f_s} respectively.

There exists a complete set of invariants $\{\phi_i\}_{i=1}^{L\leqq\infty}$ for M_{f_r} and we let $\{A_i\}_{i=1}^{L\leqq\infty}$ be the corresponding sequence of pre-supports. Thus $M_{f_r} = \bigoplus_{i=1}^{L} M_{f|_{A_i}}$ and for $1 \leqq i \leqq L$, we see that $M_{f|_{A_i}}$ has a cyclic decomposition of multiplicity $\mathcal{M}(\phi_i)$ given by $M_{f|_{A_i}} = \bigoplus_{j=1}^{\mathcal{M}(\phi_i)} M_{f|_{A_{ij}}}$ (where, if $\mathcal{M}(\phi_{i_0}) = \aleph_0$ for some i_0, then $M_{f|_{A_{i_0}}} = \bigoplus_{j=1}^{\infty} M_{f|_{A_{i_0 j}}}$).

Without loss of generality, assume that $\{\phi_i\}_{i=1}^{L}$ is countably infinite and that $\mathcal{M}(\phi_1) < \mathcal{M}(\phi_2) < \mathcal{M}(\phi_3) < \cdots$. For $j \in \mathbf{N}$, we define $B_j = \bigcup_{i\in\mathbf{N}_j} A_{ij}$, where $\mathbf{N}_j = \{i \in \mathbf{N} \mid j \leqq \mathcal{M}(\phi_i)\}$, and let f_j be $f|_{B_j}$. Then $M_{f_r} = \bigoplus_{j\in N} M_{f_j}$ and each M_{f_j} is cyclic on $L_p(A_j, \mathcal{B}(J)|_{A_j}, \nu|_{A_j})$. Also for $j \in \mathbf{N}$, we have $\sigma(M_{f_j}) \geqq \sigma(M_{f_{j+1}})$ and $\mu_j \gg \mu_{j+1}$, where μ_j is the measure associated with f_j.

Consider the set $E = \sigma(M_f) \times J$ and the measure space $(E, \mathcal{B}(E), \tau_d)$ defined as follows: for $G \in \mathcal{B}(E)$, we set $\tau_d(G) = \Sigma_{j\in\mathbf{N}} \mu_j\{\pi_1(G \cap C_j)\}$ where $C_j = \{(x, t) \in E \mid x \in \sigma(M_f); t = 1 - 1/j\}$. Then $\tau_d(G) = \Sigma_{j\in\mathbf{N}} \mu_j \times \delta_j(G \cap C_j)$. Define $\gamma_j\colon B_j \to E$ by $\gamma_j(t) = (f_j(t), 1 - 1/j)$ for $j \in \mathbf{N}$. Then we define $\gamma_d\colon A_r \to E$, where A_r is as in Theorem 3.2, by $\gamma_d(t) = (f_j(t), 1 - 1/j)$ for $t \in A_r \cap B_j \equiv B_j$. From the definition of τ_d, it follows that γ_d is a measure preserving point isomorphism from $(A_r, \mathcal{B}(J)|_{A_r}, \nu|_{A_r})$ to $(E, \mathcal{B}(E), \tau_d)$. Furthermore, we have $f|_{A_r} = \pi_1 \circ \gamma_d$ a.e. ν.

Let $A_s = J \backslash A_r$. If $p = 2$, M_{f_s} singular on $L_2(A_s, \mathcal{B}(J)|_{A_s}, \mu|_{A_s})$ implies M_{f_s} is singular on $(L_p(A_s, \mathcal{B}(J)|_{A_s}, \nu|_{A_s})$ for $p \neq 2$. We therefore assume that $p \neq 2$. There exists a surjective isometry

$$K\colon L_p\{A_s, \mathcal{B}(J)|_{A_s}, \nu|_{A_s}\} \to L_p\{E, \mathcal{B}(E), \phi_s \times \lambda\}$$

such that $K \circ M_{f_s} = M_{\pi_1} \circ K$. In addition K induces a natural measure preserving point isomorphism γ_c from $(A_s, \mathcal{B}(J)|_{A_s}, \gamma|_{A_s})$ to $(E, \mathcal{B}(E), \phi_s \times \lambda)$ such that $f|_{A_s} = \pi_1 \circ \gamma_c$ (see, e.g., **[6]** Corollary 12, p. 272). Define μ_0 to be the measure ϕ_s on $\sigma(M_f)$.

The map

$$\gamma = \begin{cases} \gamma_c & \text{on} \quad E \backslash \bigcup_{i=1}^{\infty} C_i \\ \gamma_d & \text{on} \quad \bigcup_{i=1}^{\infty} C_i \end{cases}$$

is the required point isomorphism such that $f = \pi_1 \circ \gamma$ and the result now follows.

EXAMPLE 3.1. Let $\gamma: J \to J \times J$ be a point isomorphism from the usual Borel measure space on [0, 1] the usual Borel measure space on the unit square. Then $f \equiv \pi_1 \circ \gamma$ is in $L_\infty(J, B(J), \lambda)$. There does not exist a set $B \in \mathscr{B}(J)$ of measure zero, such that $f|_{J \setminus B}$ is univalent. It follows that M_f is singular on $L_p(J, B(J), \lambda)$ for $1 \leqq p < \infty$ (see [7] Theorem 3.3).

4. A characterization theorem.

THEOREM 4.1. *Suppose (X, Σ, μ) and (Y, Φ, ν) are separable measure spaces. Then $M_f \in B(L_p(\mu))$ is isometrically equivalent to $M_g \in B(L_p(\nu))$, $p \neq 2$, if and only if the regular parts of M_f and M_g have equivalent complete sets of invariants with the same multiplicities and the singular parts of M_f and M_g have equivalent associated measures.*

Proof. ($\Leftarrow$) There exists a measure ω on $(J, \mathscr{B}(J))$ such that (X, Σ, μ) is set isomorphic to $(J, \mathscr{B}(J), \omega)$. There exists a measure ρ on $(J, \mathscr{B}(J))$ such that (Y, Φ, ν) is set isomorphic to $(J, \mathscr{B}(J), \rho)$. By an argument similar to that of the beginning of the proof of Theorem 3.2, we assume that M_f is in $B(L_p(J, \mathscr{B}(J), \omega))$ and M_g is in $B(L_p(J, \mathscr{B}(J), \rho))$. Let $(E_f, \mathscr{B}(E_f), \tau_f) \equiv \mathscr{E}_f$ and $(E_g, \mathscr{B}(E_g), \tau_g) \equiv \mathscr{E}_g$ be the measure spaces generated by f and g respectively as in Remark 3.4. Then since the invariants of the regular parts of M_f and M_g are equivalent and the singular parts have equivalent associated measures, it follows that $\mathscr{E}_f$ and $\mathscr{E}_g$ are point isomorphic under the identity mapping (although the isomorphism may not be measure preserving). Thus it follows that M_f on $L_p(\omega)$ and M_g on $L_p(\rho)$ are both equivalent to M_{π_1} on $L_p(\mathscr{E}_f)$ since the identity point isomorphism between $\mathscr{E}_f$ and $\mathscr{E}_g$ induces a surjective isometry $J: L_p(\mathscr{E}_f) \to L_p(\mathscr{E}_g)$ such that $JM_{\pi_1} = M_{\pi_1}J$.

($\Rightarrow$) Suppose $K: L_p(\mu) \to L_p(\nu)$ is a surjective isometry such that $KM_f = M_gK$. Then using the notation as in the proof of Lemma 3.2, K induces a set isomorphism $\Gamma: (X, \Sigma, \mu) \to (Y, \Phi, \nu)$ such that $K_A M_{f|_A} = M_{g|_{\Gamma(A)}} K_A$ for $A \in \Sigma$, since $p \neq 2$. Let M_{f_r} be the regular part and M_{f_s} be the singular part of M_f. Let $A_r \in \Sigma$ be as in Theorem 3.2 (i). We see that $K_{A_r} M_{f_r} = M_{g|_{\Gamma(A_r)}} K_{A_r}$ and that $K_{A_s} M_{f_s} = M_{g|_{\Gamma(A_s)}} K_{A_s}$, where $A_s = X \setminus A_r$. Thus, since K_{A_r} and K_{A_s} preserve the cyclicity of a multiplication operator, we see that $M_{g|_{\Gamma(A_r)}} \equiv M_{g_r}$ is the regular part and $M_{g|_{\Gamma(A_s)}} \equiv M_{g_s}$ is the singular part of M_g. Let ϕ_r and ψ_r be the measures associated with M_{f_r} and M_{g_r} respectively. Let ϕ_s and ψ_s be the measures associated with M_{f_s} and M_{g_s} respectively. Then we conclude that $\phi_r \approx \psi_r$ and $\phi_s \approx \psi_s$.

There exists a complete set of invariants $\{\phi_i\}_{i=1}^{L \leqq \infty}$ for M_{f_r} such that $M_{f_r} = \bigoplus_{i=1}^{L} M_{f|_{A_i}}$, where ϕ_i is the measure associated with $M_{f|_{A_i}}$, $1 \leqq i \leqq L$, and $M_{f|A_i}$ has a cyclic decomposition of cardinality $\mathcal{M}(\phi_i)$. (Here A_i is the pre-support of ϕ_i.) There exists a sequence of disjoint measurable sets of Σ, $\{A_{ij}\}_{j=1}^{\mathcal{M}(\phi_i)}$ such that $M_{f|A_i} = \bigoplus_{j=1}^{\mathcal{M}(\phi_i)} M_{f|A_{ij}}$ is a cyclic decomposition. Let ψ_i be the measure associated with $M_{g|r(A_i)}$, $1 \leqq i \leqq L$. Then we conclude that $\{\psi_i\}_{i=1}^{L}$ is a complete set of invariants for M_g, with $\psi_i \approx \phi_i$ and $\mathcal{M}(\psi_i) = \mathcal{M}(\phi_i)$ for $1 \leqq i \leqq L$.

REMARK 4.1. The "if" direction of Theorem 4.1 is true for $p = 2$. The proof is exactly the same as was presented for $p \neq 2$. However, the "only if" direction is false if $p = 2$. In fact, by standard multiplicity theory for normal operators on Hilbert space (**[4]**, Chapter 10) it is possible to construct a surjective isometry K between two L_2-spaces such that a singular multiplication operator M_f is isometrically equivalent to a regular multiplication operator M_g under K.

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