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APPROXIMATION OF COMPACT HOMOGENEOUS MAPS

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Within the class of continuous homogeneous maps between Banach spaces, it is proved that every compact map can be uniformly approximated by finite-rank maps. This result is obtained by means of the classical metric projection on Banach spaces.

The classical approximation problem is to determine those Banach spaces on which every compact continuous linear map can be uniformly approximated by finite-rank continuous linear maps. P. Enflo **[1]** recently constructed a Banach space on which this approximation is not possible.

Instead of restricting the class of underlying spaces in order to obtain the approximation property, one can expand the class of continuous maps by weakening the linearity requirement. Let E and F be Banach spaces over some field, and let $\mathscr{H}(E,F)$ denote the Banach space of all continuous homogeneous maps from E into F. The norm on $\mathscr{H}(E,F)$ is the uniform norm: $\|T\| = \sup\{\|Tx\|: \|x\| \leqq 1\}$.

THEOREM. *If $T \in \mathscr{H}(E,F)$ is compact, then for each $\epsilon > 0$ there exists a finite-rank map $P \in \mathscr{H}(E,F)$ such that $\|T - P\| < \epsilon$.*

The proof is developed through several elementary lemmas. The first two of these are well-known and can be found in Köthe **[3]**.

LEMMA 1. *If $(G, \|\cdot\|)$ is a separable Banach space, then there exists a strictly convex norm $\|\cdot\|_1$ and a number $\rho > 0$ such that $\|w\| < \rho\|w\|_1$, for all $w \in G$.*

LEMMA 2. *If G is a strictly convex Banach space and if M is a finite-dimensional subspace, then there exists a map $P_M \in \mathscr{H}(G,M)$ such that*

$$\|y - P_M y\| = \min\{\|y - z\|: z \in M\}$$

for all $y \in G$.

The map $P_M: G \to M$ is usually called the *metric projection of G onto M.* Köthe **[3]** calls it the "nearest-point mapping".

LEMMA 3. *If B is a relatively compact subset of G, then for each $\delta > 0$ there exists a finite-dimensional subspace M of G such that*

$$\sup\{\min\{\|y - z\|: z \in M\}: y \in B\} \leqq \delta.$$

Proof. Let $\{y_1, \cdots, y_k\} \subset B$ such that $y \in B \Rightarrow \|y - y_j\| < \delta$ for some $j \leqq k$. Let $M = \mathrm{sp}\{y_1, \cdots, y_k\}$ so that $\dim M \leqq k$. Then for each $y \in B$, $\min\{\|y - z\|: z \in M\} < \delta$; thus,

$$\sup\{\min\{\|y - z\|: z \in M\}: y \in B\} \leqq \delta.$$

LEMMA 4. *If C is a balanced convex subset of G, then its linear span $\mathrm{sp}C = \bigcup_{n=1}^{\infty} nC$.*

Proof. Obviously $\mathrm{sp}\, C \supset \bigcup_{n=1}^{\infty} nC$. If $w \in \mathrm{sp}C$, then since C is balanced, $w = \Sigma_{j=1}^{k} \beta_j w_j$, where all $\beta_j \neq 0$, $w_j \in C$ and $\|w_j\| \leqq 1$. Let $y = \beta^{-1} w$ where $\beta = \Sigma_{j=1}^{k} |\beta_j|$; then $y = \beta^{-1} \Sigma_{j=1}^{k} \beta_j w_j = \Sigma_{j=1}^{k} \alpha_j v_j$ where $\alpha_j = \beta^{-1} |\beta_j|$ and $v_j = |\beta_j|^{-1} \beta_j w_j$. Now each $v_j \in C$, since C is balanced, and $\Sigma_{j=1}^{k} \alpha_j = 1$. Thus $y \in C$, since C is convex. Thus $w = \beta y \in \bigcup_{n=1}^{\infty} nC$.

LEMMA 5. *If $T \in \mathscr{H}(E, F)$ is compact, then there exists a closed, separable subspace G of F such that $T \in \mathscr{H}(E, G)$.*

Proof. Let C be the convex hull of $T(U)$, the image of the closed unit ball U of E. Let $G = \mathrm{cl}(\bigcup_{n=1}^{\infty} nC)$, the closure of $\bigcup_{n=1}^{\infty} nC$ in F. Now C is balanced, since T is homogeneous. Thus by Lemma 4, $G = \mathrm{cl}(\mathrm{sp}C)$. Now $T(U)$ is relatively compact, so G is separable. Finally, $T(E) \subset G$ because $x \in E \Rightarrow Tx = T(\|x\| u) = \|x\| Tu$, where $x = \|x\| u$ for some $u \in U$, $\Rightarrow Tx \in \|x\| C \subset \bigcup_{n=1}^{\infty} nC \subset G$.

Proof of Theorem. By Lemma 5, $T \in \mathscr{H}(E, G)$, where G is separable. Then by Lemma 1, G has a strictly convex norm $\|\cdot\|_1$ and a number $\rho > 0$ such that $\|w\| < \rho \|w\|_1$, $\forall w \in G$. Let $\epsilon > 0$ and set $\delta = \rho^{-1} \epsilon > 0$. Let $B = T(U)$. Then by Lemma 3, there exists a finite-dimensional subspace M of G such that

$$\sup\{\min\{\|y - z\|_1: z \in M\}: y \in B\} \leqq \delta.$$

Then by Lemma 2, there exists a map $P_M \in \mathscr{H}(G, M)$ such that

$$\|y - P_M y\|_1 = \min\{\|y - z\|_1: z \in M\}$$

for each $y \in G$. Let $P = P_M \circ T$. Then $P \in \mathscr{H}(E, M)$, and

$$\begin{aligned} x \in U \Rightarrow \| Tx - Px \| &< \rho \| Tx - Px \|_1 \\ &= \rho \| Tx - P_M(Tx) \|_1 = \rho \min\{\| Tx - z \|_1 : z \in M\} \\ &\leqq \rho \sup\{\min\{\| y - z \|_1 : z \in M\} : y \in B\} \leqq \rho\delta = \epsilon. \end{aligned}$$

COROLLARY. *Every compact continuous linear map can be uniformly approximated by finite-rank continuous homogeneous maps.*

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