PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 1, 1976

COMPOSITION OPERATORS ON $H^p(A)$

DAVID M. BOYD

The space $H^p(A)$ is a generalization of the Hardy space H^p for functions analytic on an annulus A. This paper shows that composition operators are bounded operators on $H^p(A)$ and obtains an upper bound on the norm of the operator. The space $H^2(A)$ is given a Hilbert space structure and those composition operators that are in the Hilbert–Schmidt class of operators on $H^2(A)$ are characterized in terms of integral properties of the inducing function.

1. Introduction. If H is a space of functions analytic on a region R, and if φ is an analytic map of R into itself, the composition operator C_φ on the space H is defined by $C_\varphi(f) = f \circ \varphi$. In recent articles composition operators have been studied on various function spaces including the Hardy space H^p and the Bergman space A^2. See, for example, (2), (4), (7), and (8). In all these spaces the underlying region of analyticity was the unit disk. In this paper we show that composition operators form bounded operators on the space $H^p(A)$, $0 < p < \infty$, a generalization of the Hardy space which consists of functions analytic on an annulus. In addition, a characterization of those composition operators which form Hilbert–Schmidt operators on the Hilbert space $H^2(A)$ is derived.

2. Boundedness of composition operators. A well-known generalization of H^p, $0 < p < \infty$, was given by Rudin in (5). For r in $(0,1)$ he considered the linear space of functions f analytic on $A = (z : r < |z| < 1/r)$ with the property that $|f|^p$ has a harmonic majorant on A. He showed that for $p \geqq 1$ this space is a Banach space under the norm $\|f\|_p = (u(1))^{1/p}$, where u is the least harmonic majorant of $|f|^p$. When $0 < p < 1$, this space is an F space (i.e., a complete translation invariant metric space) when given the metric $d(f, g) = \|f - g\|_p^p$.

Another generalization for $p \geqq 1$ was introduced by Sarason in (6) as the space of functions f analytic on A with bounded integral means $M_p(f, r)$. He showed that such functions have nontangential limits almost everywhere on the boundary and that the space is a Banach space under the norm

$$\|f\|_p = (M_p^p(f, r) + M_p^p(f, 1/r))^{1/p}.$$

In (1) this author has shown that for $0 < p < 1$ this same limitation on $M_p(f, r)$ defines an F space under the metric $d(f, g) = \|f - g\|_p^p$.

That these spaces are the same and that the topologies are equivalent follows by considering a third space. If H_0^p is the subspace of H^p consisting of those functions which vanish at zero, then $H^p \oplus H_0^p$ is a Banach space ($p \geqq 1$) under the norm

$$\|(f, g)\|_p = (M_p^p(f, 1) + M_p^p(g, 1))^{1/p}.$$

Again when $0 < p < 1$, the space is an F space using the pth power of this expression as the metric.

For a function f analytic on A we write $f = f_1 + f_2$ to indicate its Laurent decomposition chosen so that f_1 is analytic on $|z| < 1/r$ and f_2 is analytic on $|z| > r$, with $f_2(\infty) = 0$. We also define $\hat{f}_2(z) = f_2(r/z)$. It can be shown (see (1) or (3)) that the map $(f_1, \hat{f}_2) \to f = f_1 + f_2$ is a continuous, linear, one-to-one map of $H^p \oplus H_0^p$ onto either Rudin's space or Sarason's space. By the Open Mapping Theorem, the inverse is continuous; therefore, all three spaces are topologically equivalent.

We choose Rudin's description as our basic definition of the space $H^p(A)$ and establish the boundedness of composition operators on $H^p(A)$.

THEOREM 1. *If $\varphi : A \to A$ is analytic function and if we define the operator $C_\varphi(f) = f \circ \varphi$ for f in $H^p(A)$, $0 < p < \infty$, then C_φ is a bounded operator on $H^p(A)$. Furthermore, if $p \geqq 1$, the norm of the operator is dominated by*

$$\left\{ \frac{e^{(\pi|t|/q)} + \tan\left(\frac{\pi}{2q}|\log|w||\right)}{1 - e^{(\pi|t|/q)}\tan\left(\frac{\pi}{2q}|\log|w||\right)} \right\}^{1/p},$$

where $\varphi(1) = w = |w|e^{it}$, $-\pi < t \leqq \pi$, and $q = -\log r$. In particular we note the following cases:

(a) *If $\varphi(1) = 1$, then $\|C_\varphi\| = 1$.*

(b) *If $|\varphi(1)| = 1$, then $\|C_\varphi\| \leqq e^{(\pi|t|/qp)} \leqq e^{(\pi^2/qp)} < \infty$.*

(c) *If $\varphi(1) > 0$, then*

$$\|C_\varphi\| \leqq \left\{ \frac{1 + \tan\left(\frac{\pi}{2q}|\log|w||\right)}{1 - \tan\left(\frac{\pi}{2q}|\log|w||\right)} \right\}^{1/p}.$$

For composition operators on the F space $H^p(A)$, $0 < p < 1$, these same estimates, raised to the power p, provide estimates on the operator norm.

Proof. Let f be in $H^p(A)$, $p \geqq 1$, and let u denote the least harmonic majorant of $|f|^p$ on A. Thus $|f(\varphi(z))|^p \leqq u(\varphi(z)) = (u \circ \varphi)(z)$. Since $u \circ \varphi$ is a harmonic majorant of $|f \circ \varphi|^p$ we know there is a least harmonic majorant and that $f \circ \varphi$ is in $H^p(A)$.

To show that C_φ is bounded it suffices to produce a constant $K = K(\varphi)$ such that $u(\varphi(1)) \leqq K \cdot u(1)$ for all positive u harmonic on A. We determine such a constant by considering a mapping Q utilized by Sarason in his study (6) of analytic function spaces of the annulus.

Let $Q(z) = \exp[(-iq/\pi)\log((1+z)/(1-z))]$, where $q = -\log r$, and the imaginary part of the logarithm has values between $-\pi$ and π. The function Q maps the unit disk U onto A in many-to-one fashion. The upper half-circle $(e^{is}: 0 < s < \pi)$ is mapped, or "coiled," onto the outer boundary of A, while the lower half-circle $(e^{is}: \pi < s < 2\pi)$ is mapped onto the inner boundary of A. The origin $z = 0$ is mapped to the point 1 in A. Furthermore, if w is in A, the set $Q^{-1}(w)$ lies on a circular arc in U with end-points $z = +1$ and $z = -1$. In particular, the diameter of U given by $-1 < z < +1$ is mapped onto the unit circle.

We write $w = |w|e^{it} = \varphi(1)$, with $-\pi < t \leqq \pi$, and consider a Poisson representation for any s in $Q^{-1}(w)$, and for any $\rho < 1$:

$$u(w) = (u \circ Q)(s) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{\frac{\rho e^{it} + s}{\rho e^{it} - s}\right\}(u \circ Q)(\rho e^{it})dt.$$

Therefore by the mean-value theorem for functions harmonic in U,

$$u(w) \leqq \left(\frac{\rho + |s|}{\rho - |s|}\right) \cdot u(Q(0)).$$

Furthermore, $Q(0) = 1$, so that if we let ρ tend to 1, we have

$$u(\varphi(1)) \leqq \left(\frac{1 + |s|}{1 - |s|}\right) u(1).$$

Since this argument holds for any s in $Q^{-1}(w)$, it is possible to choose the constant K above as

$$K = \inf\left(\frac{1 + |s|}{1 - |s|}\right) = \frac{1 + \inf|s|}{1 - \inf|s|},$$

where the infimum is taken over all s in $Q^{-1}(w)$. The problem is reduced to determining this infimum in terms of $w = \varphi(1)$.

We see next that if $Q(s) = w$ the definition of Q yields

$$s = \frac{e^{i\beta}-1}{e^{i\beta}+1} = i\tan(\beta/2),$$

$$\beta = \frac{\pi}{q}\log\left[|w|e^{i(t+2k\pi)}\right] = \frac{\pi}{q}\log(|w|) + i\frac{\pi}{q}(t+2k\pi),$$

for some integral value of k. We shall denote this expression for β by $\beta = A + iB$. With this simplification we can write

$$\begin{aligned}|s| = \tan(\beta/2) &= \left|\frac{\tan(A/2)+\tan(iB/2)}{1-\tan(A/2)\cdot\tan(iB/2)}\right| \\ &\leqq \frac{\tan(|A|/2)+\tanh(|B|/2)}{1-\tan(|A|/2)\cdot\tanh(|B|/2)}\end{aligned}$$

and this last estimate attains its minimum when $k = 0$. Hence, we can write K as

$$\frac{1-\tan(|A|/2)\cdot\tanh(|B|/2)+\tan(|A|/2)+\tanh(|B|/2)}{1-\tan(|A|/2)\cdot\tanh(|B|/2)-\tan(|A|/2)-\tanh(|B|/2)},$$

where we have chosen $k = 0$. This can be written as the norm estimate stated in the theorem.

For the special case (a) we note that K can be chosen as 1, so that $\|C_\varphi\| \leqq 1$. But $\|C_\varphi\| \geqq 1$, since any constant function has 1 an an eigenvalue.

For case (b) the general estimate reduces to $e^{(\pi|t|/qp)}$, which is dominated, independent of the inducing function φ, by the value $e^{(\pi^2/qp)} < \infty$.

For case (c) note that if $\varphi(1) > 0$ then $t = 0$; therefore, the general estimate reduces to the one desired.

3. Hilbert–Schmidt operators on $H^2(A)$.

The space $H^2(A)$ is a Hilbert space under the inner product

$$\begin{aligned}(f,g) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}/r)\overline{g(e^{it}/r)}\,dt + \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt \\ &= \sum_{-\infty}^{\infty} r^{-2n}a_n\bar{b}_n + \sum_{-\infty}^{\infty} r^{2n}a_n\bar{b}_n,\end{aligned}$$

where a_n and b_n are the Laurent coefficients of the functions f and g, respectively. With this inner product the norm of the space is given by,

$$\|f\|_2 = [M_2^2(f, r) + M_2^2(f, 1/r)]^{\frac{1}{2}}$$
$$= \left[\sum_{-\infty}^{\infty} (r^{-2n} + r^{2n})|a_n|^2\right]^{\frac{1}{2}}.$$

This norm is the one introduced by Sarason (6) and, as noted above, is topologically equivalent to Rudin's choice of norm.

Recall that on operator T on an infinite dimensional Hilbert space is called a Hilbert–Schmidt operator if there exists an orthonormal basis (g_n) such that $\Sigma_n \|Tg_n\|^2 < \infty$ and that convergence for one orthonormal basis insures convergence for every orthonormal basis. An orthonormal basis for $H^2(A)$ is given by

$$g_n(z) = z^n(r^{-2n} + r^{2n})^{-\frac{1}{2}} \qquad n = 0, \pm 1, \pm 2, \cdots.$$

The question of whether a composition operator on the Hilbert space $H^2(A)$ is Hilbert–Schmidt can be answered by testing the proximity of the function values to the boundary of A in the following way:

THEOREM 2. *The operator C_φ is Hilbert–Schmidt on $H^2(A)$ if and only if each of the following four integrals is finite*:

$$\int_0^{2\pi} (1 - r|\varphi(e^{it}/r)|)^{-1}dt, \qquad \int_0^{2\pi} (|\varphi(e^{it}/r)| - r)^{-1}dt,$$
$$\int_0^{2\pi} (1 - r|\varphi(re^{it})|)^{-1}dt, \qquad \int_0^{2\pi} (|\varphi(re^{it})| - r)^{-1}dt.$$

Proof. First note the following inequalities:
(i) If $n < 0$, $r^{2n} \leqq r^{-2n} + r^{2n} \leqq 2r^{2n}$,
(ii) If $n \geqq 0$, $r^{-2n} \leqq r^{-2n} + r^{2n} \leqq 2r^{-2n}$.
Now using the orthonormal system (g_n), we can compute

$$2\pi \cdot \sum_{-\infty}^{\infty} \|C_\varphi(g_n)\|_2^2 = 2\pi \cdot \sum_{-\infty}^{\infty} \|g_n(\varphi)\|_2^2$$
$$= \sum_{-\infty}^{\infty} \int_0^{2\pi} (|\varphi(e^{it}/r)|^{2n}/(r^{-2n} + r^{2n}))dt$$
$$+ \sum_{-\infty}^{\infty} \int_0^{2\pi} (|\varphi(re^{it})|^{2n}/(r^{-2n} + r^{2n}))dt.$$

If we divide this into four summations and use the inequalities above, we see that it exists if and only if the following sum does:

$$\int_0^{2\pi} \sum_{n=0}^{\infty} (r\,|\varphi(e^{it}/r)|)^{2n}dt + \int_0^{2\pi} \sum_{n=1}^{\infty} (r/|\varphi(e^{it}/r)|)^{2n}dt$$
$$+ \int_0^{2\pi} \sum_{n=0}^{\infty} (r\,|\varphi(re^{it})|)^{2n}dt + \int_0^{2\pi} \sum_{n=1}^{\infty} (r/|\varphi(re^{it})|)^{2n}dt.$$

Since $r < |\varphi| < 1/r$, we can interpret this sum using the Taylor Series expansions for $1/(1-z)$ and $z/(1-z)$. These integrals will exist if and only if each of the integrals in the theorem exist, that is, if and only if C_φ is a Hilbert–Schmidt operator on $H^2(A)$.

COROLLARY. *The following are equivalent*:
(a) C_φ *is Hilbert–Schmidt on* $H^2(A)$;
(b) C_ψ *is Hilbert–Schmidt on* $H^2(A)$, *for* $\psi(z) = r/\varphi(z)$;
(c) C_θ *is Hilbert–Schmidt on* $H^2(A)$, *for* $\theta(z) = \varphi(r/z)$.

REFERENCES

1. D. Boyd, *Composition Operators on the Bergman Space and Analytic Function Spaces on the Annulus*, Dissertation, University of North Carolina, 1974.
2. ———, *Composition operators on the Bergman space*, Colloquium Mathematicum XXXIV. **1** (1975), 131–140.
3. ———, *The metric space $H^p(A)$, $0 < p < 1$, and its containing Banach space*, (submitted for publication).
4. E. Nordgren, *Composition operators*, Canad. J. Math., **20** (1968), 442–449.
5. W. Rudin, *Analytic functions of class H_p*, Trans. Amer. Math. Soc., **78** (1955), 46–66.
6. D. Sarason, *The H^p spaces of an annulus*, Mem. Amer. Math. Soc., No. **56** (1965), 78 pp.
7. H. Schwartz, *Composition Operators on H^p*, Dissertation, University of Toledo, 1969.
8. J. Shapiro and P. Taylor, *Compact, nuclear, and Hilbert–Schmidt composition operators on H^2*, Indiana Univ. Math. J., **23** (1973), 471–496.

Received August 26, 1975. These results constitute a part of the author's Ph.D. thesis written at the University of North Carolina under the direction of Professors Joseph A. Cima and John A. Pfaltzgraff.

KETRON, INC., 1400 WILSON BLVD., ARLINGTON, VIRGINIA 22209