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# ARCHIMEDEAN AND BASIC ELEMENTS IN COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

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**It is known that the $bi$-prime group $B(G)$ of an $l$-group $G$ contains the basic elements of $G$. We show that every $l$-group $G$ possesses a unique, maximal, archimedean, convex $l$-subgroup $A(G)$, and that if $G$ is completely distributive and if $A(G)^{\perp}$ is representable, then $B(G)$ has a basis.**

1. Introduction. An element $s$ of a lattice-ordered group ($l$-group) $G$ is *basic* (see [4]) if $s > 0$ and the closed interval $[0, s]$ is totally ordered. An $l$-group $G$ *has a basis* if every $g > 0$ exceeds some basic element (any maximal disjoint set of basic elements is then a *basis*). An $l$-group $G$ is *completely distributive* (see [3], [4], [9], [10]) if the relation

$$\bigwedge \{\vee \{g_{ij} | j \in J\} | i \in I\} = \bigvee \{\wedge \{g_{i(if)} | i \in I\} | f \in J^I\}$$

holds whenever $\{g_{ij} | i \in I, j \in J\} \subseteq G$ is such that all the indicated joins and meets exist. By [5], p. 5.18, Theorem 5.8, every $l$-group which has a basis is completely distributive. For *archimedean* $l$-groups, i.e. those in which $a \geqq nb \geqq 0$ for all natural numbers $n$ implies $b = 0$, more can be said: viz., an archimedean $l$-group has a basis if and only if it is completely distributive ([5], p, 5.21, Theorem 5.10). In [8], we constructed, via minimal prime subgroups, the $bi$-prime group $B(G)$ of an $l$-group $G$ (see §3 below) which contains all the basic elements and which, if $G$ is completely distributive and representable, has a basis. In this note, we introduce "archimedean elements" (see §2 below) in order to investigate possible connections among the above results. Thus, in §2, we show that every $l$-group $G$ posseses a unique, maximal, archimedean, convex $l$-subgroup $A(G)$. (Kenny [7] independently proved this result for representable $l$-groups.) It follows that if $A(G)^{\perp} = \{0\}$, then $G$ is completely distributive if and only if $G$ has a basis. In §3, proving somewhat more general results, we show that $A(B(G)) = B(A(G))$ and hence that if $G$ is completely distributive and if $A(G)^{\perp}$ is representable, then $B(G)$ has a basis. In §4, we construct two examples, one of which is of completely distributive, nonrepresentable $l$-group which has a basis and for which $A(G)^{\perp}$ is representable.

NOTATION AND TERMINOLOGY. We use $\square$ for the empty set and write functions on the right. We use $N$, $Z$, and $R$ for the natural numbers, the integers and the real numbers, respectively. The cartesian product of the sets $\{S_\alpha \mid \alpha \in A\}$ is denoted by $\prod \{S_\alpha \mid \alpha \in A\}$. If $\{G_\alpha \mid \alpha \in A\}$ is a set of $l$-groups, then $|\prod| \{G_\alpha \mid \alpha \in A\} (|\sum| \{G_\alpha \mid \alpha \in A\})$ denotes their cardinal product (sum); if $A = \{1, 2\}$, we use $G_1 | \times | G_2$ for the cardinal product.

Let $G$ be an $l$-group. A subgroup $H$ of $G$ is *prime* if and only if it is a convex $l$-subgroup of $G$ such that for all $a, b \in G^+ \backslash H$, $a \wedge b \in G^+ \backslash H$ (see [5], pp. 1.13–1.16). If $g \in G \supseteqq A, B$, then $\langle A \rangle$ denotes the convex $l$-subgroup generated by $A$; $\langle A, B \rangle \equiv \langle A \cup B \rangle$; $G(g) \equiv \langle \{g\} \rangle$. For any $S \subseteqq G$, the *polar* of $S$, defined

$$S^\perp = \{g \in G \mid |g| \wedge |s| = 0 \quad \text{for all } s \in S\} ,$$

is a convex $l$-subgroup of $G$ (see [8]). The following result will prove useful.

LEMMA 1.1. *Let $H$ be a convex $l$-subgroup of an $l$-group $G$. If $\{h_\alpha\} \subseteqq H$ is such that $\bigvee_H h_\alpha$ exists in $H$, then $\bigvee_G h_\alpha$ exists in $G$ and $\bigvee_G h_\alpha = \bigvee_H h_\alpha$. The dual statement also holds.*

*Proof.* Let $\{h_\alpha\} \subseteqq H$ be such that $\bigvee_H h_\alpha \in H$. Suppose that the join of $\{h_\alpha\}$ does not exist in $G$. Then, since $\bigvee_H h_\alpha$ is an upper bound of $\{h_\alpha\}$ in $G$, there exists $b \in G$ such that $h_\beta \leqq b < \bigvee_H h_\alpha$ for all $\beta$. Since $H$ is convex, $b \in H$. This contradicts the minimality of $\bigvee_H h_\alpha$ among upper bounds of $\{h_\alpha\}$ in $H$ and hence $\bigvee_G h_\alpha \in G$. Since $\bigvee_H h_\alpha \in G$ is an upper bound of $\{h_\alpha\}$, $h_\beta \leqq \bigvee_G h_\alpha \leqq \bigvee_H h_\alpha$ for all $\beta$, and hence $\bigvee_G h_\alpha \in H$. Therefore, $\bigvee_G h_\alpha = \bigvee_H h_\alpha$. The dual property follows from the above because $G$ is an $l$-group.

For terminology left undefined, see Birkhoff [1], Fuchs [6], or Conrad [5].

## 2. Archimedean elements.

Let $G$ be an $l$-group. An element $a \in G$ is *archimedean* if $a \geqq 0$ and if for all $0 < g \leqq a$, there exists $n \in N$ such that $ng \not\leqq a$. Clearly, $G$ is archimedean if and only if every element of $G^+$ is archimedean. Let $P(G)$ be the set of all archimedean elements of $G$; let $A(G)$ be the $l$-subgroup of $G$ generated by $P(G)$.

THEOREM 2.1. $A(G)^+ = P(G)$.

*Proof.* Clearly, $0 \in P(G)$ and $P(G)$ is convex. By [5], p. 1.5, Theorem 1.3, it therefore suffices to show that $P(G)$ is a subsemigroup of $G^+$.

The proof that $P(G)$ is a subsemigroup is by contradiction.

Suppose there exist $a, b \in P(G)$ such that $a + b \notin P(G)$. Then there exists $0 < t \leqq a + b$ such that $nt \leqq a + b$ for all $n \in N$. Since $a$ is archimedean, there exists $m > 0$ such that $mt \nleqq a$. Then

$$s = (-a + mt) \vee 0 > 0 .$$

Since $nt \leqq a + b$ for all $n > 0$, $-a + nt \leqq b$ for all $n > 0$. Thus

$$(1) \qquad (-a + nt) \vee 0 \leqq b \quad \text{for all } n \in N ,$$

and in particular $0 < s \leqq b$. We will show by induction that

$$(2) \qquad ks \leqq (-a + kmt) \vee 0 \quad \text{for all} \quad k \in N .$$

Obviously,

$$s = (-a + mt) \vee 0 \leqq (-a + kmt) \vee 0$$

for all $k \in N$. Suppose $ks \leqq (-a+kmt) \vee 0$. Then

$$\begin{aligned}(k + 1)s &= (k + 1)[(-a + mt) \vee 0] \\ &= k[(-a + mt) \vee 0] + [(-a + mt) \vee 0] \\ &\leqq [(-a + kmt) \vee 0] + [(-a + mt) \vee 0] \\ &= (-a + kmt - a + mt) \vee (-a + kmt) \\ &\quad \vee (-a + mt) \vee 0 \\ &\leqq (-a + kmt + mt) \vee (-a + kmt) \vee 0 \\ &= (-a + (k + 1)mt) \vee (-a + kmt) \vee 0 \\ &= (-a + (k + 1)mt) \vee 0 .\end{aligned}$$

Then for all $k \in N$,

$$\begin{aligned}0 < ks &\leqq (-a + kmt) \vee 0 && \text{by (2)} \\ &\leqq b && \text{by (1)} .\end{aligned}$$

Therefore, $b \notin P(G)$, which contradicts our choice of $b$. Theorem 12. follows.

COROLLARY 2.2. *$A(G)$ is the unique, maximal, archimedean, convex $l$-subgroup of $G$.*

*Proof.* Since $A(G)^+ = P(G)$, $A(G)$ is archimedean. By definition of $P(G)$ any larger $l$-subgroup cannot be archimedean. That $A(G)$ is convex and unique is clear.

COROLLARY 2.3. *Let $g \in G^+$. Then $g$ is archimedean if and only if $G(g)$ is archimedean.*

*Proof.* The proof of Theorem 2.1 shows that if $g$ is archimedean, then $ng$ is archimedean for all $n \in N$. Thus, $G(g)$ is archimedean. The converse is clear.

COROLLARY 2.4. $A(G) = \{g \in G \mid G(|g|) \text{ is archimedean}\}$.

*Proof.* If $g \in A(G)$, then $|g|$ is archimedean by Theorem 2.1, and thus $G(|g|)$ is archimedean by Corollary 2.3. Conversely, if $G(|g|)$ is archimedean, Corollary 2.3 implies that $|g|$ is archimedean. Hence by Theorem 2.1, $|g| \in A(G)^+$. Since $-|g| \leqq g \leqq |g|$ and $A(G)$ is convex, $g \in A(G)$.

Kenny [7] proved independently that for every representable $l$-group $G$, $\{g \in G \mid G(|g|) \text{ is archimedean}\}$ is the unique, maximal, archimedean, convex $l$-subgroup of $G$; this follows immediately from Corollaries 2.2 and 2.4 above.

PROPOSITION 2.5. *Let $G$ be an $l$-group in which every strictly positive element exceeds a nonzero archimedean element. Then $G$ is completely distributive if and only if $G$ has a basis.*

*Proof.* By Lemma 1.1 if $G$ is completely distributive, $A(G)$ is completely distributive. Since $A(G)$ is archimedean, this implies that $A(G)$ has a basis, and then $G$ must have a basis because of the initial hypothesis. The converse follows from [5], p. 5.18, Theorem 5.8 (see §1).

**3. The bi-prime group and $A(G)$.** In [8], we defined the bi-prime group of an $l$-group $G$ as follows: Let $\{P_\phi \mid \phi \in \Phi(G)\}$ be the set of minimal prime subgroups of $G$. The *bi-prime group* of $G$ is the convex $l$-subgroup

$$B(G) = \bigcap \{\langle P_\phi, P_\omega\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} .$$

By [8], Theorem 3.1, $B(G)$ has a basis whenever $G$ is both completely distributive and representable.

The following result is an easy consequence of [2], Theorem 3.5.

LEMMA 3.1. *Let $\{0\} \neq S$ be a convex $l$-subgroup of an $l$-group $G$. If $Q$ is a minimal prime subgroup of $S$, then there exists a minimal prime subgroup $P$ of $G$ such that $Q = P \cap S$. If $P$ is a minimal prime subgroup of $G$ which does not contain $S$, then $P \cap S$ is a minimal prime subgroup of $S$.*

PROPOSITION 3.2. *Let $G$ be an $l$-group and let $H$ be a convex $l$-subgroup of $G$. Then $B(H) = B(G) \cap H$.*

*Proof.* By [5], p. 1.6, Theorem 1.4, the set of convex $l$-subgroups of an $l$-group, ordered by inclusion, is a (complete) distributive lattice. Combining this with Lemma 3.1, we have the following:

$$\begin{aligned} B(H) &= \bigcap \{\langle Q_\phi, Q_\omega\rangle \mid \phi, \omega \in \Phi(H), \phi \neq \omega\} \\ &= \bigcap \{\langle P_\phi \cap H, P_\omega \cap H\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega, P_\phi \ngeqq H \nsubseteq P_\omega\} \\ &= \bigcap \{\langle P_\phi \cap H, P_\omega \cap H\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} \\ &= \bigcap \{\langle P_\phi, P_\omega\rangle \cap H \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} \\ &= B(G) \cap H. \end{aligned}$$

COROLLARY 3.3. *For any $l$-group $G$, $B(A(G)) = A(B(G))$.*

*Proof.* By definition of $P(B(G))$(cf. §2), $P(B(G)) = P(G) \cap B(G)$. Thus,

$$\begin{aligned} A(B(G)) &= \langle P(B(G))\rangle = \langle P(G) \cap B(G)\rangle \\ &= \langle P(G)\rangle \cap B(G) = A(G) \cap B(G) . \end{aligned}$$

By Proposition 3.2,

$$A(B(G)) = A(G) \cap B(G) = B(A(G)) .$$

PROPOSITION 3.4 *Let $G$ be a completely distributive $l$-group. If $G$ has a representable convex $l$-subgroup $H$ such that $H^\perp = \{0\}$, then $B(G)$ has a basis.*

*Proof.* Since $G$ is completely distributive, $H$ is completely distributive by Lemma 1.1. Thus, since $H$ is representable, $B(H)$ has a basis by [8], Theorem 3.1. By Proposition 3.2 above, $B(H) = H \cap B(G)$. If $g \in B(G)^+\backslash\{0\}$, then since $H^\perp = \{0\}$, there exists $h \in H$ such that $g \geqq h > 0$. But since $B(G)$ is convex, $h \in B(G)$ also, and thus $h \in B(H)$. Since $B(H)$ has a basis, $h$ exceeds a basic element, and hence $g$ exceeds a basic element. Therefore, $B(G)$ has a basis.

COROLLARY 3.5. *Let $G$ be a completely distributive $l$-group. If $A(G)^\perp$ is representable, then $B(G)$ has a basis.*

*Proof.* Since $A(G)$ is archimedean, it is abelian and hence representable. Therefore, since $A(G)^\perp$ is representable, $H = \langle A(G), A(G)^\perp\rangle$ is representable (clearly $H$ is $l$-isomorphic to $A(G) \mid \times \mid A(G)^\perp$). Clearly, $H^\perp = \{0\}$, and hence by Proposition 3.4, $B(G)$ has a basis.

COROLLARY 3.6. *Let $G$ be a completely distributive $l$-group such that $A(G)^\perp$ is representable. Then $G$ has a basis if and only if $B(G)^\perp = \{0\}$.*

## 4. Examples.

EXAMPLE 4.1. We construct an abelian, completely distributive $l$-group $H$ such that $A(H) \subseteqq B(H)$ but $A(H) \neq B(H)$.

Let $V = \prod \{R \mid n \in N\}$, and $f, g \in V$; let $S(f, g) \equiv \{n \in N \mid (n)f \neq (n)g\}$. Then $V$ becomes an $o$-group under (pointwise addition and) the relation: $f \leqq g$ if and only if $f = g$ or $f \neq g$ and $(\wedge S(f, g))f \leqq (\wedge S(f, g))g$. Clearly $V$, is completely distributive and abelian. Furthermore, if $f \in V^+\backslash\{0\}$ and $h \in G$ is defined by

$$(n)h = \begin{cases} 0 & \text{if } n \leqq \wedge S(f, 0) \\ 1 & \text{otherwise ,} \end{cases}$$

then for all $k \in N$,

$$(\wedge S(f, kh))(kh) = (\wedge S(f, 0))(kh) = k(\wedge S(f, 0))(h) = 0 < (\wedge S(f, 0))f = (\wedge S(f, kh))f ,$$

and hence $f$ is not archimedean. Thus, $A(V) = \{0\}$. Let $G = |\sum|\{R|n \in N\}$. Then clearly, $G$ is completely distributive and abelian, and $A(G) = G$. It is also easy to show that any minimal prime subgroup of $G$ has the form $\{f \mid nf = 0\}$ for some $n \in N$, and thus $B(G) = G$.

Let $H = V| \times |G$. Since $V$ is an $o$-group, $V \subseteqq B(H)$; by Proposition 3.2, $G \subseteqq B(H)$. Thus, $B(H) = H$. Since $A(V) = \{0\}$ and $A(G) = G$, $A(H) = \{0\} \times G$. Thus $A(H)$ is properly contained in $B(H)$. Clearly, $H$ is completely distributive and abelian.

REMARK 4.2. If $B(G)$ is strictly contained in $G$ for some completely distributive, archimedean $l$-group $G$, then $H = V| \times |G$ (cf. Example 4.1) is an an abelian, completely distributive $l$-group for which $A(H)$ and $B(H)$ are incomparable. On the other hand, if $B(G) = G$ for all completely distributive, archimedean $l$-groups $G$, then Proposition 3.2 could be used to show that $A(G) \subseteqq B(G)$ for every completely distributive $l$-group $G$. Thus, it would be useful to have an answer to the following question: Does there exist a completely distributive, archimedean $l$-group $G$ with distinct (minimal) prime subgroups $P_1$ and $P_2$ such that $G \neq \langle P_1, P_2 \rangle$?

EXAMPLE 4.3. We construct a non-representable $l$-group $G$ which is completely distributive and has a basis and for which $A(G)^\perp$ is representable.

Let $G = Z\,Wr\,Z$ be the wreath product of $Z$ by itself. Thus,

$G = Z \times (\prod_{i \in Z} Z_i)$, where each $Z_i = Z$, and group operation on $G$ is defined as follows:

$$(i; \cdots, \alpha_j, \cdots) \oplus (k; \cdots, \beta_j, \cdots) = (i + k; \cdots, \gamma_j, \cdots) ,$$

where $\gamma_j = \alpha_{j-k} + \beta_j$. An element $(i; \cdots, \cdots, \alpha_j, \cdots)$ is positive in $G$ if $i > 0$ or if $i = 0$ and $\alpha_j \geqq 0$ for all $j$. Clearly $A(G) = \{0\} \times (\prod_{i \in Z} Z_i) \cong |\prod|_{i \in Z} Z_i$. Thus, $A(G)^{\perp} = \{0\}$; hence $A(G)^{\perp}$ is representable and $G$ satisfies the hypothesis of Proposition 2.5. Clearly, $A(G)$ has a basis so that $G$ has a basis, and thus, by Proposition 2.5, $G$ is completely distributive. It remains to show that $G$ is not representable. By [5], p. 1.20, Theorem 1.8, for this it suffices to produce $a, x \in G^+ \backslash \{0\}$ such that $a \wedge (-x \oplus a \oplus x) = 0$. For $i \in Z$, let

$$\alpha_i = \begin{cases} 1 & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0 , \end{cases} \qquad \gamma_i = \begin{cases} 1 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1 , \end{cases} \qquad \delta_i = \begin{cases} -1 & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0 . \end{cases}$$

Let $a = (0; \cdots, \alpha_i, \cdots)$ and $x = (1; \cdots, \gamma_i, \cdots)$. Then $-x = (-1; \cdots, \delta_i, \cdots)$, and hence $-x \oplus a \oplus x = (0; \cdots, \gamma_i, \cdots)$. Clearly $a \wedge (-x \oplus a \oplus x) = 0$ and $a > 0 < x$, and therefore, $G$ is not representable.

Otis Kenny has found an example which supplies an affirmative answer to the question posed at the end of Remark 4.2.

## REFERENCES


1. Garrett Birkhoff, *Lattice Theory* (American Mathematical Society Colloquium Publications **25**, Providence, 1967 (third edition)).
2. Richard D. Byrd, *M-Polars in lattice-ordered groups*, Czech. Math. J., **18** (1968), 230-239.
3. Richard D. Byrd and Justin T. Lloyd, *Closed subgroups and complete distributivity in lattice-ordered groups*, Math. Zeit., **101** (1967), 123-130.
4. Paul Conrad, *Some structure theorems for lattice-ordered groups*, Trans. Amer. Math. Soc., **99** (1966), 212-240.
5. Paul Conrad, *Lattice-ordered Groups* (Tulane University, New Orleans, 1970).
6. László Fuchs, *Partially Ordered Algebraic Systems* (Pergamon Press (Addison-Wesley Publ. Co., Inc.), New York, 1963).
7. G. Otis Kenny, *Archimedean kernel of a representable l-group*, Notices Amer. Math. Soc., **21** (1974), #7, A-590.
8. R. H. Redfield, *Bases in completely distributive lattice-ordered groups*, Mich. Math. J., (to appear).
9. Elliot Carl Weinberg, *Higher degrees of distributivity in lattices of continuous functions*, Trans. Amer. Math. Soc., **104** (1962), 334-346.
10. Elliot Carl Weinberg, *Completely distributive lattice-ordered groups*, Pacific J. Math., **12** (1962), 1131-1137.


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