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BOUNDS AND QUANTITATIVE COMPARISON THEOREMS FOR NONOSCILLATORY SECOND ORDER DIFFERENTIAL EQUATIONS

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A lower bound is given for the positive increasing solution of $y'' + 2ry' - q^2y = 0$ on the interval $[0, \infty)$ and an upper bound is given for the positive decreasing solution of this equation. These are used to estimate z/y and z_0/y_0 where y and z (respectively y_0 and z_0) are positive nonprincipal (respectively principal) solutions of nonoscillatory equations $y'' - p_1y = 0$ and $z'' - p_2z = 0$ for which $p_2 \geqq p_1$. A special case of one result is that if p_1 is bounded and if

$$\liminf \left(\int_0^x (p_2 - p_1)^{1/2}\, dt/x\right) > 0$$

as $x \to \infty$ then z/y increases exponentially and z_0/y_0 decreases exponentially.

1. Introduction. It is our objective to estimate the relative sizes of the solutions of two nonoscillatory differential equations

$$y'' - p_1 y = 0 \tag{1}$$

and

$$z'' - p_2 z = 0 \tag{2}$$

on the interval $[0, \infty)$ when $p_2 \geqq p_1$.

It is well-known that (1) has a unique, *principal*, solution y_0 with the property that for any linearly independent, *nonprincipal*, solution y, $\lim y_0(x)/y(x) = 0$ as $x \to \infty$. If p_1 is nonnegative, then y_0 may be taken to be positive and decreasing and there is a nonprincipal solution which is positive and increasing [3, p. 355].

Our first task will be to obtain a lower bound for such a nonprincipal solution and an upper bound for the principal solution. We shall actually do this in §§ 2 and 3 for the more general equation

$$y'' + 2ry' - q^2 y = 0\,. \tag{3}$$

Here r and q are required only to be real-valued and locally integrable. Note, however, from the form of (3) that we may also assume that q is nonnegative. The estimates are in terms of $\int_0^x r\,dt$, $\int_0^x q\,dt$, and an increasing function F. In the special case when $r = 0$ they take the form that if $\liminf \left(\int_0^x q\,dt/F(x)\right) > 1$ as $x \to \infty$ for suitable F, then there is an increasing solution y such that $y(x) \geqq y(x_0)e^{F(x)}$ for $x \geqq x_0$ and a positive decreasing solution y_0 such that $y_0(x) \leqq y_0(x_0)e^{-F(x)}$ for $x \geqq x_0$. The case $F(x) = cx$ of this result was discussed in [4].

For sufficiently smooth r and q satisfying appropriate conditions, asymptotic formulae have been given for the solutions of (3). (See, for instance, [1, p. 120].) Our estimates, however, require no assumptions on r and q beyond local integrability and are thus available when the asymptotic formulae are not. One such application, to the deficiency index theory of powers of formally symmetric nonoscillatory expressions, is given in [5] where it is shown for $M(y) = -(py')' + qy$ that if q is nonnegative and if $\int_0^x (q/p)^{1/2}dt \geqq K \log p(x)$ for some $K > [n(n-1)]^{1/2}$ and all x in a set of infinite measure, then M^j is limit-point for $j = 1, 2, \cdots, n$.

Here we apply these estimates in a different direction. It is essentially a form of Sturm's comparison theorem that each pair of eventually positive nonprincipal solutions y and z of (1) and (2) respectively satisfies $z'/z \geqq y'/y$ on some interval $[a, \infty)$ and that the eventually positive principal solutions y_0 and z_0 satisfy $z_0'/z_0 \leqq y_0'/y_0$ on $[a, \infty)$ [3, p. 359]. Hence for all sufficiently large x, z/y is a positive increasing function and z_0/y_0 is a positive decreasing function.

In § 5 we shall use our earlier estimates to give a lower bound for z/y and an upper bound for z_0/y_0 in terms of $p_2 - p_1$. A special case of our result is that if p_1 is bounded and if

$$\liminf \left(\int_0^x (p_2 - p_1)^{1/2} dt/x\right) > 0$$

as $x \to \infty$, then z/y increases exponentially and z_0/y_0 decreases exponentially.

In the general result the boundedness of p_1 is replaced by a growth condition on $\int_0^x p_1 dt$. We show that such a condition cannot be omitted by proving for nonnegative p_1 and p_2 that if $p_2 - p_1$ is bounded and $p_2^{1/2} - p_1^{1/2}$ approaches 0, then z/y increases more slowly than any exponential. In particular, this conclusion holds for certain functions p_1 and p_2 for which $p_2 - p_1$ is bounded away from 0.

In § 4 we shall derive an upper bound for the nonprincipal solution of (1) and a lower bound for the principal solution. These depend on $\int_0^x |p_1|\, dt$ rather than $\int_0^x |p_1|^{1/2}\, dt$ as one might hope from the other bounds. We give an example to show that this cannot be avoided.

2. A lower bound for the nonprincipal solution. We begin by deriving a lower bound for an increasing solution of (3) valid on the complement of a set of finite Lebesgue measure.

THEOREM 2.1. *Let r and q be real-valued locally integrable functions on $[0, \infty)$ with q nonnegative. Let y be any solution of (3) with $y(0) > 0$ and $y'(0) > 0$. Then for each c, $0 < c < 1$, there is a subset E_c of $[0, \infty)$ such that $m(E_c) \leqq y(0)c^2/y'(0)(1 - c^2)$ and for all x not in E_c,*

$$y(x) \geqq y(0) \exp c\left(\left[c^2\left(\int_0^x r dt\right)^2 + \left(\int_0^x q dt\right)^2\right]^{1/2} - c\int_0^x r dt\right).$$

Proof. Let y be as in the statement of the theorem. Set $f(x) = \exp 2\int_0^x r dt$ and $w = fy'/y$. Note that

$$\log y(x) = \log y(0) + \int_0^x w/f\, dt\,. \tag{4}$$

w is positive since y is increasing and satisfies the Riccati equation $w' = q^2 f - w^2/f$ or, equivalently,

$$w'/w + w/f = q^2 f/w\,. \tag{5}$$

From Schwarz's inequality and (5),

$$\begin{aligned}\left(\int_0^x q dt\right)^2 &\leqq \int_0^x w/f\, dt \int_0^x q^2 f/w\, dt \\ &\leqq \int_0^x w/f\, dt\left[\int_0^x w/f\, dt + \log(w(x)/w(0)f(x)) + 2\int_0^x r dt\right].\end{aligned} \tag{6}$$

We shall complete the proof by showing that $\log(w(x)/w(0)f(x))$ is small compared to $\int_0^x w/f\, dt$ off a set of finite measure. Fix $c < 1$ and set $b = (1 - c^2)/c^2$. Let E_c denote the set of all x for which $\log(w(x)/w(0)f(x)) \geqq b\int_0^x w/f\, dt$, that is for which

$$W(x) = [w(x)/f(x)] \exp\left(-b\int_0^x w/f\, dt\right) \geqq w(0)\,.$$

Since W is positive and is the derivative of $(-1/b)\exp\left(-b\int_0^x w/f\, dt\right)$,

$$1/b \geqq \int_0^\infty W dx \geqq \int_{E_c} w(0) dx = w(0)m(E_c)\,.$$

Recalling the definitions of b and w,

$$m(E_c) \leqq y(0)c^2/y'(0)(1 - c^2)\,.$$

For $x \notin E_c$ we have from (6) that

$$\left(\int_0^x q\,dt\right)^2 \leqq \int_0^x w/f\,dt\left[(1/c^2)\int_0^x w/f\,dt + 2\int_0^x r\,dt\right].$$

The conclusion of Theorem 2.1 now follows by solving this inequality for $\int_0^x w/f\,dt$ and substituting the result into (4).

It should be noted that the exceptional sets E_c can be unbounded and of positive measure for each $c > 0$. An example with $r = 0$ for which this occurs is given in [4].

From the basic estimate we can now quickly obtain a lower bound for nonprincipal solutions of (3).

THEOREM 2.2. *Let r and q be as in Theorem 2.1. Let F be an increasing differentiable function on $[0, \infty)$ such that $F' \leqq KF$ for some positive constant K. If*

$$\liminf_{x\to\infty}\left(\left[\left(\int_0^x q\,dt\right)^2 + \left(\int_0^x r\,dt\right)^2\right]^{1/2} - \int_0^x r\,dt\right)\Big/F(x) > 1 \tag{7}$$

then for any increasing solution y of (3),

$$y(x) \geqq y(0)e^{F(x)}$$

for all x greater than some x_0.

Proof. Choose $\delta > 0$ and x_1 so that the expression in (7) is greater than $1 + \delta$ for $x \geqq x_1$. Choose $c < 1$ so that $c^2(1 + \delta) = 1 + \delta' > 1$. Let E_c be as in Theorem 2.1. Then for x greater than x_1 and not in E_c,

$$\begin{aligned}\log y(x) &\geqq \log y(0) + c\left(\left[\left(\int_0^x q\,dt\right)^2 + c^2\left(\int_0^x r\,dt\right)^2\right]^{1/2} - c\int_0^x r\,dt\right)\\ &\geqq \log y(0) + c^2\left(\left[\left(\int_0^x q\,dt\right)^2 + \left(\int_0^x r\,dt\right)^2\right]^{1/2} - \int_0^x r\,dt\right)\\ &\geqq \log y(0) + (1 + \delta')F(x)\,.\end{aligned}$$

Now choose $\varepsilon < \delta'/K(1 + \delta')$ and x_2 so that $m(E_c \cap [x_2, \infty)) < \varepsilon$. Then for any $x \geqq x_0 = \max(x_1, x_2)$ there exists $X \in [x - \varepsilon, x]$ such that

$$\begin{aligned}\log y(x) \geqq \log y(X) &\geqq \log y(0) + (1 + \delta')F(X)\\ &= \log y(0) + F(x) + (1 + \delta')(F(X) - F(x)) + \delta' F(x)\\ &\geqq \log y(0) + F(x)\,.\end{aligned}$$

The last inequality follows from the Mean Value Theorem and the choice of δ. This completes the proof.

In particular we have the following criterion for exponential growth.

COROLLARY 2.3. *Let r and q be as in Theorem 2.1. If for some $\alpha > 0$*

$$\liminf_{x\to\infty} \left(\left[\left(\int_0^x q\,dt\right)^2 + \left(\int_0^x r\,dt\right)^2\right]^{1/2} - \int_0^x r\,dt\right)\Big/ x > \alpha \ ,$$

then for any increasing solution y of (3),

$$y(x) \geqq y(0)e^{\alpha x}$$

for all x greater than some x_0.

It is clear from the constant coefficient case that the exponent is the best possible.

3. An upper bound for the principal solution. We now take up the problem of finding an upper bound for the principal solution of (3). Our method will be slightly different than in the previous section; instead of deriving a general inequality like Theorem 2.1 we shall proceed directly to an analog of Theorem 2.2. However the proof, in part, is similar to the proof of Theorem 2.1. It is convenient to begin with a lemma.

LEMMA 3.1. *If u and v are locally integrable functions on $[0, \infty)$ with v nonnegative, u positive and $u^{-\alpha} \in L_1(0, \infty)$ for all $0 < \alpha < 1/2$, and if*

$$u(x) \geqq K \exp \int_0^x uv\,dt$$

for all sufficiently large x and some positive K, then $v^\beta \in L_1(0, \infty)$ for all $0 < \beta \leqq 1$.

Proof. For any nonnegative u and v and any positive γ,

$$uv \exp\left(-\gamma \int_0^x uv\,dt\right) = (-1/\gamma)\left[\exp\left(-\gamma \int_0^x uv\,dt\right)\right]' \in L_1(0, \infty) \ .$$

We have here that eventually $v \leqq K^{-1}uv \exp\left(-\int_0^x uv\,dt\right)$. Hence $v \in L_1(0, \infty)$. Let E be a set of finite measure such that for all x in the complement, D, of E, the hypothesis of the lemma holds and also $uv \exp\left(-1/2 \int_0^x uv\,dt\right) \leqq K^{1/2}$. Thus on D,

$$u(x) \geqq K \exp \int_0^x uv\,dt \geqq u^2(x)v^2(x) \ .$$

Then for any $0 < \beta < 1$, $u^{-\beta/2} \geqq v^\beta$ on D and so $\int_D v^\beta dt < \infty$.

Now write E as the disjoint union $E = A \cup B$ where $A = \{x \in E\colon v(x) \leqq 1\}$. Clearly $\int_A v^\beta dt < \infty$. For any $\beta < 1$ we have also $\int_B v^\beta dt \leqq \int_B v dt < \infty$. Collecting the results on A, B and D, the lemma is proved.

THEOREM 3.2. *Let r and q be as in Theorem 2.1. Let F be an increasing differentiable function such that $F' e^{\alpha F} \to \infty$ as $x \to \infty$ for each positive α. If*

$$(8) \qquad \liminf_{x \to \infty} \left(\left[\left(\int_0^x q\,dt \right)^2 + \left(\int_0^x r\,dt \right)^2 \right]^{1/2} + \int_0^x r\,dt \right) \Big/ F(x) > 1$$

then for the positive decreasing solution y_0 of (3),

$$y_0(x) \leqq y_0(0) e^{-F(x)}$$

for all x greater than some x_0.

Proof. We shall for the first part of the proof assume in addition that $q \notin L_1(0, \infty)$. Set $f(x) = \exp 2 \int_0^x r\,dt$ and $w = -f y_0'/y_0$. Then w is positive and $w' = w^2/f - q^2 f$. Also

$$\log y_0(x) = \log y_0(0) - \int_0^x w/f\; dt$$

so that we must show $\int_0^x w/f\; dt \geqq F(x)$ for $x \geqq x_0$.

We assert first that $\int_0^x w/f\; dt \geqq F(x)$ for arbitrarily large values of x. A calculation using Schwarz's lemma as in the proof of Theorem 2.1 yields

$$(9) \qquad \left(\int_0^x q\,dt \right)^2 \leqq \int_0^x w/f\; dt \left[\int_0^x w/f\; dt - 2 \int_0^x r\,dt + \log \left(f(x) w(0)/w(x) \right) \right].$$

Choose $\delta > 0$ and x_1 so that the expression in (8) is greater than $1 + \delta$ for all $x \geqq x_1$. Suppose that for some $x_2 \geqq x_1$, $\int_0^x w/f\; dt < F(x)$ for all $x \geqq x_2$. Then for $x \geqq x_2$,

$$(10) \qquad \left(\int_0^x q\,dt \right)^2 \leqq F(x) \left[F(x) - 2 \int_0^x r\,dt + \log \left(f(x) w(0)/w(x) \right) \right].$$

In order to complete the proof of the assertion we need the fact that for any real numbers A, B, and F with F nonnegative, if $(A^2 + B^2)^{1/2} + B \geqq (1 + \delta)F$, then $A^2 + 2BF - F^2 \geqq \delta F^2$. This is clear if $2B \geqq (1 + \delta)F$. Otherwise, let $B = \varepsilon F$ with $2\varepsilon < 1 + \delta$. Then $A^2 + B^2 \geqq (1 + \delta - \varepsilon)^2 F^2$ so that

$$\begin{aligned} A^2 + 2BF &\geqq (1 + 2\delta + \delta^2 - 2\delta\varepsilon)F^2 \\ &= [1 + \delta + \delta(1 + \delta - 2\varepsilon)]F^2 \\ &\geqq (1 + \delta)F^2 . \end{aligned}$$

For this inequality and (8) we have that

$$\left(\int_0^x q\,dt\right)^2 + 2F(x)\int_0^x r\,dt - F^2(x) \geqq \delta F^2(x)$$

for all $x \geqq x_1$. Thus from (10) we obtain that for $x \geqq x_2$,

$$\log\,(f(x)w(0)/w(x)) \geqq \delta F(x) \quad \text{or}$$

$$f(x)/w(x) \geqq e^{\delta F(x)}/w(0) . \tag{11}$$

To see that (11) cannot hold for all $x \geqq x_2$, set $u = f/w$. Then u is positive and

$$u'/u - 2r + u^{-1} = uq^2 . \tag{12}$$

It follows from (8) that $\left(\int_0^x q\,dt\right)^2 \geqq -2F(x)\int_0^x r\,dt$ for all $x \geqq x_1$. By integrating (12) from 0 to x and using this inequality we obtain

$$\log\,(u(x)/u(0)) + \left(\int_0^x q\,dt\right)^2\Big/F(x) + \int_0^x u^{-1}\,dt \geqq \int_0^x uq^2 dt .$$

For all x greater than some x_3, $F'(x)e^{\delta F(x)} \geqq 3w(0)$ and hence $u^{-1}(x) \leqq w(0)e^{-\delta F(x)} \leqq F'(x)/3$. Thus by the Schwarz inequality,

$$\begin{aligned} \left(\int_{x_3}^x q\,dt\right)^2 &\leqq \int_{x_3}^x u^{-1}dt\int_{x_3}^x uq^2dt \leqq \frac{1}{3}[F(x) - F(x_3)]\int_{x_3}^x uq^2dt \\ &\leqq \frac{1}{3}\,F(x)\int_0^x uq^2dt . \end{aligned}$$

Hence, since $q \notin L_1(0, \infty)$,

$$\int_0^x uq^2dt \geqq 2\left(\int_0^x q\,dt\right)^2\Big/F(x)$$

for all sufficiently large x.

For such x,

$$\log\,(u(x)/u(0)) + \int_0^x u^{-1}dt \geqq \frac{1}{2}\int_0^x uq^2dt .$$

Moreover, $u^{-\alpha} \in L_1(0, \infty)$ for each $\alpha > 0$, since for all sufficiently large x we have from (11) and the assumption on F that

$$u^{-\alpha}(x) \leqq Ce^{-\alpha\delta F(x)} \leqq F'(x)e^{-(\alpha\delta/2)F(x)} \in L_1(0, \infty) .$$

Hence for all sufficiently large x and some positive K,

$$u(x) \geqq K \exp \frac{1}{2} \int_0^x uq^2 dt .$$

Now Lemma 3.1 with $\beta = 1/2$ applied to u and $v = q^2/2$ implies that $q \in L_1(0, \infty)$. But this contradicts the extra assumption made at the beginning of the proof. Hence $\int_0^x w/f\, dt \geqq F(x)$ for arbitrarily large values of x as asserted.

Now choose x_0 so that $x \geqq x_0$ implies $F'(x) \geqq 2w(0)e^{-\delta F(x)}$. Suppose that for some $X \geqq x_0$, $\int_0^X w/f\, dt < F(X)$. Let

$$X' = \inf \left\{ x > X : \int_0^x w/f\, dt \geqq F(x) \right\} .$$

On the interval (X, X') the inequality (11) is valid, that is, $w(x)/f(x) \leqq 1/2\, F'(x)$. Hence

$$\begin{aligned} \int_0^{X'} w/f\, dt = \int_0^X w/f\, dt + \int_X^{X'} w/f\, dt &< F(X) \\ + \frac{1}{2} \int_X^{X'} F' dt &< F(X') . \end{aligned}$$

This contradiction completes the proof of the theorem when $q \notin L_1(0, \infty)$.

Now suppose $q \in L_1(0, \infty)$. Then (8) implies that

$$\liminf_{x \to \infty} 2 \int_0^x r dt / F(x) > 1 .$$

Let z be the positive decreasing solution of $z'' + 2rz' = 0$ such that $z(0) = 1$. Then

$$z(x) = k \int_x^\infty \exp\left(-2 \int_0^t r(u) du \right) dt .$$

For all sufficiently large t,

$$k \exp\left(-2 \int_0^t r(u) du \right) \leqq k \exp\left(-(1 + \delta) F(t) \right) \leqq F'(t) e^{-F(t)}$$

so that $z(x) \leqq \int_x^\infty F' e^{-F} dt = e^{-F(x)}$. But the principal solution y_0 of (3) satisfies $y_0(x) \leqq y_0(0) z(x)$ [3, p. 359] so that the conclusion is true in this case also and the proof is complete.

The hypothesis $F' e^{\alpha F} \longrightarrow \infty$ for all $\alpha > 0$ is satisfied, for example, for every positive power of x. For the function $F(x) = \log(1 + x)$, corresponding to solutions of the form x^{-r}, it is satisfied only for $\alpha > 1$. Examples of the form $y'' - k(1 + x)^{-2} y = 0$ for small positive k show that in fact the result is no longer true for this choice of F. However a slight modification of the proof of Theorem 3.2 does yield the following result for $q \notin L_1(0, \infty)$.

COROLLARY 3.3. *Let r and q be as in Theorem 2.1 with $q \notin L_1(0, \infty)$. Let F be an increasing differentiable function such that $F'e^{\alpha F} \to \infty$ as $x \to \infty$ for all α greater than some α_0. If (8) holds then*

$$y_0(x) \leqq y_0(0)e^{-\beta F(x)}$$

for all x greater than some x_0 and some $\beta > 0$.

As in the previous section we state the condition for exponential decay as a corollary of Theorem 3.2.

COROLLARY 3.4. *Let r and q be as in Theorem 2.1. If for some $\alpha > 0$,*

$$\liminf_{x\to\infty} \left(\left[\left(\int_0^x q\,dt\right)^2 + \left(\int_0^x r\,dt\right)^2\right]^{1/2} + \int_0^x r\,dt\right)\Big/ x > \alpha$$

then for the positive decreasing solution y_0 of (3),

$$y_0(x) \leqq y_0(0)e^{-\alpha x}$$

then all x greater than some x_0.

Again it is clear from the constant coefficient case that the exponent is the best possible.

4. Bounds for $y'' - py = 0$. If $q \in L_1(0, \infty)$ then certainly no upper bound for the increasing solution of $y'' - q^2y = 0$ can be made from $\exp \int_0^x q\,dt$. The following example shows that it does not help to require $q \notin L_1(0, \infty)$.

Let $\{x_k\}_{k=0}^{\infty}$ be an increasing sequence with $x_0 = 0$ chosen so that if $q(x)=0$ on each $I_n=[x_{2n}, x_{2n+1})$ and $q(x)=3n$ on each $J_n=[x_{2n+1}, x_{2n+2})$, then the solution w of $w' = q^2 - w^2$ with $w(x_0) = e$ satisfies $w(x_{2n}) = e$ and $w(x_{2n+1}) = 1$ for all n. Obviously $x_{2n+1} - x_{2n}$ does not depend on n. Solving the equations $w' = -w^2$ and $w' = 9n^2 - w^2$ it is found that $\int_{I_n} w\,dt = 1$ for each n and that for large n, $x_{2n+2} - x_{2n+1}$ is approximately $(e-1)/9n^2$ so that $\int_{J_n} q\,dt \sim (e-1)/3n$. Hence the solution $y(x) = \exp \int_0^x w\,dt$ of $y'' - q^2y = 0$ increases exponentially while $q \notin L_1(0, \infty)$ but $\int_0^x q\,dt/x \to 0$.

To overcome this difficulty we replace $\int_0^x q\,dt$ by $\left(x\int_0^x q^2dt\right)^{1/2}$. There is no advantage in restricting ourselves to nonnegative coefficients, so we state our result in terms of

$$y'' - py = 0\,. \tag{13}$$

THEOREM 4.1. *Let p be a locally integrable function on $[0, \infty)$, not identically 0, such that (13) is nonoscillatory. Let G be an increasing differentiable function such that $xG'(x)/G^2(x) \to 0$ as $x \to \infty$. If*

$$\limsup_{x\to\infty} x\int_0^x |p|\,dt/G^2(x) < 1\,,$$

then every eventually positive solution y of (13) satisfies

$$y(x_0)e^{-G(x)} \leqq y(x) \leqq y(x_0)e^{G(x)} \tag{14}$$

for all x greater than some x_0.

Proof. By applying the comparison theorem discussed in the introduction to $p_1 = p$, $p_2 = |p|$ it is clear that it suffices to establish (14) for solutions of $y'' - |p|y = 0$.

Let y be a positive increasing solution of this equation, and let $w = y'/y$. Then $w' + w^2 = |p|$. Hence, integrating from 0 to x and using the positivity of w, $\int_0^x w^2dt \leqq w(0) + \int_0^x |p|\,dt$. *If* $p \in L_1(0, \infty)$ then it is well-known that $x^{-1/2}\int_0^x wdt \to 0$ as $x \to \infty$ [2, Th. 223, p. 164]; hence $\int_0^x wdt/G(x) \to 0$ and the right hand inequality for (14) follows. If $p \notin L_1(0, \infty)$, then an application of Schwarz's inequality gives, for sufficiently large x_0,

$$\left(\int_{x_0}^x wdt\right)^2 \leqq x\left[w(0) + \int_{x_0}^x |p|\,dt\right] \leqq G^2(x)$$

for all $x \geqq x_0$ and hence the right hand side of (14) follows again. The left hand side is of course trivial in either case.

Now let y_0 be a positive decreasing solution of $y'' - |p|y = 0$ and let $w = -y_0'/y_0$. Then w is positive and $-w' + w^2 = |p|$. If $w \in L_1(0, \infty)$ then certainly for some x_0, $\int_{x_0}^x wdt \leqq G(x)$ for all $x \geqq x_0$ and the left hand side in (14) follows.

If $w \notin L_1(0, \infty)$ then for sufficiently large x_0 and some $\delta < 1$, $(1-\delta)\int_0^{x_0} wdt \geqq 1$, $2xG'(x) < (1-\delta)G^2(x)$ for $x \geqq x_0$ and

$$\left(\int_0^x wdt\right)^2\Big/x \leqq \int_0^x |p|\,dt + w(x) \leqq \delta G^2(x)/x + w(x) \tag{15}$$

for $x \geqq x_0$. Suppose that for some $x_1 \geqq x_0$, $\int_{x_0}^{x_1} wdt > G(x_1)$. If this inequality remains valid for all $x \geqq x_1$, then for such x it follows from (15) that $xw(x) \geqq (1-\delta)\left(\int_0^x wdt\right)^2$ and hence

$$w(x)\left(\int_0^x w\,dt\right)^{-2} \geqq (1-\delta)/x\ .$$

But this cannot be true for all $x \geqq x_1$ since the function on the left is in $L_1(0, \infty)$ and the function on the right is not.

Thus let $x_2 = \inf\left\{x > x_1 \colon \int_{x_0}^x w\,dt \leqq G(x)\right\}$. On (x_1, x_2) we have $w(x) \geqq (1-\delta)\left(\int_0^x w\,dt\right)^2\Big/x > \int_0^x w\,dt/x$ and also $w(x) \geqq (1-\delta)G^2(x)/x$. Hence on (x_1, x_2),

$$\begin{aligned}\left[\left(\int_{x_0}^x w\,dt\right)^2\Big/x\right]' &= \left(\int_{x_0}^x w\,dt/x\right)\left[2w(x) - \int_{x_0}^x w\,dt/x\right]\\ &\geqq w(x)\int_{x_0}^x w\,dt/x\\ &\geqq (1-\delta)G^3(x)/x^2 \geqq [G^2(x)/x]'\ .\end{aligned}$$

Then $\left(\int_{x_0}^{x_2} w\,dt\right)^2\Big/x_2 - G^2(x_2)/x_2 \geqq \left(\int_{x_0}^{x_1} w\,dt\right)^2\Big/x_1 - G^2(x_1)/x_1 > 0$, contradicting the choice of x_2. Hence we must have $\int_{x_0}^x w\,dt \leqq G(x)$ for $x \geqq x_0$ and the left hand inequality in (14) follows. The right hand inequality is clear for the decreasing solution and so the proof is complete.

5. Quantitative comparison theorems. We shall now combine the results of §§ 2, 3 and 4 to obtain some comparison theorems. Thus let p_1 and p_2 be locally integrable functions on $[0, \infty)$ with $p_2 \geqq p_1$ such that (1) and (2) are nonoscillatory. We wish to obtain a lower bound for the quotient z/y of positive nonprincipal solutions of (1) and (2) and an upper bound for the quotient z_0/y_0 of the positive principal solutions of (1) and (2).

THEOREM 5.1. *Let G and H be increasing differentiable functions such that $xG'(x)/G^2(x) \to 0$, $H'e^{\alpha H} \to \infty$ for all $\alpha > 0$ and H'/H is bounded. Suppose that*

(a) $\limsup_{x\to\infty} x\int_0^x |p_1|\,dt/G^2(x) < 1$

(b) $\liminf_{x\to\infty}\int_0^x (p_2 - p_1)^{1/2}\,dt/H(x) > 1$.

If $H(x) \geqq KG(x)$ for some positive K, then

$$z(x)/y(x) \geqq e^{\beta H(x)};\ z_0(x)/y_0(x) \leqq e^{-\beta H(x)}$$

for some $\beta > 0$ and all x greater than some x_0.

Proof. For the first assertion, set $u = z/y$. Then u satisfies $u'' + 2(y'/y)u' - (p_2 - p_1)u = 0$. For all sufficiently large x it follows from Theorem 4.1 that $\int_0^x y'/y\,dt \leqq G(x)$. Hence

$$\left[\left(\int_0^x (p_2 - p_1)^{1/2} dt\right)^2 + \left(\int_0^x y'/y\, dt\right)^2\right]^{1/2} - \int_0^x y'/y\, dt$$
$$\geqq [H^2(x) + G^2(x)]^{1/2} - G(x) \geqq \beta H(x)$$

for some $\beta > 0$ and all sufficiently large x. Thus the assertion is a consequence of Theorem 2.2. For the second assertion set $u_0 = z_0/y_0$. Then $u_0'' + 2(y_0'/y_0)u_0' - (p_2 - p_1)u_0 = 0$ and $\int_0^x y_0'/y_0\, dt \geqq -G(x)$ for all sufficiently large x. Hence

$$\left[\left(\int_0^x (p_2 - p_1)^{1/2} dt\right)^2 + \left(\int_0^x y_0'/y_0\, dt\right)^2\right]^{1/2} + \int_0^x y_0'/y_0\, dt \geqq \beta H(x)$$

for some $\beta > 0$ and all sufficiently large x and the second assertion follows from Theorem 3.2.

A variant of the above concerned with exponential growth is the following.

THEOREM 5.2. *Let G and H be as in Theorem* 5.1 *and suppose* (a) *and* (b) *are satisfied. If $H^2(x) \geqq Kx(x + G(x))$ for some positive K, then*

$$z(x)/y(x) \geqq e^{\beta x};\ z_0(x)/y_0(x) \leqq e^{-\beta x}$$

for some $\beta > 0$ and all sufficiently large x.

Proof. $\left(\int_0^x (p_2 - p_1)^{1/2} dt\right)^2 + \left(\int_0^x y'/y\, dt\right)^2 \geqq H^2(x) + G^2(x) \geqq [\beta x + G(x)]^2$ for some $\beta > 0$ and all sufficiently large x. Thus the first assertion is a consequence of Theorem 2.2 as in the previous theorem. The second assertion follows similarly from Theorem 3.2.

Choosing G and H to be multiples of x in either theorem yields a slightly more general result than that mentioned in the introduction.

COROLLARY 5.3. *If $\int_0^x |p_1|\, dt \leqq Mx$ for some M, and if*

$$\liminf_{x\to\infty} \int_0^x (p_2 - p_1)^{1/2}\, dt/x > 0\ ,$$

then

$$z(x)/y(x) \geqq e^{\beta x};\ z_0(x)/y_0(x) \leqq e^{-\beta x}$$

for some $\beta > 0$ and all x greater than some x_0.

If p_1 and p_2 are nonnegative, then an application of Schwarz's lemma shows that (b) in Theorem 5.1, is in the presence of the other hypotheses of that theorem, equivalent to the in general more restrictive

(b′) $\liminf_{x\to\infty} \int_0^x \sqrt{p_2} - \sqrt{p_1}\, dt/H(x) > 0$.

The same is then true of Corollary 5.3. From the form of Theorems 2.2 and 3.2 when $r = 0$ one might expect this condition to be more closely related than (b) to the behavior of z/y. Our final theorem may also be viewed in this way, for it implies that z/y can fail to increase exponentially even when $p_2 - p_1$ is bounded away from 0 provided that $\sqrt{p_2} - \sqrt{p_1}$ approaches 0.

THEOREM 5.5. *Suppose that p_1 and p_2 are nonnegative locally integrable functions such that $|p_2 - p_1| \leqq M$ for some M and $\sqrt{p_2} - \sqrt{p_1} \to 0$ as $x \to \infty$. Then for any positive nonprincipal solutions y and z of* (1) *and* (2),

$$(\log z/y)' \longrightarrow 0 \quad as \quad x \longrightarrow \infty .$$

It then follows immediately that for any positive α,

$$\log (z(x)/y(x)) - \alpha x \longrightarrow -\infty$$

as $x \to \infty$ and so, exponentiating, $[z(x)/y(x)]e^{-\alpha x} \to 0$ as $x \to \infty$.

Proof. We may assume $p_2 \geqq p_1$, for the hypotheses are still valid for the functions $\min(p_1, p_2)$ and $\max(p_1, p_2)$ and the quotient of nonprincipal solutions of the equations with p_1 and p_2 replaced by these functions is greater than z/y. We shall for the present assume further that $p_1(x) \geqq 1$ for all x.

It suffices to extablish the theorem for the solutions y and z of (1) and (2) with $y(0) = y'(0) = z(0) = z'(0) = 1$. Set $u = z'/z$ and $v = y'/y$ so that $u' = p_2 - u^2$ and $v' = p_1 - v^2$. Note that $p_1 \geqq 1$ implies $v \geqq 1$, for if $v(x_0) < 1$ then $v(x_1) < 1$, $v'(x_1) < 0$ for some $x_1 \in (0, x_0)$, and this is impossible. Similarly $u \geqq 1$. Finally, set $w = u - v$. Then

$$w' = p_2 - p_1 - (u + v)w;\ w(0) = 0 . \tag{16}$$

Let $\varepsilon > 0$ be given. Choose $c < 1$ so that $K = c^2/(1 - c^2)$ satisfies $MK < \varepsilon/3$. Choose x_0 so that $\sqrt{p_2(x)} - \sqrt{p_1(x)} < c\varepsilon/3$ for $x \geqq x_0$. Repeating the argument in the proof of Theorem 2.1 for the equation $u' = p_2 - u^2$ on the interval $[t, \infty)$ yields that $\int_t^x u ds \geqq c \int_t^x \sqrt{p_2}\, ds$ for all $x \geqq t$ not in a set $E_{c,t}$ such that $m(E_{c,t}) \leqq c^2/u(t)(1 - c^2) \leqq K$. Hence for any $x \geqq t + K$ there is some $X \in [x - K, x]$ such that

$$\int_t^x u ds \geqq \int_t^X u ds \geqq c \int_t^X \sqrt{p_2}\, ds \geqq c \int_t^{x-K} \sqrt{p_2}\, ds.$$

Similarly $\int_t^x v ds \geqq c\int_t^{x-K} \sqrt{p_1}\, ds$ whenever $x - K \geqq t$.

For all $x \geqq x_0 + K$ we have from (16) that

$$w(x) = \int_0^x \exp\left(-\int_t^x u + v\, ds\right)(p_2(t) - p_1(t))dt$$
$$= \int_0^{x_0} + \int_{x_0}^{x-K} + \int_{x-K}^{x} = I_1 + I_2 + I_3 .$$

Since $u + v \geqq 2$,

$$I_1 \leqq Mx_0 \exp\left(-\int_{x_0}^x u + v\, ds\right) \leqq Mx_0 \exp(-2(x - x_0)) < \varepsilon/3$$

for all x greater than some x_1.

By the choice of x_0 we have next

$$I_2 \leqq (c\varepsilon/3)\int_{x_0}^{x-K} \exp\left(-c\int_t^{x-K} \sqrt{p_2} + \sqrt{p_1}\, ds\right)\left(\sqrt{p_2(t)} + \sqrt{p_1(t)}\right) dt$$
$$= (c\varepsilon/3)(1/c) \exp\left(-c\int_t^{x-K} \sqrt{p_2} + \sqrt{p_1}\, ds\right)\Big|_{x_0}^{x-K} < \varepsilon/3 .$$

Finally, by the choice of K, $I_3 \leqq KM < \varepsilon/3$. Hence $w(x) < \varepsilon$ for all $x \geqq x_1$. Also $p_2 \geqq p_1$ and $u(0) = v(0)$ implies $w = u - v \geqq 0$. Since ε was arbitrary we have that $w = u - v = (\log z/y)' \to 0$ as $x \to \infty$ and the theorem is proved when $p_1 \geqq 1$.

Now let p_1 and p_2 be as in the statement of the theorem and consider the functions y_1 and z_1 such that $y_1(0) = y_1'(0) = z_1(0) = z_1'(0) = 1$ and

$$y_1'' - (p_1 + 1)y = 0$$
$$z_1'' - (p_2 + 1)z = 0 .$$

Note that $(\sqrt{p_2 + 1} - \sqrt{p_1 + 1})(\sqrt{p_2 + 1} + \sqrt{p_1 + 1}) = p_2 - p_1 = (\sqrt{p_2} - \sqrt{p_1})(\sqrt{p_2} + \sqrt{p_1})$ so that $|\sqrt{p_2 + 1} - \sqrt{p_1 + 1}| \leqq |\sqrt{p_2} - \sqrt{p_1}|$. Hence the special case already proved can be applied to z_1/y_1.

Let y and z be the solutions of (1) and (2) such that

$$y(0) = y'(0) = z(0) = z'(0) = 1 .$$

Then $w = y'/y$ and $w_1 = y_1'/y_1$ satisfy $w' = p_1 - w^2$, $w_1' = p_1 + 1 - w_1^2$, and $w(0) = w_1(0)$. Also $(w + 1)' = p_1 - (w + 1)^2 + 1 + 2w \geqq p_1 + 1 - (w + 1)^2$. Hence $y'/y \leqq y_1'/y_1 \leqq y'/y + 1$.

Set $s = z/y$ and $s_1 = z_1/y_1$. Then

$$s'' + 2(y'/y)s' - (p_2 - p_1)s = 0$$
$$s_1'' + 2(y_1'/y_1)s_1' - (p_2 - p_1)s_1 = 0$$

and by the proof so far we know $(\log s_1)' = s_1'/s_1 \to 0$ as $x \to \infty$. Now $r = s/s_1$ is an increasing solution of

$$r'' + 2[y'/y + s_1'/s_1]r' - 2(y_1'/y_1 - y'/y)(s_1'/s_1)r = 0$$

since $r(0) = 1$, $r'(0) = 0$ and the coefficient of r is nonnegative. Also $r'/r = s'/s - s_1'/s_1$. Thus to establish that $(\log z/y)' = s'/s \to 0$ it remains, finally, only to verify that if f and g are nonnegative functions such that $g \to 0$ then every increasing solution of $y'' + fy' - gy = 0$ satisfies $y'/y \to 0$. To see this note that $w = y'/y$ satisfies $w' + w^2 = g - fw \leqq g$. It is well-known (and easy to see) that if $g \to 0$ then every solution of $u' + u^2 = g$ on $[0, \infty)$ does also. If u is the solution with $u(0) = w(0)$, then $0 \leqq w \leqq u$ so that $y'/y = w$ does approach 0 and the proof of Theorem 5.5 is complete.

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