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KOROVKIN APPROXIMATIONS IN L_p-SPACES

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The main result is a characterization of finite Korovkin sets for positive operators in l_p. It follows that a finite set containing a positive function is a Korovkin set in l_p if and only if it is a Korovkin set in c_0. The methods also show:

PROPOSITION. Let X be a compact subset of R^n. Let K be a subspace of $C(X)$ containing the constants. If K is a Korovkin set in $C(X)$, then K is Korovkin set in $L_p(X)$.

Several related results are also given. For example a question of G. G. Lorentz about the restrictions of Korovkin set in $C(X)$ to a subset $Y \subseteqq X$ is answered.

Let $\mathscr{L}$ be a class of operators on a Banach space E. A subset $K \subseteqq E$ is an *$\mathscr{L}$-Korovkin set* if whenever

(i) $\{L_i\}$ is a bounded sequence in $\mathscr{L}$, and

(ii) $L_i k \rightarrow k$ for each $k \in K$;

we have

(iii) $L_i f \rightarrow f$ for each f in E.

Let $\mathscr{L}^1$ be the class of norm one operators on E. If E is also a lattice, let $\mathscr{L}^+$ denote the positive operators on E; and, $\mathscr{L}^{1,+} = \mathscr{L}^1 \cap \mathscr{L}^+$.

After Korovkin showed that $\{1, x, x^2\}$ is an $\mathscr{L}^+$-Korovkin set in $C[0, 1]$, interest in this field has been in characterizing the Korovkin subsets of the classic Banach spaces.

Papers by Berens and Lorentz [3], Franchetti [8, 9], Krasnosilskii and Lifsic [13], Lorentz [14], Saskin [18], Scheffold [19], and Wulbert [22] identified the various types of Korovkin sets in $C(X)$ spaces. Berens and Lorentz [3] have essentially characterized the $\mathscr{L}^{1,+}$-Korovkin subsets of L_1 spaces (see §3 of this article, also see [Lorentz, 14] and [Wulbert, 22]), and Dzjadyk [7] has shown that $\{1, \sin x, \cos x\}$ is an $\mathscr{L}^+$-Korovkin set in $L_p[0, 2\pi]$. (See also [James, 11], and [Zaricka, 24].)

The results here are related to identifying $\mathscr{L}^+$-Korovkin subsets of L_p-spaces. A sufficient condition is presented that encompasses the known (and the suspected) $\mathscr{L}^+$-Korovkin sets. For example each $\mathscr{L}^+$-Korovkin set in $C[a, b]$ that contains constants is also an $\mathscr{L}^+$-Korovkin set in $L_p[a, b]$. The main result given is a characterization of finite $\mathscr{L}^+$-Korovkin sets in l_p. A consequence of this characterization is that the l_p spaces have the same finite $\mathscr{L}^+$-Korovkin sets. That is, if K is a finite subset of both l_r and l_s, and K contains a positive sequence, then K is $\mathscr{L}^+$-Korovkin in l_p if and only if K is $\mathscr{L}^+$-Korovkin in l_s.

We use the last two sections of the paper to give short direct generalizations of some related Korovkin theorems. For example, a recent result by Bernau and Lacey [5] enables the removal of the last conditions from the characterization of $\mathscr{L}^{1,+}$-Korovkin subsets of L_p-spaces with an easy argument.

G. G. Lorentz [14] proved that if X is a compact metric space, and K is $\mathscr{L}^+$-Korovkin set in $C(X)$ containing a constant, then for each closed subset $Y \subseteqq X$, $K|_Y$ is an $\mathscr{L}^+$-Korovkin set in $C(Y)$. Lorentz asked if the property was true for any compact Hausdorff space X. A counterexample is given in section two.

NOTATION. If X is a compact Hausdorff space $C(X)$ is the space of continuous real functions on X. For $x \in X$, $\xi(x)$ is the linear functional on $C(X)$ given by $\xi(x)(f) = f(x)$. If K is a linear subspace of $C(X)$, we say $x \in cb\, K$, the choquet boundary of K, if the only positive linear functional on $C(X)$ that agrees with $\xi(x)$ on K is $\xi(x)$ itself. If F is a subset of a set Y, ψ_F is the characteristic function of F. We use $f|_F$ to denote the restriction of a function f to the domain F, and for a set of functions K, $K|_F = \{k|_F : k \in K\}$. The dual of a normed space E is written E^*.

As usual, c denotes the space of convergent sequences with the sup norm,

$$c_0 = \{x(i) \in c : \lim x(i) = 0\}, \quad \text{and}$$
$$l_p = \{x(i) \in c_0 : \|x\|_p = \sqrt[p]{\Sigma |x(i)|^p} < \infty\} .$$

The norm on l_p is assumed to be $\|\cdot\|_p$ as given above. We will frequently view these sequence spaces as spaces of continuous functions on the one point compactification of the integers.

Let $\mathscr{L}$ be a class of linear operators on a normed space E. Let K be a subset of E. A member $f \in E$ is in the *$\mathscr{L}$-shadow of K* if $L_n f \to f$ for each bound sequence $\{L_n\} \subseteqq \mathscr{L}$ such that $L_n k \to k$ for each $k \in K$. Hence K is an $\mathscr{L}$-Korovkin set if the $\mathscr{L}$-shadow of K is E. Since the $\mathscr{L}$-shadow of K is the same as the $\mathscr{L}$-shadow of the span of K we will often assume that K is already a linear subspace of E.

1. $\mathscr{L}^+$-Korovkin sets in L_p-spaces. The main result of this section is the characterization of finite $\mathscr{L}^+$-Korovkin subsets of l_p-spaces. The condition is sufficient in general, and provides an accessible class of $\mathscr{L}^+$-Korovkin sets in L_p-spaces.

We also show that an $\mathscr{L}^+$-Korovkin set of an $\mathscr{L}_p$-space contains three functions. The interest in this fact comes from the surprising observation that that $\{1, x\}$ is $\mathscr{L}^{1,+}$-Korovkin in $L_p[0, 1]$ (see §3).

Let K be a linear subspace of a normed linear lattice E. Let $f \in E$. Two sets of vectors $\{u_i\}_{i=1}^n$ $\{l_i\}_{i=1}^m$ is an *ε-trap* for f if there is a vector e such that:

1. $-e + \bigvee_{i=1}^m l_i \leqq f \leqq e + \bigwedge_{i=1}^n u_i$,
2. $\bigwedge_{i=1}^n u_i - \bigvee_{i=1}^m l_i + 2e\| < \varepsilon$, and
3. $\|e\| < \varepsilon$.

DEFINITION. *K traps f* if for each $\varepsilon > 0$, K contains an ε-trap for f.

PROPOSITION 1.1. *If K traps f, then f is in the $\mathscr{L}^+$-shadow of K.*

Proof. Let L_i be a sequence of positive operators such that $L_i k \to k$ for all k in K and $\|L_i\| < B$. Then for k sufficiently large,

$$\left\|\bigwedge_{i=1}^n L_k(u_i) - \bigwedge_{i=1}^n u_i\right\| < \varepsilon \text{ , and } \left\|\bigvee_{i=1}^m L_k(l_i) - \bigvee_{i=1}^n l_i\right\| < \varepsilon \text{ .}$$

We also have,

$$\begin{aligned} -L_k(e) + \bigvee_{i=1}^m L_k(l_i) &\leqq -L_k(e) + L_k\Big(\bigvee_{0=1}^m l_i\Big) \\ &\leqq L_k(f) \\ &\leqq L_k(e) + L_k\Big(\bigwedge_{i=1}^n u_i\Big) \\ &\leqq L_k(e) + \bigwedge_{i=1}^n L_k(u_i) \text{ .} \end{aligned}$$

Since,

$$\left\|\bigwedge_{i=1}^n L_k(u_i) - \bigvee_{i=1}^m L_k(l_i) + 2L_k(e)\right\| \leqq \varepsilon B \text{ ,}$$

we have,

$$\begin{aligned} \|L_k f - f\| \leqq \Big\|L_k f - L_k(e) - \bigwedge_{i=1}^n L_k(u_i)\Big\| & \\ + \|L_k e\| + \Big\|\bigwedge_{i=1}^n L_k(u_i) - \bigwedge_{i=1}^n u_i\Big\| & \\ + \Big\|\bigwedge_{i=1}^n u_i - f\Big\| & \\ \leqq 2\varepsilon(B + 1) \text{ .} & \end{aligned}$$

We need the following known result. [Alfsen, 1, Cor. 1.5.10].

Let X be a compact Hausdorff space. Let K be a linear subspace of $C(X)$ that contains the constants and separates the points of X.

LEMMA 1.2. *If* $f \in C(X)$ *and* $x \in cbK$ *then*

$$f(x) = \inf \{k(x): k \in K, k \geqq f\} .$$

COROLLARY 1.3. *Let* X *and* K *be as above. Let* μ *be a positive finite, regular Borel measure on* X. *If the support of* μ *is contained in* $cb\ K$, *then* K *is an* $\mathscr{L}^+$-*Korovkin set in* $L_p(X, \mu)$, $1 \leqq p < \infty$.

Proof. From the lemma and Dini's theorem K traps every continuous function. Since the $\mathscr{L}^+$-shadow of K is closed, and the continuous functions are dense in $L_p(X, \mu)$, the corollary is proved.

COROLLARY 1.4. *Let* X, K, *and* μ *be as above. If* $cb\ K = X$ *then* K *is an* $\mathscr{L}^+$-*Korovkin set in* $L_p(X, \mu)$. *In particular if* X *is metrizable and* K *is* $\mathscr{L}^+$-*Korovkin in* $C(X)$, *then* K *is* $\mathscr{L}^+$-*Korovkin in* $L_p(X, \mu)$.

Proof. If X is metrizable the Choquet boundary of an $\mathscr{L}^+$-Korovkin set is X [14]. (Also see §2.)

EXAMPLE 1.5. (a) (*Dzjadyk*) $\{1, \sin x, \cos x\}$ is an $\mathscr{L}^+$-Korovkin set in $L_p[0, 2\pi]$.

(b) $\{1, x, x^2\}$ is an $\mathscr{L}^+$-Korovkin set in $L_p[0, 1]$.

(c) $\{1, x, y, x^2, y^2\}$ is an $\mathscr{L}^+$-Korovkin set in $L_p([0, 1] \times [0, 1])$.

In the above corollaries the ε-traps constructed are exact in the sense that $e \equiv 0$. Unfortunately such ε-traps cannot generally be constructed.

PROPOSITION 1.6. *If* K *is a finite dimensional subspace of an infinite dimensional* L_p *space, then there is an* $f \in L_p$ *which cannot be bounded above by any* $k \in K$.

Proof. Let $k_1, \cdots, k_n$ be a basis for K, and let $w = \Sigma^n |k_i|$.

If $k \geqq f$ then there is a multiple of w which also bounds f.

If w has a finite range a.e., then the infinite dimensionality of L_p can be used to construct an $f \in L_p$ which cannot be bounded by w. Otherwise looking at level sets we can find a countable family of disjoint measurable sets $A(n)$ such that

$$0 < \int_{A(n)} w^p \leqq \left(\frac{1}{n^3}\right)^p .$$

Let

$$f(x) = \begin{cases} nw(x) & \text{on} \quad A(n) \\ 0 & \text{otherwise} \end{cases}$$

then $f \in L_p$ and cannot be bounded by w.

DEFINITION. For the remainder of this section let V be either c_0 or l_p for some $1 \leqq p < \infty$.

With a series of lemmas we will prove a characterization theorem for finite dimensional $\mathscr{L}^+$-Korovkin sets in V.

DEFINITION. $K \subseteqq V$ contains *essentially positive members* if for every $\varepsilon > 0$, and every integer x there is a $k \in K$ for which

(1) $$k(x) \geqq 1, \quad \text{and}$$

(2) $$\|k \wedge 0\| < \varepsilon .$$

(for example—if K contains a strictly positive function, K contains essentially positive members.)

THEOREM 1.7. *Let K be a finite dimensional subspace of V then*:

(1) *K is an $\mathscr{L}^+$-Korovkin set, and*

(2) *K contains essentially positive members*

if and only if

(3) *K traps every member of V.*

Proposition 1.1 proved that (3) implies (1), and it is trivial that (3) implies (2).

Let K be a linear subspace of V.

Let

$$T = \{f \in V: K \quad \text{traps} \quad f\} .$$

LEMMA 1.8. *T is a closed linear space.*

Proof. Clearly K traps f, implies K traps αf, for all $\alpha \in \boldsymbol{R}$.

Suppose k traps f and g.

Since it is always true that

$$x \wedge y + z = (x + z) \wedge (y + z) ,$$

it follows that

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{s} (u_i + v_j) = \bigwedge_{i=1}^{n} u_i + \bigwedge_{j=1}^{s} v_j .$$

Therefore if $\{u_i\}^n$, $\{l_i\}_{i=1}^m$ and $\{v_j\}_{j=1}^s$, $\{h_j\}_{j=1}^t$ are ε-traps for f and g, then

$$\{u_i + v_j: i = 1, \cdots, n, j = 1, \cdots, s\}$$
$$\{l_i + h_j: i = 1, \cdots, m, \cdots, t\}$$

is a 2ε-trap for $f + g$.

It is also easy to see that T is closed.

LEMMA 1.9. *Let K be an $\mathscr{L}^+$-Korovkin subspace of V. If $p \in V^*$ is nonnegative and $p(k) = (i)$ for some integer i and all k in K then $p = \xi(i)$.*

Proof. Suppose p is as above. Let

$$(Pf)(j) = \begin{cases} f(j) & j \neq i \\ p(f) & j = i \,. \end{cases}$$

Then P carries k onto k for all $k \in K$. Hence P is the identity and $p = \xi(i)$.

In particular K separates the integers.

LEMMA 1.10. *Let K be a subspace of V for which $cbK = \{1, 2, 3, \cdots\}$. For each integer i there is a $k \in K$ for which $k(i) < k(j)$ for all $j \neq i$, and $k(i) < 0$.*

Proof. Let K' be the span of K and 1 in (c). From Lemma 1.2 there is an $\alpha \in \boldsymbol{R}$ and a k in K such that

$$(1) \qquad k(j) + \alpha \geqq 0 \quad \text{for} \quad j \neq i$$

$$(2) \qquad k(i) + \alpha < -1 \,.$$

Since $\lim_{j\to\infty} k(j) = 0$, $\alpha \geqq 0$. Hence this k has the desired properties,

LEMMA 1.11. *Le K be a finite dimensional $\mathscr{L}^+$-Korovkin set in V. Let $w(i)$ be a strictly positive sequence such that $wk \in (c_0)$ for all k in K. Then each integer i is in $cb(wK)$.*

Proof. Let p be a nonnegative sequence in l_1, such that $p(wk) = w(i)k(i)$ for each $k \in K$. Let $g \in V$. Using Caratheodory's theorem, the Hahn-Banach theorem, and the characterization of the extreme points of the unit ball of $(c)^*$, there is a finite set of integers $\{x_j\}_{j=1}^n$ and nonnegative numbers $\{\lambda_j\}_{j=0}^n$ such that,

$$p(f) = \lambda_0 f(\infty) + \sum_{j=1}^{n} \lambda_j f(x_j) \quad \text{for all} \quad f \in wK \oplus g \oplus 1$$

where ∞ denotes the point at infinity.

Let

$$q(t) = \begin{cases} \lambda_j w(x_j)/w(x_i): & \text{for} \quad t = x_j,\ j = 1, \cdots, n \\ 0 & : \ \text{otherwise} \end{cases}$$

Now Lemma 1.9 applies to q, and $p(g) = q(g) = g(i)$. Since g was arbitrary the lemma is proved.

LEMMA 1.12. *Let K be a finite dimensional subspace of V. There is a sequence p such that*

$$(1) \qquad p > 0\,, \quad (2) \quad pK \subseteqq c_0\,, \quad \text{and} \quad (3) \quad \frac{1}{p} \varepsilon V\,.$$

Proof. Let $k_1, \cdots, k_n$ be a basis for K. Let

$$w(x) = \sum_{i=1}^{n} |k_i(x)|\,.$$

It suffices to consider the case in which w has no zeros. It follows that $k(x)/w(x)$ is bounded for each $k \in K$. Thus if there is a $q \in c_0$ such that $w/q \in V$, then

$$p = q \Big/ \sum_{i=1}^{n} |k_i|$$

is the desired function.

To find such a q when V is an l_p space, let $N(\varepsilon)$ be the smallest integer such that

$$\sum_{j>N(\varepsilon)} w(j)^p \leqq \varepsilon\,, \quad \text{and let}$$

$$q(j) = \left(\frac{1}{n}\right)^{1/p} \quad \text{for} \quad N\left(\frac{1}{n^3}\right) \leqq j < N\left(\frac{1}{(n+1)^3}\right).$$

If $V = c_0$, let $N(\varepsilon)$ be the smallest integer such that

$$\sup_{j>N(\varepsilon)} \{|w(j)|\} < \varepsilon\,,$$

then let

$$q(j) = \frac{1}{n} \quad \text{for} \quad N\left(\frac{1}{n^2}\right) \leqq j < N\left(\frac{1}{(n+1)^2}\right).$$

LEMMA 1.13. *Let K be a finite dimensional $\mathscr{L}^+$-Korovkin subspace of V.*

(a) *For each integer i and each $\varepsilon > 0$ there is a $k \varepsilon K$ such that*

$$(1) \qquad k(i) = -1\,, \quad \textit{and}$$

$$(2) \qquad \|k \wedge 0\| < 1 + \varepsilon\,.$$

(b) *If in addition each member of K is also in l_q then the norm in* (2) *can be taken to be the l_q norm.*

Proof. For Lemma 1.12 there is a positive sequence p such that

$pK \subseteqq c_0$ and $1/p \in V$ ($1/p \in l_q$, resp.). We may also assume that $\|1/p\| = 1$ ($\|1/p\|_q = 1$ resp.). Let

$$w(j) = \begin{cases} p(j)/\varepsilon & j \neq i \\ 1 & j = i \end{cases}$$

By Lemma 1.10 and Lemma 1.11 there is a $k \in K$ such that

$$-1 = (wk)(i) < (wk)(j) \quad (j \neq i) .$$

Thus

$$k(i) = -1 , \quad \text{and} \quad k(j) \geqq 1/w(j) .$$

LEMMA 1.14. *Let K be a subspace of V that contains essentially positive function and which satisfies the conclusion of Lemma* 13(a), *then for each i, K traps $\psi_{\{i\}}$.*

Proof. Let $0 < \varepsilon < 1/2$. The lower sequence $\{l_i\}$ for the definition of an ε-trap for $\psi_{\{i\}}$ is guaranteed by hypothesis.

Since K contains essentially positive functions for each integer j there is a $k_j \in K$ such that

$$(1) \qquad k_j(i) = 1 , \quad \text{and}$$

$$(2) \qquad \|k_j \wedge 0\| < \varepsilon/2^{j+1} .$$

Let $m_j \in K$ be a function (guaranteed by hypothesis) such that

$$(3) \qquad m_j(j) = -k_j(j) \wedge 0 , \quad \text{and}$$

$$(4) \qquad \|m_j \wedge 0\| < (\varepsilon/2^{j+1} - m_j(j)) .$$

For $j \neq i$ let,

$$u_j = (k_j + m_j)/[(k_j + m_j)(i)] ,$$

then there is an n for which $\{u_j\}_{j=1, j\neq i}^{n}$ forms the upper sequence in the definition of an ε-trap for $\psi_{\{i\}}$.

Proof of Theorem 1.7. The theorem is now immediate from Lemma 1.14, Lemma 1.13 and Lemma 1.8.

THEOREM 1.15 *Let K be a finite dimensional subspace of l_p that contains a strictly positive function. Then K is $\mathscr{L}^+$-Korovkin if and only if it is an $\mathscr{L}^+$-Korovkin subspace of c_0.*

Proof. The necessity is immediate from Theorem 1.7. The sufficiency follows from Lemma 1.13(b), Lemma 1.14 and Lemma 1.8.

EXAMPLE 1.16. Let $X = \{1/i\}_{i=1}^{\infty} \cup \{0\}$, and let K' be a finite dimensional subspace of $C(X)$ that contains the constants and such that $\{1/i\}_{i=1}^{\infty} \subseteqq cbK$. Let $w \in l_p$.

For $k \in K'$ let

$$(Tk)(i) = w(i)k\left(\frac{1}{i}\right).$$

Then $Tk \in lp$. Let $K = \{Tk: k \in K'\}$. Then in view of Lemma 1.2, K satisfies the conclusion of Lemma 1.13(a) (even with $\varepsilon = 0$). Hence Lemma 1.14 implies that K is an $\mathscr{L}^+$-Korovkin set in l_p. For example, this shows that $K = \{1/i^2, 1/i^3, 1/i^4\}$ is $\mathscr{L}^+$-Korovkin in each l_p, by letting $w(i) = i^2$ and $K' = \{1, x, x^2)\}$.

PROPOSITION 1.17. *If $L_p(X, \Sigma, \mu)$ contains a two-dimensional $\mathscr{L}^+$-Korovkin set, then $L_p(X, \Sigma, \mu)$ is two dimensional.*

Proof. We again use several lemmas. For these let K be a two-dimensional subspace of $L_p = L_p(X, \Sigma, \mu)$.

LEMMA 1.18. *If there exists positive functionals ϕ_1 and ϕ_2 on L_p and a set Y of positive measure such that:*

1. *if $k \in K$, $\phi_1(k) \geqq 0$, and $\phi_2(k) \geqq 0$ then $k \geqq 0$ on Y*
2. *for each pair of real numbers r_1, r_2 there is a $k \in K$ such that $\phi_i(k) = r_i$ and*
3. $\dim L_p|_Y \geqq 3$,

then K is not $\mathscr{L}^+$-Korovkin.

Proof. For f in L_p let Lf be the unique member k of K such that

$$\phi_i(f) = \phi_i(k) \qquad i = 1, 2.$$

Now simply let

$$Pf(x) = \begin{cases} f(x) & x \notin Y \\ (Lf)(x) & x \in Y. \end{cases}$$

Then P is a nontrivial positive operator which acts as the identity on K.

LEMMA 1.19. *Let g be a measurable positive function that is bounded and bounded away from zero. Let*

$$K' = \{gk: k \in K\}$$

then K is $\mathscr{L}^+$-Korovkin if and only if K' is $\mathscr{L}^+$-Korovkin.

Proof. If suffices to show that if K is $\mathscr{L}^+$-Korovkin then K' is also. Let L_n be a bounded sequence of positive operators, such that

$$L_n(k') \longrightarrow k' \quad \text{for each} \quad k' \in K' .$$

Let

$$P_n f = g^{-1} L_n(gf) .$$

Since

$$P_n k \longrightarrow k \quad \text{for all} \quad k \in K ,$$
$$P_n(g^{-1}f) \longrightarrow g^{-1}f \quad \text{for all} \quad f \in L_p .$$

Hence

$$L_n f \longrightarrow f \quad \text{for all} \quad f \in L_p .$$

LEMMA 1.20. *Let $F \subsetneqq X$ be a set of positive measure which is not an atom. If K is $\mathscr{L}^+$-Korovkin then* $\dim K|_F = 2$.

Proof. Again one easily constructs a nontrivial positive operator that is the identity on K.

LEMMA 1.21. *A two-dimensional subspace H of $\boldsymbol{R}^3$ that does not contain a positive vector, has a nonnegative annihilator.*

Proof. Let $a = (a_1 a_2 a_3)$ be an annihilator of H. If H does not have a nonnegative annihilator we may assume that $a_1 > 0 > a_2$. Let $h = (h_1, h_2, h_3)$ be a member in H such that $h_3 = 0$. Then $a(h) = 0$ implies $\operatorname{sgn} h_1 = \operatorname{sgn} h_2$. Since H also contains some vector whose third coordinate is positive, H contains a vector with all positive coordinates.

LEMMA 1.22. *If K is $\mathscr{L}^+$-Korovkin then there is an $F \subsetneqq X$ and a $k \in K$ such that*

1. $\dim L_p|_F \geqq 3$, *and*
2. *k is bounded, positive and bounded away from zero on F.*

Proof. If X is not purely atomic the lemma follows from Lemma 1.20. If X is purely atomic the lemma follows from Lemmas 1.20 and 1.21, since if p is a nonnegative annihilator of K, $Pf = f + p(f)\psi F$ is a positive operator for any set F of finite measure.

Proof of the proposition. Suppose K is $\mathscr{L}^+$-Korovkin. From Lemmas 1.19 and 1.22 we may assume that there is a set $F \subsetneqq X$

such that $\dim(L_p|_F) \geqq 3$, that K is spanned by functions k_1, and k_2, and that k_1 is identically 1 on F. From Lemma 1.20 we can find subsets F_1, F_2 and F_3 of positive finite measure such that

$$\max k_2|_{F_1} < \min k_2|_{F_3} \leqq \max k_2|_{F_3} < \min k_2|_{F_2} \quad \text{a.e.}$$

Furthermore if F is not purely atomic we may assume that $\dim L_p|_{F_3} \geqq 3$. Hence letting $\phi_i f = \int_{F_i} f$ $(i = 1, 2)$, and $Y = F_3$ contradicts Lemma 1.18. If F is purely atomic we may assume that each F_i is an atom, and then letting $\phi_i f = f(F_i)$ and $Y = \bigcup_{i=1}^3 \{F_i\}$ would also contradict Lemma 1.18.

2. Korovkin sets in $C(X)$. Let X be metrizable, and let K be a subspace of $C(X)$ that contains the constants. G. G. Lorentz [14] showed that K is $\mathscr{L}^+$-Korovkin in $C(X)$ if and only if $cbK = X$. It follows that if Y is a closed subset of X then $K|_Y$ is $\mathscr{L}^+$-Korovkin in $C(Y)$. Answering a question by Lorentz, we will give examples of a compact Hausdorff space X, and an $\mathscr{L}^+$-Korovkin sets $K \subseteqq C(X)$ whose restrictions to closed subsets of X fail to be Korovkin. The examples also extend a result by E. Sheffold [19].

DEFINITION. $K \subseteqq C(X)$ is *$\mathscr{L}$-Korovkin for nets* if every bounded net of operators in $\mathscr{L}$ that converges strongly to the identity on K, also converges strongly to the identity on $C(X)$.

LEMMA 2.1. *Let X be a compact Hausdorff space, K is $\mathscr{L}^+$-Korovkin for nets if and only if cb $K = X$.*

Proof. This is a minor variant of known results. The sufficiency can be obtained from the method of proof of Lemma 1 in [Wulbert, 22]. The necessity follows from the following known construction [Lorentz, 14]. Let $\{U_\alpha\}$ be a neighborhood base for a point $x \in X$. Suppose μ is a positive measure in $C(X)^*$ such that

$$k(x) = \int k d\mu \quad \text{for all} \quad k \in K .$$

Let g_α be a continuous function that is 1 at x and vanishes off U_α. Let

$$L_\alpha(f) = (1 - g_\alpha)f + \left(\int f d_\mu\right)g .$$

Then

$$L_\alpha(k) \longrightarrow k \quad \text{for all} \quad k$$

but also

$$(L_\alpha f)(x) \longrightarrow \int f d_\mu .$$

The following is also a variant of the proof in [Wulbert, 22].

LEMMA 2.2. *Let $\{L_n\}$ be a bounded sequence of positive operators on $C(X)$ such that $L_n k \to k$ for all $k \in K \subseteq C(X)$. If Y is a countably compact subset of cbK, then for each $f \in C(X)$, $L_n f$ converges uniformly to f on Y.*

COROLLARY 2.3. *Let X be an open countably compact dense subset of a compact Hausdorff space Y. Assume that $Y - X$ contains two points, and let*

$$K = \{f \in C(Y): f \quad \text{constant on} \quad Y - X\} .$$

Then K is $\mathscr{L}^+$-Korovkin, but not $\mathscr{L}^+$-Korovkin for nets.

EXAMPLES 2.4. (1) Let X be locally compact and countably compact. Let $Y = \beta X$ be the Stone-Čech compactification of X. If $Y - X$ contains two points then X and Y satisfy the conditions of the corollary.

(2) Let W be the space of ordinals less than the first uncountably ordinal. Let $X = W \times W$, then X and $Y = \beta X$ satisfy the properties of part (1) above.

(3) Let Y be an F-space. Let G be a finite subset of Y containing two points, and let $X = Y - G$. Then X and Y satisfy the conditions of the corollary. (See [Gillman and Jerison, 10, p. 215].)

(4) In N denotes the integers then $\beta N - N$ is an F-space.

EXAMPLE 2.5. Let X, Y and K be as in the corollary then K is $\mathscr{L}^+$-Korovkin in $C(Y)$, but $k|_{Y-K}$ is not $\mathscr{L}^+$-Korovkin in $C(Y - X)$.

REMARK 2.5. Let X and Y be as in the corollary and let J be the ideal of continuous functions vanishing on $Y - X$. Let $y \in Y - X$. Since the operator P given by

$$(Pf)(x) = f(x) + f(y)$$

is a positive mapping that acts as the identity on J, J is not an $\mathscr{L}^+$-Korovkin set in $C(Y)$. However it only requires minor modification to show that J is an $\mathscr{L}^1$-Korovkin set, although it is not $\mathscr{L}^1$-Korovkin for nets.

E. Sheffold [19] gave the first example of a set that was an

$\mathscr{L}^1$-Korovkin set but not $\mathscr{L}^1$-Korovkin for nets. Using a different method Sheffold showed that if Y is an F-space, and J is the ideal of all continuous functions vanishing at a single point, then J has the above properties.

R. M. Minkova [15] has proved a Korovkin type theorem involving convergence of the higher order derivatives for functions in $C^r[0, 1]$. Indeed let X be an open-bounded subset of $\boldsymbol{R}^n$. Let Y be the closure of X and let $C^r(X)$ be the continuous real-valued functions on Y, with r bounded, continuous (Frechet) derivatives on X. Let the norm on $C^r(X)$ be the sum of the uniform norms of the derivatives

$$\|f\| = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^n\|_\infty .$$

An operator T on $C(X)$ is *r-smooth* if $T(C^r(X)) \subseteqq C^r(X)$ and T is continuous on $C^r(X)$.

PROPOSITION 2.6. *Let K be a subspace of $C(X)$ that contains the constants and for which cb K is dense in X. Let $\{T_i\}$ be a bounded sequence of positive r-smooth operators on $C(X)$ such that*

(1) *$\{T_i\}$ is uniformly bounded as operators on $C^r(X)$, and*

(2) *$T_i k \to k$ for all $k \in K$,*

then

(3) *$T_i f^{(j)} \to f^{(j)}$ uniformly for each $f \in C^r(X)$, and for each $j = 0, 1, 2, \cdots, r-1$.*

Proof. This easily follows by induction from Ascoli's theorem since in this setting $(T_i f)(x) \to f(x)$ for all $x \in cbK$ (Lemma 2.2).

Minkova used a delicate estimate of Landau to bound the derivative of a function with bounds for the function and its second derivative, and proved the case of the above proposition obtained when X is a compact interval of the line, and K is an $\mathscr{L}^+$-Korovkin set.

3. $\mathscr{L}^{1,+}$-Korovkin sets in L_p. Let (X, Σ, μ) be a finite measure space, and let K be a subspace of $L_1(X, \Sigma, \mu)$ that contains the constants. Let E be the closed linear sublattice generated by K. Since the conditional expectation operator is a contractive projection of L_1 onto E, the $\mathscr{L}^{1,+}$-shadow of K is contained in E. Berens and Lorentz [3] have in fact shown that E is the $\mathscr{L}^{1,+}$-shadow of K. Bernau and Lacey [5] have announced that every closed sublattice of an L_p-space is the range of a contractive projection. Hence the restrictions in the Berens-Lorentz theorem can be removed.

THEOREM 3.1. *Let K be a subset of L_p. The $\mathscr{L}^{1,+}$-shadow of*

K is the closed linear sublattice of L_p generated by K.

Proof. Let S be the $\mathscr{L}^{1,+}$-shadow of K. It is obvious that S is closed. To show S is a lattice it suffices to show that $f \vee g \in S$ when both $f \in S$ and $g \in S$. Let L_i be a sequence of positive contractive on L_p such that $L_p k \to k$ for all $k \in K$. Since $f \vee g$ dominates both f and g

$$L_i(f \vee g) \geqq L_i(f) \vee L_i(g) .$$

We also know that $\|f \vee g\| \geqq \|L_i(f \vee g)\|$ and that

$$L_i(f) \vee L_i(g) \longrightarrow f \vee g .$$

Hence if $f \vee g \geqq 0$, $\lim L_i(f \vee g) = f \vee g$. Indeed, if we are working in L_1, this limit is found by inspecting the integral $\|L_i(f \vee g) - f \vee g\|$. Otherwise the statement follows from the uniform convexity of L_p. Therefore if f and g are arbitrary members of S, $|f| \vee |g| \in S$, and

$$f \vee g + |f| \vee |g| = (f + |f| \vee |g|) \vee (g + |f| \vee |g|) \in S ,$$

thus $f \vee g \in S$.

The $\mathscr{L}^{1,+}$-shadow of K, therefore, contains the closed lattice generated by K. The converse statement is immediate from the result of Bernau and Lacey mentioned before the theorem.

REMARK 3.2. Let X be a compact metric space, and let K be a subspace of $C(X)$ containing the constants. The lattice characterization of the $\mathscr{L}^{1,+}$-shadow of K does not apply. In particular the space spanned by 1 and x is not an $\mathscr{L}^{1,+}$-Korovkin set. However, it does follow from the proof of Lemma 2.1, and Lemma 1.2. that if K is a Korovkin set then, the closed sublattice generated by K is all of $C(X)$.

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