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THE EXISTENCE OF NATURAL FIELD STRUCTURES FOR FINITE DIMENSIONAL VECTOR SPACES OVER LOCAL FIELDS

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Let K be a local field (e.g., a p-adic or p-series field) and n a positive integer. Let K' be the unique (up to isomorphism) unramified extension of K. It is shown that the natural (modular) norm of K' is the nth power of the usual (l^∞) vector space norm of K' when K' is viewed as an n-dimensional vector space over K. Further, the two distinct descriptions of the dual of K' (which is isomorphic to K') that arise from the field model and vector space model are isomorphic under a K-linear isomorphism of K' as a vector space over K, and the isomorphism is norm preserving.

1. If $\boldsymbol{R}^n$ is n-dimensional Euclidean space and $n > 1$, then the only case for which $\boldsymbol{R}^n$ has a (commutative) field structure is $n = 2$. In that case $\boldsymbol{R}^2$ can be identified as the additive group of $\boldsymbol{C}$, the complex numbers, and the norms for $\boldsymbol{R}^2$ and $\boldsymbol{C}$ are compatible in the following sense: Let $(x, y) \in \boldsymbol{R}^2$ and consider the correspondence $(x, y) \leftrightarrow z = x + iy$. The norm of $(x, y) \in \boldsymbol{R}^2$ is $|z|_{R^2} = |(x, y)|_{R^2} = (x^2 + y^2)^{1/2}$. Let dz be Haar measure on $\boldsymbol{C}$. We define $N_C(\mathrm{w}) = w\bar{w}$ and $\mathrm{mod}_C(w)$ by the relation $d(wz) = \mathrm{mod}_C(w)dz$. We obtain, as is well known: $|z|_{R^2}^2 = N_C(z) = \mathrm{mod}_C(z)$.

We will show that if K is a local field (e.g., if K is a p-adic field) and n is an integer greater than 1, then K^n, the n-dimensional vector space over K, has a field structure, as a local field, which is compatible with the usual vector space norm of K^n, in the same sense as above.

The reader is referred to [3; Ch. I] for a review of the basic facts about local fields and to [4; Chs. I-II] for many details and proofs.

2. Let K be a local field; which is to say a locally compact, nondiscrete field that is not connected. The K is totally disconnected. Such a field is either a p-adic field, a finite algebraic extension of a p-adic field or the field of formal Laurent series over a finite field. The ring of integers, $\mathfrak{O}$, in K is the unique maximal compact subring of K. The prime ideal, $\mathfrak{O}$, in $\mathfrak{D}$, is a maximal ideal that is principal, $\mathfrak{O}/\mathfrak{P} \cong GF(q)$, a finite field. There is a norm on K, $|\cdot|_K: K^* \to [0, \infty)$, such that $|x + y|_K \leqq \max[|x|_K, |y|_K]$. (This is known as the ultrametric inequality.) $\mathfrak{O} = \{|x|_K \leqq 1\}$. $\mathfrak{P} = \{|x|_K < 1\}$.

The group of units, $\mathfrak{O}^*$, in K^* (the multiplicative group of K) is $\{|x|_K = 1\}$. The norm, $|\cdot|_K$, arises naturally since $|y|_K = \text{mod}_K(y)$ where $\text{mod}_K(y)$ is the module of the endomorphism $x \to xy$; that is, $\text{mod}_K(0) = 0$ and if $y \neq 0$ then $d(yx) = \text{mod}_K(y)dx$, where dx is Haar measure on K^+, the additive group of K. The n-dimensional vector space over K, K^n, is endowed with a norm as follows: $x = (x_1, \cdots, x_n) \in K^n$, $|x|_{K^n} = \max_k |x_k|_K$. As Weil points out [4, Ch. II § 1], this norm is "natural" in the sense that any K-homogeneous, ultrametric norm on K^n gives rise to the same topology on K^n as $|\cdot|_{K^n}$.

Let n be a positive integer, $n \geqq 2$. If $x \in K^*$ then $|x|_K = q^k$ for some $k \in \boldsymbol{Z}$. Furthermore, the principal ideal $\mathfrak{P}$ is generated by $\mathfrak{p} = \mathfrak{P}$, $|\mathfrak{p}|_K = q^{-1}$. The polynomial $x^n - \mathfrak{p}$ is clearly irreducible over K since if x is a root $|x|_K = q^{-1/n}$, which is impossible. Thus, there is is an algebraic field extensions of K of degree n for all n.

Let $K[\tau]$ be a given finite algebraic field extension of K of degree n. $K[\tau]$ is a local field and is endowed with an (analytically) natural norm, $\text{mod}_{K[\tau]}(\cdot)$. We note that if $y \in K$ then $\text{mod}_{K[\tau]}(y) = |y|_K^n$ [4; p. 6]. If $K[\tau]$ is normal over K then $K[\tau]$ is also endowed with an (algebraically) natural norm as follows: Let A be the automorphism group of $K[\tau]$ over K. Then one defines the norm function $N(y) = \prod_{\alpha \in A} \alpha(y)$. $N(y) \in K$ for all $y \in K[\tau]$ and the norm is defined by $x \to |N(x)|_K$. Clearly, if $x \in K$, $|N(x)|_K = |x|_K^n$. In fact, as is well known, $|N(x)|_K = \text{mod}_{K[\tau]}(x)$ for all $x \in K[\tau]$. This follows easily from the observation that if $x \in K[\tau]$ and $\alpha \in A$, $\text{mod}_{K[\tau]}(\alpha(x)) = \text{mod}_{K[\tau]}(x)$ since automorphisms of local fields have module 1 [4; p. 14].

$$
\begin{aligned}
|N(x)|_K &= \{\text{mod}_{K[\tau]}(N(x))\}^{1/n} \\
&= \{\text{mod}_{K[\tau]}(\textstyle\prod_{\alpha \in A} \alpha(x))\}^{1/n} \\
&= \{\textstyle\prod_{\alpha \in A} \text{mod}_{K[\tau]}(\alpha(x))\}^{1/n} \\
&= \{\text{mod}_{K[\tau]}(x)\}^{n \cdot 1/n} = \text{mod}_{K[\tau]}(x) .
\end{aligned}
$$

If $x \in K[\tau]$, $x = x_1 + x_2\tau + \cdots + x_n\tau^{n-1}$, $x_k \in K$. The correspondence $x_1 + \cdots + x_n\tau^{n-1} \leftrightarrow (x_1, \cdots, x_n)$ is a linear isomorphism of $K[\tau]$ and K^n as vector spaces over K. Using that isomorphism we will denote each element in the corresponding pair with the single symbol x. It would be nice to find an extension $K[\tau]$ of degree n such that $\text{mod}_{K[\tau]}(x) = |x|_{K^n}^n = \max_k |x_k|_K^n$. (Note that this holds for all $x \in K$.)

We can do this with the aid of Corollaries 2-3 in Chapter III § 4 of Weil's book, Basic Number Theory [4]. According to these results, if K is a local field, $n \geqq 2$ is an integer and $\mathfrak{O}/\mathfrak{P} \cong GF(q)$ where q is a power of a prime p, then there is a field K' which is the unique (up to isomorphism) unramified extension of K of degree n, and K' is a cyclic Galois extension of K, $K' = K[\tau]$ where τ is a root of unity (of order prime to p).

We denote $\mathfrak{O}$, $\mathfrak{O}'$ the rings of integers of K and K'; $\mathfrak{P}$, $\mathfrak{P}'$ the prime ideals of $\mathfrak{O}$ and $\mathfrak{O}'$ and we let $\mathfrak{k} = \mathfrak{O}/\mathfrak{P}$, $\mathfrak{k}' = \mathfrak{O}'/\mathfrak{P}'$. From the two corollaries we obtain that $\mathfrak{k}' = \mathfrak{k}[\rho'(\tau)]$ where ρ' is the canonical homomorphism of K' onto $\mathfrak{k}'$ and that $\mathfrak{k}'$ is an extension of $\mathfrak{k}$ of degree n.

THEOREM. *Let $K' = K[\tau]$ be the unramified extension of K of degree n. Then $|N(x)|_K = \text{mod}_{K'}(x) = |x|^n_{K^n}$ for all $x \in K'$.*

It has been suggested that this theorem is well-known to experts. However, no one has yet been able to give a reference for the second of the two equalities. Since this is needed for the applications in § 3 I will sketch a proof.

Proof. Since K' is normal over K we only need to show the second equality; namely,

$$\text{mod}_{K'}(x_1 + x_2\tau + \cdots + x_n\tau^{n-1}) = \max_k [\text{mod}_K(x_k)]^n .$$

(a) $\forall x \in K$, $\text{mod}_{K'}(x) = [\text{mod}_K(x)]^n$. See [4; p. 6]

(b) $\text{mod}_{K'}(\tau) = 1$. Note that τ is a root of unity.

(c) $\text{mod}_{K'}(x) \leqq \max_k [\text{mod}_K(x_k)]^n$. Use the fact that $\text{mod}_{K'}(\cdot)$ is ultrametric and apply (a) and (b).

(d) We may assume, without loss of generality, that $\max_k [\text{mod}_K(x_k)] = 1$ and that at least two coefficients x_k, x_l, $k \neq l$ are such that $\text{mod}_K(x_k) = \text{mod}_K(x_l) = 1$.

The reduction to $\max_k [\text{mod}_k(x_k)] = 1$ is by homogeneity. If there is only one coefficient x_k (say $k = 1$) such that $\text{mod}_K(x_k) = 1$ then the result follows from the ultrametric inequality. For suppose $\text{mod}_K(x_1) = 1$ and $\text{mod}_K(x_k) < 1$, $k \neq 1$. Then from (c) $\text{mod}_{K'}(x_2\tau + \cdots + x_n\tau^{n-1}) < 1$ and from (a) $\text{mod}_{K'}(x_1) = 1$. An easy consequence of the ultrametric inequality is that if $|y_1| \neq |y_2|$ then $\text{mod}_{K'}(y_1 + y_2) = \max [\text{mod}_{K'}(y_1), \text{mod}_{K'}(y_2)]$. Thus $\text{mod}_{K'}(x) = \text{mod}_{K'}(x_1) = 1$.

Hence our result is proved if we show, under the assumptions of (d) that $\text{mod}_{K'}(x) < 1$ will lead to a contradiction.

(e) $\text{mod}_{K'}(x) < 1$ iff $\rho'(x) = 0$. Use the characterization: $\mathfrak{P}' = \{x\colon \text{mod}_{K'}(x) < 1\}$.

(f) $\rho'(x)$ is a polynomial in $\rho'(\tau)$ with coefficients in $\mathfrak{k}$, it is of degree less than n and has at least two nonzero coefficients. This follows from (d) and the remarks preceding the theorem.

(g) The desired contradiction follows from (e) and (f). If $\text{mod}_{K'}(x) < 1$ then $\rho'(\tau)$ is the root of a monic polynomial over $\mathfrak{k}$ of degree less than n. This implies that $[\mathfrak{k}':\mathfrak{k}] < n$, but $[\mathfrak{k}':\mathfrak{k}] = n$. Hence $\text{mod}_{K'}(x) = 1$, which proves the theorem.

3. We now give a few simple consequences of the theorem in § 2.

Throughout this section K is a fixed local field with norm: $|x|_K = \text{mod}_K(x)$, n is an integer greater than 1, $K' = K[\tau]$ is the unramified extension of K of degree n with norm: $|x|_{K'} = \text{mod}_{K'}(x)$, K^n is the n-dimensional vector space over K with norm $|x|_{K^n} = \max_k |x_k|_K$, $x = (x_1, \cdots, x_n)$, $x_k \in K$. As in § 2 if $x \in K' = K[\tau]$ we have $x = x_1 + \cdots + x_n\tau^{n-1}$ and we identify

$$(x \in K') \longleftrightarrow (x = (x_1, \cdots, x_n) \in K^n) \text{ so that } |x|_{K'} = |x|^n_{K^n} .$$

We recall that if $\mathfrak{O}$ is the ring of integers in K, and $\mathfrak{P}$ is the prime ideal in $\mathfrak{O}$ then $\mathfrak{O}/\mathfrak{P} \cong GF(q)$, a finite field. We also have the fractional ideals $\mathfrak{P}^k = \{|x|_K \leqq q^{-k}\}$, $k \in \boldsymbol{Z}$.

In K' we proceed in the same way. Let R be the ring of integers in K', P the prime ideal in R so $R/P \cong GF(q^n)$. The fractional ideals are $P^k = \{|x|_{K'} \leqq (q^n)^{-k}\}$. We note that $R = P^0$, $P = P^1$. Details may be found in [3; Ch. I § 5].

For the vector space K^n one defines a neigeborhood system at 0, with the collection of balls with centers at the origin. Namely, we set $P_1^k = \{|x|_{K^n} \leqq q^{-k}\}$ and then let $R_1 = P_1^0$ and $P_1 = P_1^1$. From the fact that $|x|_{K'} = |x|^n_{K^n}$ it follows that $P_1^k = P^k$ for all $k \in \boldsymbol{Z}$ and hence $R_1 = R$, $P_1 = P$. Consequently we drop the subscripts. See [3; ch. III § 1] for details of this construction for K^n.

As additive groups (and as n-dimensional vector spaces over K), K' and K^n agree so additive harmonic analysis, Haar measure, etc., all agree on these two different models for K^n. We now examine the two different descriptions of the dual of K^n that arise from the two models.

We fix a character on K^+ that is trivial on $\mathfrak{O}$, but is nontrivial on $\mathfrak{P}^{-1}$. This character is denoted χ. (See [3; Ch. I § 5] for details.) The dual of K^n is put into a linear isomorphism with K^n, as a vector space over K, by the identification $y \leftrightarrow \chi_y^1$, $\chi_y^1(x) = \chi(x \cdot y) = \chi(x_1y_1 + \cdots + x_ny_n)$.

The dual of K' (as an additive group) is put into a linear isomorphism with the additive group of K' as follows: One first defines the trace function, $Tr(x) = \sum_{\alpha \in A} \alpha(x)$, where A is the automorphism group of K' over K. It is known that Tr maps K' onto K [4; p. 139] and since K' is unramified over K we have that Tr maps P^k onto $\mathfrak{P}^k$ for all k [4; p. 141]. The dual of K' is then identified with K' by the correspondence $y \leftrightarrow \chi_y^2$, $\chi_y^2(x) = \chi(Tr(xy))$.

Thus, given any $y \in K'$, there is an $L(y) \in K'$ such that $\chi_y^1 = \chi_{L(y)}^2$, which is to say

$$\chi(x_1y_1 + \cdots + x_2y_2) = \chi(Tr(xL(y))) \text{ for all } x \in K' ,$$

and the map $y \leftrightarrow L(y)$ is a K-linear isomorphism of K' (or, more properly, of the dual of the additive group of K'). Moreover, this linear map preserves the norm of y; that is, $|L(y)|_{K'} = |y|_{K'}$ for all $y \in K'$.

We first note that $\chi_y^1 \equiv 1$ iff $y = 0$ and if $|y|_{K'} = q^{kn}$, then χ_y^1 is trivial on P^k but is nontrivial on P^{k-1}. (See [3; Ch. III § 1].) From the fact that Tr maps P^k onto $\mathfrak{P}^k$ and the fact that χ is trivial on $\mathfrak{O}$ but is nontrivial on $\mathfrak{P}^{-1}$ we see that $\chi_{L(y)}^2 \equiv 1$ iff $L(y) = 0$ and that if $|L(y)|_{K'} = q^{ln}$, then $\chi_{L(y)}^2$ is trivial on P^l but is nontrivial on P^{l-1}. Thus, $|L(y)|_{K'} = |y|_{K'}$.

Therefore, these two representations of the dual of K' as an additive group have the same induced norm and hence the same induced metric.

Note also that the prime ideal P is generated by *any* element $p \in P$ such that $|p|_{K'} = q^{-n}$. $\mathfrak{P}$ is generated by $\mathfrak{p} \in \mathfrak{P}$, where $|\mathfrak{p}|_K = q^{-1}$. But $\mathfrak{p} \in P$ and $|\mathfrak{p}|_{K'} = |\mathfrak{p}|_K^n = q^{-n}$, so P is generated in R by the same element, $\mathfrak{p}$, that generates $\mathfrak{P}$ in $\mathfrak{O}$.

These last few results are simply the working out of notational consequences of the identity $|x|_{K^n}^n = |x|_{K'}$.

When we study Calderón-Zygmund kernels on K we look at functions of the form $\Omega(x)/|x|_K$ where $\Omega(x)$ is homogeneous of degree 0 in the sense that $\Omega(\mathfrak{p}^k x) = \Omega(x)$, $\forall x \in K$, $k \in \boldsymbol{Z}$ [3; Ch. VI § 4]. Thus, on K' we examine functions of the form $\Omega(x)/|x|_{K'}$ where Ω is homogeneous of degree 0 in the sense that $\Omega(\mathfrak{p}^k x) = \Omega(x)$ for all $x \in K'$, $k \in \boldsymbol{Z}$ and "$\mathfrak{p}^k x$" is multiplication of $x \in K'$ by $\mathfrak{p}^k \in K'$.

When we examine such kernels on K^n, the functions are of the form $\Omega(x)/|x|_{K^n}^n$ where Ω is homogeneous of degree zero in the sense that $\Omega(\mathfrak{p}^k x) = \Omega(x)$ for all $x \in K^n$, $k \in \boldsymbol{Z}$ and "$\mathfrak{p}^k x$" is scalar multiplication of $x \in K^n$ by $\mathfrak{p}^k \in K$. But these two "multiplications" agree and since $|x|_{K^n}^n = |x|_{K'}$ the classes of kernels that would arise from these two approaches to K^n are the same class.

We will continue the analysis of these kernels a little further. Note that $R^* = \{|x|_{K'} = 1\}$ is a multiplicative group. It is the group of units in $(K')^*$. We consider (as in [2] and [3; Ch. II § 4]) the collection $\{\pi_{kl}\}_{k=0,\, l=0}^{\infty,\, l_k}$ of unitary multiplicative characters on R^*, where π_{kl} is ramified of degree k and $l_k = q^{kn}(1 - q^{-n})^2$, $k \geqq 2$, $l_0 = 1$, $l_1 = q^n - 2$. $\{(1 - q^{-n})\pi_{kl}\}$ is a complete orthonormal system on R^* and π_{kl} is the local field analogue of a spherical harmonic of degree k.

Consider $\Omega(x)/|x|^n_{K^n}$ as above with $\int_{R^*} \Omega(x)dx = 0$. Then Ω can be considered as a function on R^* and we may write, formally,

$$\Omega(x) \sim \sum_{k=1}^{\infty} \sum_{l=1}^{l_k} c_{kl}\pi_{kl}(x) \text{ and so}$$
$$\Omega(x)/|x|^n_{K^n} \sim \sum\sum c_{kl}\pi_{kl}(x)/|x|_{K'} .$$

The Fourier transform of the principal value distribution induced by $\Omega(x)/|x|_{K'}$ is a function which is homogeneous of degree zero. Call that function $\hat{\Omega}$. Using the results for the gamma function [3; Ch. II § 5] it is easy to see that $\hat{\Omega}(x) \sim \sum\sum c_{kl}\Gamma(\pi_{kl})\pi_{kl}^{-1}(x)$. That is, the map $\Omega \to \hat{\Omega}$ is essentially, a multiplier transform on the group R^* and the behaviour of the operator depends on the properties of the distribution $M(x) \sim \sum_{k>1} \Gamma(\pi_{kl}^{-1})\pi_{kl}(x)$.

If convolution by the principal value distribution induced by $\Omega(x)/|x|_{K'}$ is a bounded operator on any L^p space, then it is bounded on L^2 and this implies that $\hat{\Omega}$ is bounded. What conditions on Ω imply that $\hat{\Omega}$ is bounded? By the usual arguments for multipliers we see that $\hat{\Omega}$ is bounded whenever $\Omega \in L^2(R^*)$ implies that $M \in L^2(R^*)$. But $|\Gamma(\pi_{kl})| = q^{-kn/2}$ [3; Ch. II § 5] and since $l_k = q^{kn}(1 - q^{-n})^{-2}$, $k \geqq 2$, we see that $M \notin L^2(R^*)$. (See [2] for details and extensions.)

Similarly $\hat{\Omega}$ is bounded whenever $\Omega \in L^\infty(R^*)$ implies that M is a finite Borel measure. When q is odd, a careful examination shows that M is not a finite Borel measure and thus the singular integral operator $f \to f*(\text{P.V. } \Omega(x)/|x|_{K'})$ is not necessarily bounded on $L^2(K')$ when $\Omega \in L^\infty(R^*)$. The same result also follows for Ω continuous on R^*. (This is the essential part of Daley's argument in [1].)

As a final example, we state an especially simple *F.* and *M.* Riesz theorem for K^n. Let q be odd, $\mathfrak{O}/\mathfrak{P} \cong GF(q)$ and n be any positive integer. Then there is a singular integral operator of the Calderón-Zygmund type, $f \to \tilde{f} = f*(P.V.\ \Omega(x)/|x|^n_{K^n})$ with the following property. If μ is a finite Borel measure and $\tilde{\mu}$ is a finite Borel measure, then μ is absolutely continuous. Viewed from the perspective of K' we choose $\Omega(x) = \pi(x)$ where π is any unitary character on R^*, π ramified of degree 1, homogeneous of degree 0 and odd. This was shown by Chao for $n = 1$ [3; Ch. VII § 3].

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