

BOUNDS FOR NUMBERS OF GENERATORS OF COHEN-MACAULAY IDEALS

JUDITH D. SALLY

Let (R, \underline{m}) be a local Cohen-Macaulay ring of dimension d and multiplicity $e(R) = e$. A natural question to ask about an \underline{m} -primary ideal I is whether there is any relation between the number of generators of I and the least power t of \underline{m} contained in I . (t will be called the nilpotency degree of R/I) It is quite straight forward to obtain a bound for $v(I)$, the number of generators in a minimal basis of I , in terms of t and e . However, there are several interesting applications. The first is the existence of a bound for the number of generators of any Cohen-Macaulay ideal I , i.e. any ideal I such that R/I is Cohen-Macaulay, in terms of $e(R/I)$, $e(R)$ and height I . The second application is a bound in terms of d and e for the reduction exponent of \underline{m} .

1. \underline{m} -primary ideals. In this section we will use only the standard facts about the existence and properties of superficial elements. However, later we will need a result stronger than the usual existence theorem for these elements so we take this opportunity to recall the definition and prove this special form of the existence theorem.

DEFINITION. Let (R, \underline{m}) be a local ring. An element x in \underline{m} is superficial for \underline{m} if there is an integer $c > 0$ such that

$$(\underline{m}^n : x) \cap \underline{m}^c = \underline{m}^{n-1} \quad \text{for all } n > c.$$

It is a standard fact that x is superficial for \underline{m} if and only if there is an integer $c > 0$ such that $0 \neq \bar{X} \in \underline{m}/\underline{m}^2 = G_1$ and

$$(0 : \bar{x}G) \cap G_n = 0$$

for $n \geq c$ where $G_n = \underline{m}^n/\underline{m}^{n+1}$, and $G = G_0 \oplus G_1 \oplus \dots$.

LEMMA 1.1. *Let (R, \underline{m}) be a local ring with R/\underline{m} infinite. Let I, J_1, \dots, J_s be distinct ideals of R which are also distinct from \underline{m} . Then there is an element x in R such that*

- (1) $x \notin J_i$, $i = 1, \dots, s$
- (2) x is a superficial element for \underline{m}

and

- (3) the image of x in a superficial element for \underline{m}/I .

Proof. Let $G = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \dots$ and $\bar{G} = R/\underline{m} \oplus \underline{m}/\underline{m}^2 + I \oplus$

$\underline{m}^2 + I/\underline{m}^3 + I \oplus \dots$. Then $\bar{G} = G/K$ where K is a homogeneous ideal of G . Let $\text{Ass } G = \{P_1, \dots, P_l\}$, $\text{Ass } \bar{G} = \{Q_1/K, \dots, Q_t/K\}$, where the Q_i are primes in G . Suppose that $P_i = \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \dots$ and $Q_i = P_i/K$. The following subspaces are all proper R/\underline{m} -subspaces of $\underline{m}/\underline{m}^2$:

$$J_i + \underline{m}^2/\underline{m}^2, i = 1, \dots, n; P_i \cap (\underline{m}/\underline{m}^2), i = 1, \dots, t - 1$$

$$Q_i \cap (\underline{m}/\underline{m}^2), i = 1, \dots, l - 1; I + \underline{m}^2/\underline{m}^2.$$

Since R/\underline{m} is infinite there is a nonzero \bar{x} in $\underline{m}/\underline{m}^2$ such that \bar{x} is not in any of these subspaces. We claim that if x is any element of \underline{m} which maps to \bar{x} , then x is superficial for \underline{m} and the image of x in R/I is superficial for \underline{m}/I . We need $(0: \bar{G}\bar{x}) \cap \underline{m}^n/\underline{m}^{n+1} = 0$ and $(0: \bar{G}(x + I/I)) \cap \underline{m}^n + I/\underline{m}^{n+1} + I = 0$ for large n . Let $0 = N_1 \cap \dots \cap N_{t-1} \cap N_t$ with N_i P_i -primary be a primary decomposition of 0 in G . Then $(0: G\bar{x}) \subseteq N_1 \cap \dots \cap N_{t-1}$. But $(\underline{m}/\underline{m}^2)^c \subseteq N_t$ for some c , hence $(0: G\bar{x}) \cap \underline{m}^n/\underline{m}^{n+1} = 0$ for $n \geq c$. The same reasoning shows that the image of x in R/I is a superficial element for \underline{m}/I .

THEOREM 1.2. *Let (R, \underline{m}) be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an \underline{m} -primary ideal and t the nilpotency degree of R/I . Then*

$$v(I) \leq t^{d-1}e(R) + d - 1.$$

Proof. The proof is by induction on d . If $d = 1$, the theorem is well-known (cf. [6] or [7]) but we include a proof for completeness. We may assume that R/\underline{m} is infinite so that \underline{m} has a superficial element x which is also a nonzero divisor. Since $d = 1$, $x\underline{m}^n = \underline{m}^{n+1}$ for some $n > 0$. We have $\lambda(R/xR) = \lambda(I/xI) = e(R)$, where $\lambda(B)$ denotes the length of an R -module B . The exact sequence

$$0 \longrightarrow \underline{m}I/xI \longrightarrow I/xI \longrightarrow I/\underline{m}I \longrightarrow 0$$

gives $\lambda(I/\underline{m}I) = \lambda(I/xI) - \lambda(\underline{m}I/xI) = e(R) - \lambda(\underline{m}I/xI)$.

Assume $d > 1$. Again assuming that R/\underline{m} is infinite as we may, there is a nonzero divisor x such that x is a superficial element for \underline{m} . Pass to the $d - 1$ dimensional Cohen-Macaulay ring R/x^t . I/x^t is \underline{m}/x^t -primary so, by induction,

$$v(I/x^t) \leq t^{d-2}e(R/x^t) + d - 2.$$

Hence $v(I) \leq v(I/x^t) + 1 \leq t^{d-2}te(R) + d - 1$.

REMARKS. 1. If (R, \underline{m}) is regular local and $d \geq 2$ then $v(I) \leq gt^{d-2} + d - 1$, where g is the degree of I , i.e. $I \subseteq \underline{m}^g \setminus \underline{m}^{g+1}$.

2. (1.2) generalizes a result of Abhyankar [1]. In [1], Abhyankar

shows that the Cohen-Macaulay hypothesis is necessary.

If (R, \underline{m}) is a d -dimensional local ring, the Hilbert function H of \underline{m} is defined as follows:

$$H(n) = v(\underline{m}^n) = \lambda(\underline{m}^n/\underline{m}^{n+1})$$

for integers $n \geq 0$. For large n , $H(n)$ is a polynomial of degree $d - 1$. If we apply (1.2) to $I = \underline{m}^n$ we obtain, for Cohen-Macaulay rings, a polynomial of degree $d - 1$ which bounds $H(n)$ for all n .

COROLLARY 1.3. *Let (R, \underline{m}) be a Cohen-Macaulay local ring of multiplicity e and dimension $d > 0$. Then, if $P(n)$ is the polynomial, $P(n) = en^{d-1} + d - 1$, $H(n) \leq P(n)$ for all $n > 0$.*

Using a trick of Kirby [4] we have

COROLLARY 1.4. *Let I be an ideal ideal of a d -dimensional Cohen-Macaulay local ring (R, \underline{m}) , $d > 0$. Define the Artin-Rees number of I , $a(I)$, to be the least integer a such that $I \cap \underline{m}^a \subseteq I\underline{m}$. Then*

$$v(I) \leq a^{d-1}e(R) + d - 1 .$$

Proof. $I/I\underline{m} = I/I\underline{m} + I \cap \underline{m}^a = I/I \cap I\underline{m} + \underline{m}^a \cong I + \underline{m}^a/I\underline{m} + \underline{m}^a$. Hence $v(I) \leq v(I + \underline{m}^a) = \lambda(I + \underline{m}^a/\underline{m}(I + \underline{m}^a))$. So by (1.2), $v(I) \leq a^{d-1}e(R) + d - 1$.

Note that if R/I is Cohen-Macaulay, then $a(I) \leq e(R/I) + 1$ but in general $a(I)$ is not bounded by $e(R/I)$.

2. Applications. If (R, \underline{m}) is a local ring, an ideal I is a Cohen-Macaulay ideal if R/I is a Cohen-Macaulay ring.

THEOREM 2.1. *Let (R, \underline{m}) be a d -dimensional Cohen-Macaulay local ring. Let I be a Cohen-Macaulay ideal of height $h > 0$. Then*

$$v(I) \leq e(R/I)^{h-1}e(R) + h - 1 .$$

Proof. We may assume R/\underline{m} is infinite. The proof is by induction on $s = \dim R/I$. If $s = 0$ then by (1.2) it is sufficient to note that $e(R/I) = \lambda(R/I) \geq \text{nilpotency degree of } R/I$.

Assume $s > 0$. By (1.1) there is a nonzero divisor x in \underline{m} such that x is superficial for \underline{m} , and the image of x in R/I is a nonzero divisor in R/I and is a superficial element for \underline{m}/I . We pass to the $d - 1$ dimensional Cohen-Macaulay ring R/x and to the height h Cohen-Macaulay ideal $(I, x)/x$. By induction $v(I) = v((I, x)/x) \leq e(R/(I, x))^{h-1}e(R/x) + h - 1 = e(R/I)^{h-1}e(R) + h - 1$.

REMARKS 1. For height 1 ideals I , (2.1) gives Rees' theorem [6] stating that $v(I) \leq e(R)$. If height $I = 2$, Rees [6] has the result $v(I) \leq e(R) + e(R/I)$ which gives a better bound than (2.1) except when R or R/I is regular.

2. If R is an equicharacteristic regular local ring, Becker [2] has results similar to (2.1).

Another application of (1.2) gives a bound for what we will call the reduction exponent $r(\underline{m})$ of \underline{m} , where \underline{m} is the maximal ideal of a d -dimensional Cohen-Macaulay local ring (R, \underline{m}) . Assume that R/\underline{m} is infinite. $r(\underline{m})$ is the least integer r such that there exists a system of parameters $\bar{x}_1, \dots, \bar{x}_d$ of degree 1 in $G = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \dots$ with $\mathcal{M}^r \subseteq (\bar{x}_1, \dots, \bar{x}_d)$ where $\mathcal{M} = \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \dots$.

THEOREM 2.2. *Let (R, \underline{m}) be a d -dimensional local Cohen-Macaulay ring of multiplicity $e(R)$ with R/\underline{m} infinite and $d > 0$. Then*

$$r(\underline{m}) \leq d! e(R) - 1.$$

Proof. In our case the main theorem of Eakin and Sathaye in [3] states that $H(n) < \binom{n+d}{d}$ implies $r(\underline{m}) \leq n$. By Corollary 1.3, $H(n) \leq n^{d-1}e(R) + d - 1$. So it is sufficient to note that

$$(d! e(R))^{d-1} e(R) + d - 1 < \binom{d! e(R) + d}{d}.$$

This generalizes a result in [7] where R was assumed to be of dimension 1.

REFERENCES

1. S. Abhyankar, *Local rings of high embedding dimension*, A. J. M., **89** (1967), 1073-1077.
2. J. Becker, *On the boundedness and unboundedness of the number of generators of ideals and multiplicity*, preprint.
3. P. Eakin and A. Sathaye, *Prestable ideals*, to appear in J. of Algebra.
4. D. Kirby, *The reduction number of a one dimensional local ring*, J. London Math. Soc., (2) **10** (1975), 471-481.
5. D. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Camb. Phil. Soc., **50** (1954), 145-158.
6. D. Rees, *Estimates for the minimum number of generators of modules over Cohen-Macaulay local rings*, preprint.
7. J. Sally and W. Vasconcelos, *Stable rings*, J. of Pure and Appl. Algebra, **4** (1974), 319-336.

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NORTHWESTERN UNIVERSITY